

# Natural solutions of rational Stieltjes moment problems

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*Report TW 575, August 2010*



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## Abstract

In the strong or two-point Stieltjes moment problem, one has to find a positive measure on  $[0, \infty)$  for which infinitely many moments are prescribed at the origin and at infinity. Here we consider a multipoint version in which the origin and the point at infinity are replaced by sequences of points that may or may not coincide. In the indeterminate case, two natural solutions  $\mu_0$  and  $\mu_\infty$  exist that can be constructed by a limiting process of approximating quadrature formulas. The supports of these natural solutions are disjoint (with possible exception of the origin). The support points are accumulation points of sequences of zeros of even and odd indexed orthogonal rational functions. These functions are recursively computed and appear as denominators in approximants of continued fractions. They replace the orthogonal Laurent polynomials that appear in the two-point case. In this paper we consider the properties of these natural solutions and analyse the precise behaviour of which zero sequences converge to which support points.

**Keywords :** Stieltjes moment problem, orthogonal polynomials, orthogonal rational functions, continued fractions.

**MSC :** Primary : 30D15,

# Natural solutions of rational Stieltjes moment problems

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## Abstract

In the strong or two-point Stieltjes moment problem, one has to find a positive measure on  $[0, \infty)$  for which infinitely many moments are prescribed at the origin and at infinity. Here we consider a multipoint version in which the origin and the point at infinity are replaced by sequences of points that may or may not coincide. In the indeterminate case, two natural solutions  $\mu_0$  and  $\mu_\infty$  exist that can be constructed by a limiting process of approximating quadrature formulas. The supports of these natural solutions are disjoint (with possible exception of the origin). The support points are accumulation points of sequences of zeros of even and odd indexed orthogonal rational functions. These functions are recursively computed and appear as denominators in approximants of continued fractions. They replace the orthogonal Laurent polynomials that appear in the two-point case. In this paper we consider the properties of these natural solutions and analyse the precise behaviour of which zero sequences converge to which support points.

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## 1. Introduction

The *Stieltjes transform*  $S(z, \mu)$  of a finite measure  $\mu$  on the real line  $(-\infty, \infty)$  is here defined as

$$S(z, \mu) = \int_{-\infty}^{\infty} \frac{d\mu(t)}{z - t}. \quad (1.1)$$

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The function  $S(z, \mu)$  is holomorphic outside the support  $\text{supp } \mu$  of  $\mu$  and maps the open upper half-plane into the lower closed half-plane.

The *Stieltjes moment problem* for a sequence  $\{c_n\}_{n=0}^{\infty}$  of real numbers consists of finding positive measures  $\mu$  with  $\text{supp } \mu \subset [0, \infty)$  such that

$$\int_0^{\infty} t^n d\mu(t) = c_n \quad \text{for } n = 0, 1, 2, \dots \quad (1.2)$$

An important tool in the treatment of the moment problem is the study of *orthogonal polynomial sequences*  $\{\varphi_n\}_{n=0}^{\infty}$  associated with the moments. It is known that the zeros  $x_{n,k}$  of  $\varphi_n$  are simple and contained in  $(0, \infty)$ . When the zeros are numbered such that  $x_{n,1} < x_{n,2} < \dots < x_{n,n}$ , then the zeros of  $\varphi_n$  separate the zeros of  $\varphi_{n+1}$  in the sense that

$$x_{n+1,1} < x_{n,1} < x_{n+1,2} < \dots < x_{n,n} < x_{n+1,n+1}. \quad (1.3)$$

It follows that each sequence  $\{x_{n,k}\}_n$  is decreasing to a value  $\xi_k \in [0, \infty)$ , while each sequence  $\{x_{n,n-k}\}_n$  is increasing to a value  $\eta_k \in (0, \infty]$ ,  $k = 0, 1, 2, \dots$

The polynomials  $\varphi_n$  are the canonical denominators of a continued fraction (equivalent to a *real Jacobi fraction* or a *Grommer fraction*). The approximants of the continued fraction converge to  $S(z, \mu_0)$  for a solution  $\mu_0$  of the moment problem. When the moment problem is *indeterminate* (i.e., has more than one solution), the polynomials  $\varphi_n$  suitably normalized converge locally uniformly in the complex plane  $\mathbb{C}$  to an entire function  $\Phi_0$ , which has as its zeros exactly the values  $\xi_k$  (while  $\eta_k = \infty$  for all  $k$ ). Furthermore the set  $\cup_{k=0}^{\infty} \{\xi_k\}$  constitutes the support of  $\mu_0$ .

The real Jacobi fraction is the even contraction of another continued fraction (equivalent to a *Stieltjes fraction*). The canonical denominators  $\psi_n$  of the odd contraction of this continued fraction have similar properties to those of the  $\varphi_n$ , and give rise to a solution  $\mu_{\infty}$  of the moment problem. The approximants of this continued fraction converge to  $S(z, \mu_{\infty})$ . The measures  $\mu_0$  and  $\mu_{\infty}$  are called *natural solutions* of the Stieltjes moment problem.

For relevant discussions of the Stieltjes moment problem we refer to [3], [9, Ch. 1-2], [16, Sections 42-44], [17, Chapters I,IV,VII].

The *strong* (or *two-point*) *Stieltjes moment problem* for a sequence  $\{c_n\}_{n=-\infty}^{\infty}$  of real numbers consists of finding positive measures  $\mu$  with  $\text{supp } \mu \subset [0, \infty)$  such that

$$\int_0^{\infty} t^n d\mu(t) = c_n \quad \text{for } n = 0, \pm 1, \pm 2, \dots \quad (1.4)$$

This concept was introduced in [11], where basic results on existence and uniqueness were obtained. For material relevant to the discussion of strong Stieltjes moment problems below, we refer to [1], [2], [3], [10], [11], [15], [14].

In a way similar to orthogonal polynomials in the classical situation, *orthogonal Laurent polynomials*  $\varphi_n$  play a central role in the theory of strong moment problems. As in the classical case, the zeros of  $\varphi_n$  are simple and contained in  $(0, \infty)$ , and the zeros of  $\varphi_n$  separate those of  $\varphi_{n+1}$ . When numbered as above, each sequence  $\{x_{n,k}\}_n$  decreases to a value  $\xi_k \in [0, \infty)$  and each sequence  $\{x_{n,n-k}\}_n$  increase to a value  $\eta_k \in (0, \infty]$ ,  $k = 0, 1, 2, \dots$

The functions  $\varphi_n$  are the canonical denominators of a continued fraction (equivalent to a *positive Thron fraction*). The even and odd approximants of the continued fraction converge to  $S(z, \mu_0)$

and  $S(z, \mu_\infty)$ , where  $\mu_0$  and  $\mu_\infty$  are solutions of the moment problem. The measures  $\mu_0$  and  $\mu_\infty$  are called *natural solutions* of the strong Stieltjes moment problem. When the problem is *indeterminate*, the sequences  $\{\varphi_{2m}\}$  and  $\{\varphi_{2m+1}\}$ , suitably normalized, converge locally uniformly in  $\mathbb{C} \setminus \{0\}$  to distinct functions  $\Phi_0$  and  $\Phi_\infty$ . The zeros of  $\Phi_0$  and the zeros of  $\Phi_\infty$  are disjoint. The zeros of  $\Phi_0$  together with the origin constitute  $\text{supp } \mu_0$ , while the zeros of  $\Phi_\infty$  together with the origin constitute  $\text{supp } \mu_\infty$ . The connection between the limits of zero sequences of  $\{\varphi_n\}$  and the zeros of  $\Phi_0$  and  $\Phi_\infty$  is radically different from that in the classical situation. Because of the disjointness of the supports of  $\mu_0$  and  $\mu_\infty$  (apart from the origin), we have  $\xi_k = 0$  and  $\eta_k = \infty$  for all  $k$ . The support of  $\mu_0$  (resp.  $\mu_\infty$ ) consists of all accumulation points of zeros of  $\varphi_{2m}$  (resp.  $\varphi_{2m+1}$ ), but in general no results are known concerning *how* the zeros of the orthogonal functions approach the support points of the natural solutions.

When the recurrence relation for the orthogonal Laurent polynomials (equivalently for the positive Thron fraction) has a special form, it has been shown that every sequence of zeros of  $\varphi_{2m}$  (resp.  $\varphi_{2m+1}$ ) keeping a fixed position relative to the middle zeros of  $\varphi_{2m+1}$  converges to the zeros of  $\Phi_0$  (resp.  $\Phi_\infty$ ), i.e., to the support points of  $\mu_0$  (resp.  $\mu_\infty$ ) in  $(0, \infty)$ . For these results we refer to [1], [2], [3].

In the present paper we consider certain *rational Stieltjes moment problems* (which in general split in a weaker and a stronger form). Here polynomials or Laurent polynomials are replaced by rational functions with two sequences  $\{\alpha_n\}$  and  $\{\beta_n\}$  of prescribed poles in  $(-\infty, 0)$  (the  $\{\alpha_n\}$  playing the role of the origin and the  $\{\beta_n\}$  playing the role of infinity in the two-point case). The *orthogonal rational functions*  $\varphi_n$  associated with the moment problem are the canonical denominators of a continued fraction (a *multipoint Padé continued fraction*). The even and the odd approximants also in this situation converge to  $S(z, \mu_0)$  and  $S(z, \mu_\infty)$ , where the measures  $\mu_0$  and  $\mu_\infty$  are solutions related to the moment problem. These measure  $\mu_0$  and  $\mu_\infty$  are called *natural solutions* for the moment problem.

We show that in the *indeterminate* case the orthogonal rational functions  $\varphi_{2m}$  and  $\varphi_{2m+1}$ , suitably normalized, converge under certain conditions locally uniformly outside the closure of the set of interpolation points  $\{\alpha_n\}, \{\beta_n\}$ . Furthermore the zeros of the limit functions  $\Phi_0$  and  $\Phi_\infty$  constitute together with the origin the support of  $\mu_0$  and  $\mu_\infty$ . Finally we prove a result on the convergence of sequences of zeros of  $\varphi_{2m}$  and  $\varphi_{2m+1}$  to the support points of  $\mu_0$  and  $\mu_\infty$ , analogous to the result in the two-point situation.

For the theory of rational moment problems and orthogonal rational functions in general and in the Stieltjes situation (support in  $[0, \infty)$ ) in particular we refer to [5], [6], [7], [8] and further references found there.

We shall in this paper mainly use notation as in [8], with some deviations.

## 2. Rational Stieltjes moment problems

In [8] a general theory of rational moment problems on the non-negative real axis is presented, with underlying assumptions gradually strengthened during the exposition, according to the need in the various arguments. Cf. also [6], [7]. In this and the following sections we repeatedly refer to concepts and results in [8], but we shall consider one and the same setting in the whole paper.

For more general expositions of rational moment problems on the real line and on the unit circle, we refer to [5, Chapter 11], and further references found there.

Our assumptions are the following. Let two sequences  $\{\alpha_n\}_{n=1}^{\infty}$  and  $\{\beta_n\}_{n=1}^{\infty}$  of real numbers be given, and set  $r_{2m}(z) = (\beta_m - z)$  for  $m = 1, 2, \dots$ ,  $r_{2m+1}(z) = (\alpha_{m+1} - z)$  for  $m = 0, 1, 2, \dots$ ,  $r_0 = 1$ ,  $r_{-1} = 1$ . Furthermore we assume there are two numbers  $\alpha, \beta$  in  $(-\infty, 0]$  such that  $-\infty < \beta_j \leq \beta < \alpha \leq \alpha_k \leq 0$  for all  $j, k$ . We set  $A = \{a \in \mathbb{R} : \alpha_k = a \text{ for some } k\}$ ,  $B = \{b \in \mathbb{R} : \beta_k = b \text{ for some } k\}$ ,  $S = \text{cl}(A) \cup \text{cl}(B)$  where  $\text{cl}(\cdot)$  denotes closure of the set. We use  $\bar{\cdot}$  to denote the complex conjugate.

We may look upon points  $\alpha_n$  as playing the role of the origin and the points  $\beta_n$  as playing the role of infinity in the two-point situation. In [8] values  $\beta_k = -\infty$  and  $\beta = -\infty$  are allowed. Since the development then becomes more cumbersome, we shall not include this possibility in this paper. Thus for purely technical reasons, the theory of strong or two-point moment problems will not be formally covered in this presentation. Even more, we shall assume that all  $\beta_k$  stay away from  $-\infty$ , i.e., we assume that  $S$  is compact.

We set

$$D_n(z) = r_1(z)r_2(z) \cdots r_n(z) \text{ for } n = 1, 2, \dots; \quad D_0 = 1. \quad (2.1)$$

Let  $\Pi_n$  denote the space of polynomials of degree at most  $n$ , and define the spaces  $\mathcal{L}_n$  and  $\mathcal{L}$  by

$$\mathcal{L}_n = \left\{ \frac{p(z)}{D_n(z)} : p \in \Pi_n \right\}, \quad \mathcal{L} = \bigcup_{n=0}^{\infty} \mathcal{L}_n. \quad (2.2)$$

We shall also make use of the space  $\mathcal{L} \cdot \mathcal{L}$  consisting of all products of two functions in  $\mathcal{L}$ . We note that if for every  $a \in A$ ,  $b \in B$ , we have  $\alpha_k = a$ ,  $\beta_k = b$  for infinitely many  $k$ , then  $\mathcal{L} \cdot \mathcal{L} = \mathcal{L}$ . This is in particular the case of each of the sets  $A, B$  consists of a finite number of points repeated periodically.

Let  $M$  be a positive definite Hermitian linear functional on  $\mathcal{L} \cdot \mathcal{L}$ . Thus  $\overline{M[g]} = M[\bar{g}]$  for every  $g \in \mathcal{L} \cdot \mathcal{L}$  and  $M[f \cdot \bar{f}] > 0$  for every  $f \in \mathcal{L}$ ,  $f \neq 0$ . We shall for convenience normalize such that  $M[1] = 1$ . A measure  $\mu$  with  $\text{supp } \mu \subset [0, \infty)$  is said to solve the *rational Stieltjes moment problem on  $\mathcal{L}$*  if all functions in  $\mathcal{L}$  are absolutely integrable with respect to  $\mu$  and

$$M[g] = \int_0^{\infty} g(t) d\mu(t), \text{ for all } g \in \mathcal{L}. \quad (2.3)$$

The measure is said to solve the *rational Stieltjes moment problem on  $\mathcal{L} \cdot \mathcal{L}$*  if all functions in  $\mathcal{L} \cdot \mathcal{L}$  are absolutely integrable with respect to  $\mu$  and

$$M[h] = \int_0^{\infty} h(t) d\mu(t) \text{ for all } h \in \mathcal{L} \cdot \mathcal{L}. \quad (2.4)$$

An equivalent formulation which may make clear the use of the expression *moment problem* can be described as follows. Let  $\{\Omega_n\}_{n=0}^{\infty}$  be some basis for  $\mathcal{L}$  where  $M[\Omega_n]$  is real for all  $n$ , for example the basis described in [8, Section 13]. Define the *moments*  $c_{j,k}$  as

$$c_{j,k} = M[\Omega_j \Omega_k], \text{ for } j, k = 0, 1, 2, \dots \quad (2.5)$$

Note the symmetry  $c_{j,k} = c_{k,j}$ . Then  $\mu$  solves the moment problem on  $\mathcal{L}$  if and only if

$$\int_0^\infty \Omega_j(t) d\mu(t) = c_{j,0} \text{ for } j = 0, 1, 2, \dots, \quad (2.6)$$

and  $\mu$  solves the moment problem on  $\mathcal{L} \cdot \mathcal{L}$  if and only if

$$\int_0^\infty \Omega_j(t) \Omega_k(t) d\mu(t) = c_{j,k} \text{ for } j, k = 0, 1, 2, \dots \quad (2.7)$$

We denote by  $\mathcal{M}_+(\mathcal{L})$  the set of all solutions of the moment problem on  $\mathcal{L}$ , and by  $\mathcal{M}_+(\mathcal{L} \cdot \mathcal{L})$  the set of all solutions of the moment problem on  $\mathcal{L} \cdot \mathcal{L}$ . A rational moment problem is said to be *determinate* if it has a unique solution, *indeterminate* if it has more than one solution.

### 3. Orthogonal rational functions

Let  $M$  be a given positive linear functional on  $\mathcal{L} \cdot \mathcal{L}$ , and assume that the rational Stieltjes moment problem on  $\mathcal{L} \cdot \mathcal{L}$  is solvable.

We define the inner product  $\langle \cdot, \cdot \rangle$  on  $\mathcal{L}$  by

$$\langle g, h \rangle = M[g \cdot \bar{h}], \quad g, h \in \mathcal{L}. \quad (3.1)$$

Equivalently  $\langle g, h \rangle = \int_0^\infty g(t) \overline{h(t)} d\mu(t)$ , where  $\mu$  is any measure in  $\mathcal{M}_+(\mathcal{L} \cdot \mathcal{L})$ . We can construct an *orthonormal sequence*  $\{\varphi_n\}_{n=1}^\infty$  in  $\mathcal{L}$  such that  $\varphi_0 \in \mathcal{L}_0$ ,  $\varphi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$  for  $n = 1, 2, \dots$  (for example by Gram-Schmidt orthonormalization of the basis  $\{\Omega_n\}$  in [8, Section 13]). Note that  $\varphi_n$  is unique up to a unimodular constant factor.

We may write

$$\varphi_n(z) = \frac{P_n(z)}{D_n(z)}, \quad \text{where } P_n \in \Pi_n \setminus \Pi_{n-1}. \quad (3.2)$$

All the zeros of the polynomial  $P_n$  are simple and contained in  $(0, \infty)$ . We may then normalize  $\varphi_n$  such that  $P_n(x) < 0$  for  $x \in (-\infty, 0)$ .

The functions  $\sigma_n$  of the second kind are defined by

$$\sigma_n(z) = M_t \left[ \frac{\varphi_n(t) - \varphi_n(z)}{t - z} \right], \quad \text{for } n = 0, 1, 2, \dots \quad (3.3)$$

(Note that  $\varphi(t) = \frac{\varphi_n(t) - \varphi_n(z)}{t - z}$  belongs to  $\mathcal{L}_n$  and that  $M_t$  means that the linear functional  $M$  operates on the variable  $t$ ). Thus

$$\sigma_n(z) = \int_0^\infty \frac{\varphi_n(t) - \varphi_n(z)}{t - z} d\mu(t) \quad (3.4)$$

for any  $\mu \in \mathcal{M}_+(\mathcal{L})$ .

The functions  $\sigma_n, \varphi_n$  satisfy a three-term recurrence relation of the form

$$\begin{bmatrix} \sigma_0 & \sigma_{-1} \\ \varphi_0 & \varphi_{-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} \sigma_n(z) \\ \varphi_n(z) \end{bmatrix} = b_n(z) \begin{bmatrix} \sigma_{n-1}(z) \\ \varphi_{n-1}(z) \end{bmatrix} + a_n(z) \begin{bmatrix} \sigma_{n-2}(z) \\ \varphi_{n-2}(z) \end{bmatrix}, \quad n = 1, 2, \dots \quad (3.5)$$

where

$$a_1(z) = \frac{W_1}{r_1(z)}, \quad b_1(z) = \frac{Q_1 z + R_1}{r_1(z)}, \quad a_n(z) = \frac{W_n r_{n-2}(z)}{r_n(z)}, \quad b_n(z) = \frac{Q_n r_{n-2}(z) + R_n r_{n-1}(z)}{r_n(z)}, \quad n = 2, 3, \dots \quad (3.6)$$

Thus

$$\begin{aligned} a_1(z) &= \frac{W_1}{\alpha_1 - z}, & b_1(z) &= \frac{Q_1 z + R_1}{\alpha_1 - z}, \\ a_2(z) &= \frac{W_2}{\beta_1 - z}, & b_2(z) &= \frac{Q_2 + R_2(\alpha_1 - z)}{\beta_1 - z}, \\ a_{2m}(z) &= \frac{W_{2m}(\beta_{m-1} - z)}{\beta_m - z}, & b_{2m}(z) &= \frac{Q_{2m}(\beta_{m-1} - z) + R_{2m}(\alpha_m - z)}{\beta_m - z}, \quad m = 2, 3, \dots \\ a_{2m+1}(z) &= \frac{W_{2m+1}(\alpha_m - z)}{\alpha_{m+1} - z}, & b_{2m+1}(z) &= \frac{Q_{2m+1}(\alpha_m - z) + R_{2m+1}(\beta_m - z)}{\alpha_{m+1} - z}, \quad m = 1, 2, \dots \end{aligned} \quad (3.7)$$

Here  $Q_n, R_n, W_n$  are constants. (Note that  $Q_1$  and  $R_1$  have different meaning from that in [8], these coefficients corresponding to  $U_1, V_1$  in [8].)

The signs of the coefficients are as follows (see [8, Section 9]):

$$\begin{aligned} Q_{2m} &< 0, \quad Q_{2m+1} > 0, \quad \text{for all } m, \\ R_{2m} &> 0, \quad R_{2m+1} < 0, \quad \text{for all } m, \\ W_n &< 0, \quad \text{for all } n. \end{aligned} \quad (3.8)$$

(Note that the result is obvious for  $R_1$  and  $Q_1$  from our normalization which assumes that the numerator of  $\varphi_n = P_n/D_n$  satisfies  $P_n(x) < 0$  if  $x \in (-\infty, 0)$ .)

It follows from (3.8) that  $b_{2m} < 0, b_{2m+1} > 0$  for  $x \in (\beta, \alpha)$ .

The *determinant formula* for the orthonormal functions has the form

$$\sigma_n(z) \varphi_{n-1}(z) - \sigma_{n-1}(z) \varphi_n(z) = (-1)^{n-1} a_1(z) a_2(z) \cdots a_n(z). \quad (3.9)$$

This means

$$\begin{aligned} \sigma_{2m}(z) \varphi_{2m-1}(z) - \sigma_{2m-1}(z) \varphi_{2m}(z) &= -\frac{W_1 W_2 \cdots W_{2m}}{(\beta_m - z)(\alpha_m - z)}, \\ \sigma_{2m+1}(z) \varphi_{2m}(z) - \sigma_{2m}(z) \varphi_{2m+1}(z) &= \frac{W_1 W_2 \cdots W_{2m+1}}{(\beta_m - z)(\alpha_{m+1} - z)}. \end{aligned} \quad (3.10)$$

The *confluent Christoffel-Darboux formula* has the form

$$r_n(z) \varphi_n(z) [r_{n-1}(z) \varphi_{n-1}(z)]' - [r_n(z) \varphi_n(z)]' r_{n-1}(z) \varphi_{n-1}(z) = (-1)^{n-1} W_1 W_2 \cdots W_n \sum_{k=0}^{n-1} \varphi_k(z)^2. \quad (3.11)$$

This means

$$\begin{aligned}
& (\beta_m - z)\varphi_{2m}(z)[(\alpha_m - z)\varphi_{2m-1}(z)]' - [(\beta_m - z)\varphi_{2m}(z)]'(\alpha_m - z)\varphi_{2m-1}(z) \\
& \qquad \qquad \qquad = -W_1W_2\cdots W_{2m} \sum_{k=0}^{2m-1} \varphi_k(z)^2, \\
& (\alpha_{m+1} - z)\varphi_{2m+1}(z)[(\beta_m - z)\varphi_{2m}(z)]' - [(\alpha_{m+1} - z)\varphi_{2m+1}(z)]'(\beta_m - z)\varphi_{2m}(z) \\
& \qquad \qquad \qquad = W_1W_2\cdots W_{2m+1} \sum_{k=0}^{2m} \varphi_k(z)^2.
\end{aligned} \tag{3.12}$$

For all these formulas, see [8].

#### 4. Continued fractions

For general concepts and results concerning continued fractions we refer to [12], [16].

The recurrence relation (3.5) defines a continued fraction  $\mathbf{K}_{n=1}^{\infty} \left( \frac{a_n}{b_n} \right)$ , with approximants  $\frac{\sigma_n}{\varphi_n}$ . (When convenient, we omit the argument  $z$  in formulas in the following.) The *odd contraction*  $\mathbf{K}_{m=1}^{\infty} \left( \frac{u_m}{v_m} \right)$  of  $\mathbf{K}_{n=1}^{\infty} \left( \frac{a_n}{b_n} \right)$  has approximants  $\frac{\sigma_{2m+1}}{\varphi_{2m+1}}$  and satisfies the recurrence relation

$$\begin{bmatrix} \sigma_{2m+1} \\ \varphi_{2m+1} \end{bmatrix} = v_m \begin{bmatrix} \sigma_{2m-1} \\ \varphi_{2m-1} \end{bmatrix} + u_m \begin{bmatrix} \sigma_{2m-3} \\ \varphi_{2m-3} \end{bmatrix}, \tag{4.1}$$

where

$$u_m = -\frac{a_{2m-1}a_{2m}b_{2m+1}}{b_{2m-1}}, \quad v_m = \frac{b_{2m-1}b_{2m}b_{2m+1} + a_{2m}b_{2m+1} + a_{2m+1}b_{2m-1}}{b_{2m-1}} \tag{4.2}$$

for  $m = 1, 2, \dots$ . See e.g., [12, Section 2.2.4].

In particular, this gives in our situation

$$u_m = -\frac{(\alpha_{m-1} - z)(\beta_{m-1} - z)b_{2m+1}(z)}{(\alpha_m - z)(\beta_m - z)b_{2m-1}(z)}. \tag{4.3}$$

We shall make use of these formulas in Section 8. Note that for each  $m$ ,  $b_{2m+1}(z)$  has at most one zero, which is real. The odd contraction is defined only for those  $z$  which are not a zero for any  $b_{2m+1}(z)$ .

A continued fraction  $\mathbf{K}_{n=1}^{\infty} \left( \frac{c_n}{d_n} \right)$  is said to be *equivalent* to the continued fraction  $\mathbf{K}_{n=1}^{\infty} \left( \frac{a_n}{b_n} \right)$  if the  $n$ th approximant of  $\mathbf{K}_{n=1}^{\infty} \left( \frac{c_n}{d_n} \right)$  equals the  $n$ th approximant of  $\mathbf{K}_{n=1}^{\infty} \left( \frac{a_n}{b_n} \right)$ . It will be convenient in the development of our results to make use of a continued fraction of the form  $\mathbf{K}_{n=1}^{\infty} \left( \frac{1}{d_n} \right)$ , equivalent to  $\mathbf{K}_{n=1}^{\infty} \left( \frac{a_n}{b_n} \right)$ . The elements (recurrence coefficients)  $d_n(z)$  have the form

$$d_n = \rho_n b_n, \tag{4.4}$$

where

$$\rho_1 = 1, \quad \rho_{2m+1} = \frac{a_2 a_4 \cdots a_{2m}}{a_1 a_3 \cdots a_{2m+1}}, \quad \rho_{2m} = \frac{a_1 a_3 \cdots a_{2m-1}}{a_2 a_4 \cdots a_{2m}}, \quad m = 1, 2, \dots \quad (4.5)$$

See e.g., [12, Section 2.2.2]. We shall denote the canonical numerators and denominators of  $\mathbf{K}_{n=1}^\infty \left( \frac{1}{d_n} \right)$  as  $\Sigma_n(z)$  and  $\Phi_n(z)$ . Then

$$\Phi_n = \rho_1 \rho_2 \cdots \rho_n \varphi_n, \quad \Sigma_n = \rho_1 \rho_2 \cdots \rho_n \sigma_n. \quad (4.6)$$

In our case this gives

$$d_{2m} = \frac{W_1 W_3 \cdots W_{2m-1} r_{2m}}{W_2 W_4 \cdots W_{2m} r_{2m-1}} b_{2m}, \quad d_{2m+1} = \frac{W_2 W_4 \cdots W_{2m} r_{2m+1}}{W_1 W_3 \cdots W_{2m+1} r_{2m}} b_{2m+1}. \quad (4.7)$$

Written out this formula gives

$$d_{2m} = \frac{W_1 W_3 \cdots W_{2m-1}}{W_2 W_4 \cdots W_{2m}} \cdot \frac{Q_{2m}(\beta_{m-1} - z) + R_{2m}(\alpha_m - z)}{\alpha_m - z}, \quad (4.8)$$

$$d_{2m+1} = \frac{W_2 W_4 \cdots W_{2m}}{W_1 W_3 \cdots W_{2m+1}} \cdot \frac{Q_{2m+1}(\alpha_m - z) + R_{2m+1}(\beta_m - z)}{\beta_m - z},$$

$$\Phi_{2m}(z) = \frac{\beta_m - z}{W_2 W_4 \cdots W_{2m}} \varphi_{2m}(z), \quad \Sigma_{2m}(z) = \frac{\beta_m - z}{W_2 W_4 \cdots W_{2m}} \sigma_{2m}(z) \quad (4.9)$$

$$\Phi_{2m+1}(z) = \frac{\alpha_{m+1} - z}{W_1 W_3 \cdots W_{2m+1}} \varphi_{2m+1}(z), \quad \Sigma_{2m+1}(z) = \frac{\alpha_{m+1} - z}{W_1 W_3 \cdots W_{2m+1}} \sigma_{2m+1}(z).$$

Note that  $\Phi_n$  and  $\varphi_n$  have the same zeros, and  $\Sigma_n$  and  $\sigma_n$  have the same zeros, since  $\beta_m - z$  is a factor in  $D_{2m}(z)$  and  $\alpha_{m+1} - z$  is a factor in  $D_{2m+1}(z)$ .

From the established relationships between  $\varphi_n$  and  $\Phi_n$  and  $\sigma_n$  and  $\Sigma_n$ , the determinant formula (3.10) takes the form

$$\begin{aligned} \Sigma_{2m} \Phi_{2m-1} - \Sigma_{2m-1} \Phi_{2m} &= -1 \\ \Sigma_{2m+1} \Phi_{2m} - \Sigma_{2m} \Phi_{2m+1} &= 1, \end{aligned} \quad (4.10)$$

while the confluent Christoffel-Darboux formula (3.12) takes the form

$$\begin{aligned} \Phi_{2m} \Phi'_{2m-1} - \Phi'_{2m} \Phi_{2m-1} &= - \sum_{k=0}^{2m-1} \varphi_k^2 \\ \Phi_{2m+1} \Phi'_{2m} - \Phi'_{2m+1} \Phi_{2m} &= \sum_{k=0}^{2m} \varphi_k^2. \end{aligned} \quad (4.11)$$

We state a few crucial facts as propositions.

**Proposition 4.1.** *Let  $x \in (\alpha, \beta)$ . Then  $d_n(x) > 0$  for all  $n$ .*

PROOF. This result follows from (3.8), (4.8) and the assumption  $\alpha_k \geq \alpha, \beta_k \leq \beta$ . ■

**Proposition 4.2.** *The following pairs of functions have no common zero:*

- (i)  $\Phi_n$  and  $\Phi_{n-1}$
- (ii)  $\Sigma_n$  and  $\Sigma_{n-1}$
- (iii)  $\Phi_n$  and  $\Sigma_n$ .

PROOF. This follows immediately from (4.10). ■

**Proposition 4.3.** *Between two consecutive zeros of  $\Phi_{n+1}$ , there is exactly one zero of  $\Phi_n$ .*

PROOF. This result follows by a standard argument from (4.11), where we take into account the intermediate value theorem for continuous functions, the facts that the zeros of all  $\Phi_n$  are simple and contained in  $(0, \infty)$ , and the fact that  $\sum_{k=0}^{\infty} \varphi_k(x)^2$  is always positive for real  $x$ . ■

## 5. Natural solutions

For all the results in this section we refer especially to [8, Sections 12-13]. See also [6], [7].

Let  $x_{n,1}, x_{n,2}, \dots, x_{n,n}$  be the zeros of  $\varphi_n$ . Then there exist positive weights  $\lambda_{n,1}, \lambda_{n,1}, \dots, \lambda_{n,n}$  such that

$$\int_0^{\infty} f(t) d\mu(t) = \sum_{k=1}^n \lambda_{n,k} f(x_{n,k}) \quad (5.1)$$

for all  $f \in \mathcal{L}_n \cdot \mathcal{L}_{n-1}$ ,  $\mu \in \mathcal{M}_+(\mathcal{L} \cdot \mathcal{L})$ . For a given  $z$ , the function  $\varphi(t) = \frac{\varphi_n(t) - \varphi_n(z)}{t - z}$  belongs to  $\mathcal{L}_n$  and therefore to  $\mathcal{L}_n \cdot \mathcal{L}_{n-1}$ . Hence the above quadrature formula implies that  $\sigma_n(z) = \sum_{k=1}^n \lambda_{n,k} \frac{\varphi_n(x_{n,k}) - \varphi_n(z)}{x_{n,k} - z}$  (recall the definition (3.3)-(3.4)), and thus

$$\sigma_n(z) = \varphi_n(z) \sum_{k=1}^n \frac{\lambda_{n,k}}{z - x_{n,k}}, \quad (5.2)$$

since  $\varphi_n(x_{n,k}) = 0$ . Let  $\mu_n$  be the discrete measure with mass of size  $\lambda_{n,k}$  at the point  $x_{n,k}$  for  $k = 1, 2, \dots, n$ . Then we may write (5.2) as

$$\frac{\Sigma_n(z)}{\Phi_n(z)} = \frac{\sigma_n(z)}{\varphi_n(z)} = \int_0^{\infty} \frac{d\mu_n(t)}{z - t} = S(z, \mu_n). \quad (5.3)$$

By using the determinant formula (3.10) and the formula (see e.g., [12, (1.2.10)])

$$\frac{\sigma_n(z)}{\varphi_n(z)} - \frac{\sigma_{n-2}(z)}{\varphi_{n-1}(z)} = b_n(z) \frac{\varphi_{n-1}(z)}{\varphi_n(z)} \left[ \frac{\sigma_{n-1}(z)}{\varphi_{n-1}(z)} - \frac{\sigma_{n-2}(z)}{\varphi_{n-2}(z)} \right] \quad (5.4)$$

it can be shown that the sequence  $\left\{ \frac{\Sigma_{2m}(x)}{\Phi_{2m}(x)} \right\} = \left\{ \frac{\sigma_{2m}(x)}{\varphi_{2m}(x)} \right\}$  decreases on  $(\beta, \alpha)$ , and the sequence  $\left\{ \frac{\Sigma_{2m+1}(x)}{\Phi_{2m+1}(x)} \right\} = \left\{ \frac{\sigma_{2m+1}(x)}{\varphi_{2m+1}(x)} \right\}$  increases on  $(\beta, \alpha)$ . Furthermore it follows from (3.10) that  $\frac{\sigma_{2p}(x)}{\varphi_{2p}(x)} > \frac{\sigma_{2q+1}(x)}{\varphi_{2q+1}(x)}$  for any  $p, q$  and  $x \in (\beta, \alpha)$ . In particular  $\left\{ \frac{\sigma_{2m}(x)}{\varphi_{2m}(x)} \right\}$  is bounded below on  $(\beta, \alpha)$  and  $\left\{ \frac{\sigma_{2m+1}(x)}{\varphi_{2m+1}(x)} \right\}$  is bounded above on  $(\beta, \alpha)$ . Consequently the two sequences converge on  $(\beta, \alpha)$ . By an argument using properties of normal families, taking into account the quadrature formula

(5.2), it follows that  $\left\{ \frac{\sigma_{2m}(z)}{\varphi_{2m}(z)} \right\}$  and  $\left\{ \frac{\sigma_{2m+1}(z)}{\varphi_{2m+1}(z)} \right\}$  converge locally uniformly in  $\mathbb{C} \setminus [0, \infty)$  to holomorphic functions  $F_0(z)$  and  $F_\infty(z)$ . By a standard compactness argument applied to the sequence of measures  $\{\mu_n\}$ , it follows that the functions  $F_0$  and  $F_\infty$  are Stieltjes transforms of measures  $\mu_0$  and  $\mu_\infty$  with support in  $[0, \infty)$  representing the functional  $M$  on  $\mathcal{L}$ . I.e.,

$$F_0(z) = S(z, \mu_0), \quad F_\infty(z) = S(z, \mu_\infty) \quad (5.5)$$

where  $\mu_0, \mu_\infty \in \mathcal{M}_+(\mathcal{L})$ .

The measures  $\mu_0$  and  $\mu_\infty$  are called *natural solutions* of the moment problem on  $\mathcal{L}$ .

Our aim is to investigate structure and properties of the support of  $\mu_0$  and  $\mu_\infty$ . We state as a proposition a preliminary result in this direction.

We define

$$Z_n = \text{The set of all zeros of } \varphi_n. \quad (5.6)$$

**Proposition 5.1.** *Every point in  $\text{supp } \mu_0$  (resp.  $\text{supp } \mu_\infty$ ) is an accumulation point for sequences in  $\cup_{m=1}^\infty Z_{2m}$  (resp.  $\cup_{m=0}^\infty Z_{2m+1}$ ).*

PROOF. This is a direct consequence of the construction of  $\mu_0$  and  $\mu_\infty$  through weak star convergence of  $\{\mu_{2m}\}$  and  $\{\mu_{2m+1}\}$ . ■

We also state as a proposition a fact that will be crucial in the discussion in Section 6.

**Proposition 5.2.** *For any  $\mu \in \mathcal{M}_+(\mathcal{L} \cdot \mathcal{L})$  we have*

$$F_\infty(z) = S(x, \mu_\infty) \leq S(x, \mu) \leq S(x, \mu_0) = F_0(x) \quad (5.7)$$

for  $x \in (\beta, \alpha)$ .

PROOF. The result follows from [8, Theorem 13.2]. ■

The functions  $F_0(z) = S(z, \mu_0)$  and  $F_\infty(z) = S(z, \mu_\infty)$  map the open upper half-plane into the open lower half-plane (none of the functions being a constant). Consequently the singularities of  $F_0$  and  $F_\infty$  in  $(0, \infty)$  are simple poles in  $\mathbb{R}$ .

## 6. Indeterminate problems

We shall in this section consider a rational Stieltjes moment problem on  $\mathcal{L} \cdot \mathcal{L}$  which is indeterminate, i.e.,  $\mathcal{M}_+(\mathcal{L} \cdot \mathcal{L})$  contains more than one element. We shall make essential use of the continued fraction  $\mathbf{K}_{n=1}^\infty \left( \frac{1}{d_n} \right)$  introduced in Section 4. Recall that the even and odd approximants  $\frac{\Sigma_{2m}(z)}{\Phi_{2m}(z)}$  and  $\frac{\Sigma_{2m+1}(z)}{\Phi_{2m+1}(z)}$  converge to the Stieltjes transforms  $S(z, \mu_0)$  and  $S(z, \mu_\infty)$ .

**Proposition 6.1.** *For  $x \in (\beta, \alpha)$  the series  $\sum_{n=1}^\infty d_n(x)$  converges.*

PROOF. Since the moment problem on  $\mathcal{L} \cdot \mathcal{L}$  is indeterminate, it follows from Proposition 5.2 that the functions  $S(x, \mu_0)$  and  $S(x, \mu_\infty)$  do not coincide for  $x \in (\beta, \alpha)$ . This implies that the sequence  $\left\{ \frac{\Sigma_n(x)}{\Phi_n(x)} \right\}$  does not converge. In other words, the continued fraction  $\mathbf{K}_{n=1}^\infty \left( \frac{1}{d_n(x)} \right)$  does not converge. Since  $d_n(x) > 0$  for  $x \in (\beta, \alpha)$  by Proposition 4.1, the series  $\sum_{n=1}^\infty d_n(x)$  converges according to *Van Vleck's criterion*. see e.g., [12, p. 142].  $\blacksquare$

It follows from (4.8) and Proposition 6.1 that for  $x \in (\beta, \alpha)$  we have

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{W_1 W_3 \cdots W_{2m-1}}{W_2 W_4 \cdots W_{2m}} \left[ Q_{2m} \frac{\beta_{m-1} - x}{\alpha_m - x} + R_{2m} \right] &< \infty, \\ \sum_{m=1}^{\infty} \frac{W_2 W_4 \cdots W_{2m}}{W_1 W_3 \cdots W_{2m+1}} \left[ Q_{2m+1} \frac{\alpha_m - x}{\beta_m - x} + R_{2m+1} \right] &< \infty. \end{aligned} \quad (6.1)$$

We wish to establish convergence of the sequence  $\{\Phi_{2m+1}(z)\}$ ,  $\{\Sigma_{2m+1}(z)\}$ ,  $\{\Phi_{2m}(z)\}$ , and  $\{\Sigma_{2m}(z)\}$ . We shall show that this holds true for an indeterminate problem for  $z$  in compact subsets of  $\mathbb{C} \setminus S$  with  $S$  compact. This will be a generalization of the treatment of the two-point situation (see [1], [2], [3]).

We shall use the following simple observation. Let  $K$  be a compact subset of  $\mathbb{C} \setminus S$  and define

$$\delta = \inf\{|z - \gamma| : z \in K, \gamma \in S\} \quad \text{and} \quad \Delta = \sup\{|z - \gamma| : z \in K, \gamma \in S\}. \quad (6.2)$$

then it is clear that for  $z \in K$

$$\frac{\delta}{\Delta} \leq \left| \frac{\beta_{m-1} - z}{\alpha_m - z} \right| \leq \frac{\Delta}{\delta} \quad \text{and} \quad \frac{\delta}{\Delta} \leq \left| \frac{\alpha_m - z}{\beta_m - z} \right| \leq \frac{\Delta}{\delta}. \quad (6.3)$$

The following proposition will be an essential tool.

**Proposition 6.2.** *Suppose that all  $\beta_k$  are in a compact subset of  $(-\infty, \beta]$ , then the rational Stieltjes moment problem on  $\mathcal{L} \cdot \mathcal{L}$  is indeterminate if and only if there exist finite numbers  $\Gamma_o$ ,  $\Gamma_e$ ,  $\Lambda_o$  and  $\Lambda_e$  such that*

$$\sum_{m=1}^{\infty} \left| R_{2m} \frac{W_1 W_3 \cdots W_{2m-1}}{W_2 W_4 \cdots W_{2m}} \right| \leq \Gamma_e < \infty, \quad \sum_{m=1}^{\infty} \left| R_{2m+1} \frac{W_2 W_4 \cdots W_{2m}}{W_1 W_3 \cdots W_{2m+1}} \right| \leq \Gamma_o < \infty, \quad (6.4)$$

and

$$\sum_{m=1}^{\infty} \left| Q_{2m} \frac{W_1 W_3 \cdots W_{2m-1}}{W_2 W_4 \cdots W_{2m}} \right| \leq \Lambda_e < \infty, \quad \sum_{m=1}^{\infty} \left| Q_{2m+1} \frac{W_2 W_4 \cdots W_{2m}}{W_1 W_3 \cdots W_{2m+1}} \right| \leq \Lambda_o < \infty. \quad (6.5)$$

PROOF. For  $x \in (\beta, \alpha)$  it holds by (3.8) that

$$\left| Q_{2m} \frac{\beta_{m-1} - x}{\alpha_m - x} + R_{2m} \right| \geq |R_{2m}| \quad \text{and} \quad \left| Q_{2m+1} \frac{\alpha_m - x}{\beta_m - x} + R_{2m+1} \right| \geq |R_{2m+1}|.$$

So that for an indeterminate problem, (6.1) immediately implies

$$\sum_{m=1}^{\infty} \left| R_{2m} \frac{W_1 W_3 \cdots W_{2m-1}}{W_2 W_4 \cdots W_{2m}} \right| \leq \Lambda_e < \infty \quad \text{and} \quad \sum_{m=1}^{\infty} \left| R_{2m+1} \frac{W_2 W_4 \cdots W_{2m}}{W_1 W_3 \cdots W_{2m+1}} \right| \leq \Lambda_o < \infty.$$

From (6.1) and (6.3), it follows that for  $z \in K \cap (\beta, \alpha)$

$$\sum_{m=1}^{\infty} \left| Q_{2m} \frac{W_1 W_3 \cdots W_{2m-1}}{W_2 W_4 \cdots W_{2m}} \frac{\delta}{\Delta} \right| \leq \sum_{m=1}^{\infty} \left| Q_{2m} \frac{W_1 W_3 \cdots W_{2m-1}}{W_2 W_4 \cdots W_{2m}} \right| \left| \frac{\beta_{m-1} - z}{\alpha_m - z} \right| < \infty$$

This means that there exists a finite  $\Gamma_e$  such that

$$\sum_{m=1}^{\infty} \left| Q_{2m} \frac{W_1 W_3 \cdots W_{2m-1}}{W_2 W_4 \cdots W_{2m}} \right| \leq \Gamma_e < \infty,$$

and similarly, there is a finite constant  $\Gamma_o$  such that

$$\sum_{m=1}^{\infty} \left| Q_{2m+1} \frac{W_2 W_4 \cdots W_{2m}}{W_1 W_3 \cdots W_{2m+1}} \right| \leq \Gamma_o < \infty.$$

This concludes the proof in one direction.

In the opposite direction we should prove that (6.4) and (6.5) imply the indeterminacy of the rational Stieltjes moment problem on  $\mathcal{L} \cdot \mathcal{L}$ . Using (6.3), (6.4) and (6.5) it is easy to show that for  $z \in K$

$$\sum_{m=1}^{\infty} d_{2m}(z) \leq \sum_{m=1}^{\infty} R_{2m} \frac{W_1 W_3 \cdots W_{2m-1}}{W_2 W_4 \cdots W_{2m}} + \sum_{m=1}^{\infty} \left| Q_{2m} \frac{W_1 W_3 \cdots W_{2m-1}}{W_2 W_4 \cdots W_{2m}} \right| \frac{\Delta}{\delta} \leq \Lambda_e + \Gamma_e \frac{\Delta}{\delta} = D_e < \infty.$$

Similarly

$$\sum_{m=1}^{\infty} d_{2m+1}(z) \leq \Lambda_o + \Gamma_o \frac{\Delta}{\delta} = D_o < \infty.$$

Hence also  $\sum_n d_n(z) < D_e + D_o$  will converge for  $z \in K$ . Thus  $\mathbf{K} \left( \frac{1}{d_n(z)} \right)$  diverges [12, p. 142]

while we know that the two sequences  $\left\{ \frac{\Sigma_{2m+1}}{\Phi_{2m+1}} \right\}$  and  $\left\{ \frac{\Sigma_{2m}}{\Phi_{2m}} \right\}$  converge on subsets of  $\mathbb{C} \setminus [0, \infty)$  to  $F_\infty$  and  $F_0$  respectively. Since the continued fraction diverges, these two limiting functions must be distinct, which means that the moment problem is indeterminate.  $\blacksquare$

The proof of the next theorem now follows closely the arguments used in [4, Theorem 2.8].

**Theorem 6.3.** *Let the rational Stieltjes moment problem on  $\mathcal{L} \cdot \mathcal{L}$  be indeterminate and assume that the set of poles is contained in a compact set  $S = \text{cl}(A) \cup \text{cl}(B)$ . Then the sequences  $\{\Phi_{2m}(z)\}$ ,  $\{\Phi_{2m+1}(z)\}$ ,  $\{\Sigma_{2m}(z)\}$  and  $\{\Sigma_{2m+1}(z)\}$  converge locally uniformly in  $\mathbb{C} \setminus S$  to holomorphic functions  $\Phi_0(z)$ ,  $\Phi_\infty(z)$ ,  $\Sigma_0(z)$  and  $\Sigma_\infty(z)$  and it holds that*

$$\Sigma_\infty(z) \Phi_0(z) - \Sigma_0(z) \Phi_\infty(z) = 1, \quad (6.6)$$

and

$$\Phi_\infty(z) \Phi_0'(z) - \Phi_\infty'(z) \Phi_0(z) = \sum_{k=0}^{\infty} \varphi_k(z)^2. \quad (6.7)$$

PROOF. Let  $K$  be a compact set in  $\mathbb{C} \setminus S$ , then, according to Proposition 6.2, we have  $\sum_{n=1}^{\infty} d_n(z) \leq D_e + D_o = D < \infty$  for  $z \in K$ .

Next we use the recurrence relations

$$\begin{bmatrix} \Sigma_{-1}(z) & \Sigma_0(z) \\ \Phi_{-1}(z) & \Phi_0(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} \Sigma_n(z) \\ \Phi_n(z) \end{bmatrix} = d_n(z) \begin{bmatrix} \Sigma_{n-1}(z) \\ \Phi_{n-1}(z) \end{bmatrix} + \begin{bmatrix} \Sigma_{n-2}(z) \\ \Phi_{n-2}(z) \end{bmatrix}, \quad n \geq 1,$$

which yield

$$|\Phi_n(z)| \leq \prod_{k=1}^n (1 + d_k(z)) \quad \text{and} \quad |\Sigma_n(z)| \leq \prod_{k=1}^n (1 + d_k(z)).$$

Taking the logarithm, and using  $\log(1 + d) < d$  for  $d > 0$  we get from this that

$$|\Phi_n(z)| \quad \text{and} \quad |\Sigma_n(z)| \quad \text{are bounded by} \quad \exp \left\{ \sum_{k=1}^n d_k(z) \right\} = e^D.$$

Thus both

$$\sum_{n=1}^{\infty} |d_n \Sigma_n(z)| \quad \text{and} \quad \sum_{n=1}^{\infty} |d_n \Phi_n(z)| \quad \text{are bounded by} \quad De^D.$$

Hence  $\sum_{n=1}^{\infty} d_n \Sigma_n(z)$  and  $\sum_{n=1}^{\infty} |d_n \Phi_n(z)|$  converge uniformly and absolutely on  $K$ .

Using the recurrence relation again, we deduce

$$\Sigma_{2m}(z) = \sum_{k=1}^m d_{2k}(z) \Sigma_{2k-1}(z), \quad \Phi_{2m}(z) = 1 + \sum_{k=1}^m d_{2k}(z) \Phi_{2k-1}(z), \quad m = 1, 2, \dots$$

and

$$\Sigma_{2m+1}(z) = 1 + \sum_{k=1}^m d_{2k+1}(z) \Sigma_{2k}(z), \quad \Phi_{2m+1}(z) = \sum_{k=1}^m d_{2k+1}(z) \Phi_{2k}(z), \quad m = 1, 2, \dots$$

so that all the sequences  $\{\Sigma_{2m}(z)\}_{m=1}^{\infty}$ ,  $\{\Sigma_{2m+1}(z)\}_{m=1}^{\infty}$ ,  $\{\Phi_{2m}(z)\}_{m=1}^{\infty}$ ,  $\{\Phi_{2m+1}(z)\}_{m=1}^{\infty}$  converge uniformly on  $K$ .

The equalities (6.6) and (6.7) follow from the determinant relation (4.10) and the confluent Christoffel-Darboux formula (4.11) by letting  $m$  tend to  $\infty$ . ■

## 7. Support of natural solutions

We continue to assume the conditions and hence also the results of the foregoing section. We then have

$$F_0(z) = S(z, \mu_0) = \frac{\Sigma_0(z)}{\Phi_0(z)}, \quad F_{\infty}(z) = S(z, \mu_{\infty}) = \frac{\Sigma_{\infty}(z)}{\Phi_{\infty}(z)}. \quad (7.1)$$

Consequently  $F_0$  and  $F_{\infty}$  are meromorphic functions in  $\mathbb{C} \setminus S$ . We have established in Section 5 that the singularities of  $F_0$  and  $F_{\infty}$  in  $(0, \infty)$  are simple poles. Since the support of  $\mu_0$  and of  $\mu_{\infty}$  are contained in  $[0, \infty)$ , it follows that  $\text{supp } \mu_0$  consists of the poles of  $F_0$  in  $(0, \infty)$  and possibly the origin, while  $\text{supp } \mu_{\infty}$  consists of the poles of  $F_{\infty}$  in  $(0, \infty)$  and possibly the origin.

From (6.6) we conclude that each of the pairs  $(\Phi_0, \Phi_\infty)$ ,  $(\Sigma_0, \Sigma_\infty)$ ,  $(\Phi_0, \Sigma_0)$  and  $(\Phi_\infty, \Sigma_\infty)$  have no common zero. Since the poles of  $F_0$  and  $F_\infty$  are simple, this implies that the zeros of  $\Phi_0$  and  $\Phi_\infty$  are simple zeros. Arguing as in Section 4 we conclude from (6.7) that between consecutive zeros of  $\Phi_0$  there is a zero of  $\Phi_\infty$  and *vice versa*.

For the sake of clarity we state the following result as a proposition.

**Proposition 7.1.** *supp  $\mu_0 \cap (0, \infty)$  consists of all zeros of  $\Phi_0$  and supp  $\mu_\infty \cap (0, \infty)$  consists of all zeros of  $\Phi_\infty$ .*

PROOF. This follows immediately from the fact that supp  $\mu_0 \cap (0, \infty)$  (resp. supp  $\mu_\infty \cap (0, \infty)$ ) consists of all poles in  $(0, \infty)$  of  $F_0$  (resp.  $F_\infty$ ), and these poles are exactly the zeros of  $\Phi_0$  (resp.  $\Phi_\infty$ ).

■

We shall henceforth denote the zeros of  $\Phi_{2m}$  by  $x_k^{(2m)}$ ,  $k = -m, \dots, -1, 1, \dots, m$ , and the zeros of  $\Phi_{2m+1}$  by  $x_k^{(2m+1)}$ ,  $k = -m, \dots, -1, 0, 1, \dots, m$ . They are ordered by size in the following sense:

$$\begin{aligned} x_{-m}^{(2m)} < x_{-m+1}^{(2m)} < \dots < x_{-1}^{(2m)} < x_1^{(2m)} < \dots < x_{m-1}^{(2m)} < x_m^{(2m)}, \quad \text{and} \\ x_{-m}^{(2m+1)} < x_{-m+1}^{(2m+1)} < \dots < x_{-1}^{(2m+1)} < x_0^{(2m+1)} < x_1^{(2m+1)} < \dots < x_{m-1}^{(2m+1)} < x_m^{(2m+1)}. \end{aligned} \quad (7.2)$$

Note that the smallest and the largest zero of  $\Phi_n$  can be written as  $x_{-\lfloor \frac{n}{2} \rfloor}^{(n)}$  and  $x_{\lfloor \frac{n}{2} \rfloor}^{(n)}$ . From the separation properties of the zeros it follows that for each  $k = 0, 1, 2, \dots$ , the sequence  $\{x_{-\lfloor \frac{n}{2} \rfloor + k}^{(n)}\}_{n \geq 2k+1}$  decreases to a values  $\xi_k \in [0, \infty)$  and the sequence  $\{x_{\lfloor \frac{n}{2} \rfloor - k}^{(n)}\}_{n \geq 2k+1}$  increases to a value  $\eta_k \in (0, \infty]$ .

We shall now concentrate on the central zero of  $\Phi_{2m+1}$ . For  $\Phi_1$ , this central zero is clearly

$$x_0 = -\frac{R_1}{Q_1} \quad (7.3)$$

since there is only one. It follows from (3.8) that  $x_0 \in (0, \infty)$ . Furthermore we make the crucial assumption that

$$\frac{\alpha_m Q_{2m+1} + \beta_m R_{2m+1}}{Q_{2m+1} + R_{2m+1}} = x_0 \quad \text{for all } m. \quad (7.4)$$

Observe that the condition (7.4) is equivalent to  $b_{2m+1}(x_0) = 0$  for all  $m$  where the  $b_{2m+1}$  are as defined in (3.7).

This assumption will fix  $x_0$  as being the central zero of all subsequent  $\Phi_{2m+1}$  as shown below.

**Proposition 7.2.** *The middle zero of  $\Phi_{2m+1}$  equals  $x_0$ , i.e.,*

$$x_0^{(2m+1)} = x_0 \quad \text{for } m = 0, 1, 2, \dots \quad (7.5)$$

PROOF. From (3.5) and (3.7) we find that  $\varphi_1(z) = \frac{Q_1 z + R_1}{\alpha_1 - z}$ . By the definition of  $x_0$  (formula (7.3)) we then get  $\varphi_1(x_0) = 0$ . Since the condition (7.4) is equivalent to  $b_{2m+1}(x_0) = 0$  for all  $m$ , we conclude by induction, using (3.5) and (3.7), that  $\varphi_{2m+1}(x_0) = 0$ , hence  $\Phi_{2m+1}(x_0) = 0$  for all  $m$ .

By separation properties of zeros we have  $x_{-1}^{(2)} < x_0^{(1)} < x_1^{(2)}$  and  $x_0^{(1)} = x_0$  since  $x_0$  is the only zero of  $Q_1$ . It follows that  $x_{-1}^{(3)} < x_1^{(2)} < x_0 < x_1^{(2)} < x_1^{(3)}$ , which means that  $x_0^{(3)} = x_0$ . Arguing step by step in this way, we find that  $x_0^{(2m+1)} = x_0$  for all  $m$ . ■

It follows from Proposition 7.2 that  $x_0$  is a zero of  $\Phi_\infty$ . We denote the zeros of  $\Phi_0$  by  $x_p^{(0)}$ ,  $p = \pm 1, \pm 2, \dots$ , and the zeros of  $\Phi_\infty$  by  $x_p^{(\infty)}$ ,  $p = 0, \pm 1, \pm 2, \dots$ , with  $x_0^{(\infty)} = x_0$ , ordered such that

$$\dots < x_{-2}^{(0)} < x_{-1}^{(\infty)} < x_{-1}^{(0)} < x_0^{(\infty)} < x_1^{(0)} < x_1^{(\infty)} < x_2^{(0)} < \dots \quad (7.6)$$

(recall that the zeros of  $\Phi_0$  and  $\Phi_\infty$  separate each other). I.e.,

$$x_{-p}^{(0)} < x_{-p+1}^{(\infty)} < x_{-p+1}^{(0)} \quad \text{and} \quad x_p^{(0)} < x_p^{(\infty)} < x_{p+1}^{(0)} \quad \text{for } p = 2, 3, \dots, \text{ and } x_{-1}^{(0)} < x_0^{(\infty)} < x_1^{(0)}. \quad (7.7)$$

(Cf. the discussion just before Proposition 7.1.)

We recall that  $Z_n$  denotes the set of zeros of  $\Phi_n$ . Furthermore the support of  $\mu_0$  (resp.  $\mu_\infty$ ) consists of the zeros of  $\Phi_0$  (resp.  $\Phi_\infty$ ), plus possibly the origin. Except for the origin,  $\text{supp } \mu_0$  and  $\text{supp } \mu_\infty$  are disjoint.

**Proposition 7.3.**  $\text{supp } \mu_0 \cap (0, \infty)$  consists of all accumulation points in  $(0, \infty)$  of the set  $\cup_{m=1}^\infty Z_{2m}$  and  $\text{supp } \mu_\infty \cap (0, \infty)$  consists of all accumulation points in  $(0, \infty)$  of the set  $\cup_{m=0}^\infty Z_{2m+1}$ .

PROOF. We already have from Proposition 5.1 that in the general situation every point in  $\text{supp } \mu_0 \cap (0, \infty)$  (resp.  $\text{supp } \mu_\infty \cap (0, \infty)$ ) is an accumulation point of  $\cup_{m=1}^\infty Z_{2m}$  (resp.  $\cup_{m=0}^\infty Z_{2m+1}$ ). It follows from *Hurwitz' theorem* (see e.g., [13, Part II, p.49]) that every accumulation point in  $(0, \infty)$  of  $\cup_{m=1}^\infty Z_{2m}$  (resp.  $\cup_{m=0}^\infty Z_{2m+1}$ ) is a zero of  $\Phi_0$  (resp.  $\Phi_\infty$ ), hence belongs to  $\text{supp } \mu_0$  (resp.  $\text{supp } \mu_\infty$ ). ■

Let  $\xi_k$  be any of the limits discussed earlier in this section. It follows from Proposition 7.3 that if  $\xi_k \in (0, \infty)$ , then  $\xi_k$  belongs to the support of both  $\Phi_0$  and  $\Phi_\infty$ . This is a contradiction, and hence we conclude that  $\xi_k = 0$  for all  $k$ . Similarly we find that  $\eta_k = \infty$  for all  $k$ . This is in contrast to the classical situation.

Thus the sequences  $\{x_{-m+k}^{(2m)}\}_m$  and  $\{x_{m-k}^{(2m)}\}_m$  do not converge to support points for  $\mu_0$  in  $(0, \infty)$ , and the sequences  $\{x_{-m+k}^{(2m+1)}\}_m$  and  $\{x_{m-k}^{(2m+1)}\}_m$  do not converge to support points for  $\mu_\infty$  in  $(0, \infty)$ .

On the other hand, according to Proposition 7.3, every point in  $\text{supp } \mu_0 \cap (0, \infty)$  is an accumulation point for zeros of  $\{\Phi_{2m}\}$  and every point in  $\text{supp } \mu_\infty \cap (0, \infty)$  is an accumulation point for zeros of  $\{\Phi_{2m+1}\}$ . We may then pose the question: Can anything be said about *how* zeros of the orthogonal functions approach zeros of the limit functions? In general no such results are known, but in the two final sections we shall identify sequences of zeros of  $\Phi_{2m}$  (resp.  $\Phi_{2m+1}$ ) which converge to each zero of  $\Phi_0$  (resp.  $\Phi_\infty$ ) in the setting we have assumed from Section 6 on.

## 8. Monotonicity of zero sequences

We assume as before that the rational Stieltjes moment problem on  $\mathcal{L} \cdot \mathcal{L}$  is indeterminate. We furthermore assume that (7.4) is satisfied with  $x_0$  as in (7.3).

The arguments in this and the next section are strongly influenced by ideas from the papers [3], [2], which again have their origin in the thesis [1]. Our proofs of Theorem 8.4 and Theorem 9.1 are quite similar to the proofs of the corresponding results in the two-point situation in [3]. For the sake of completeness, we carry out the arguments.

Recall that the zeros of  $\Phi_n$  coincide with zeros of  $P_n$ , where as before  $\varphi_n(z) = \frac{P_n(z)}{D_n(z)}$ .

**Proposition 8.1.**

$$\lim_{x \rightarrow \infty} P_{2m}(x) = -\infty, \quad \lim_{x \rightarrow \infty} P_{2m+1}(x) = \infty. \quad (8.1)$$

PROOF. We established in Section 3 that  $P_n(x) < 0$  for  $x \in (-\infty, 0)$  for all  $n$ . The limit values in (8.1) then follow from the fact that all the zeros of  $P_n$  are simple and contained in  $(0, \infty)$ . ■

Recall that  $Q_{2m+1} > 0$ ,  $R_{2m+1} < 0$  (cf. (3.8)).

**Proposition 8.2.** *We have*

$$|R_{2m+1}| < Q_{2m+1} \quad (8.2)$$

PROOF. According to (7.4) we have

$$\frac{\alpha_m Q_{2m+1} + \beta_m R_{2m+1}}{Q_{2m+1} + R_{2m+1}} = x_0.$$

Since  $x_0$  is finite,  $|R_{2m+1}| \neq Q_{2m+1}$ . If  $|R_{2m+1}| > Q_{2m+1}$ , then the denominator in the equation is negative while the numerator is positive (recall  $|\beta_m| > |\alpha_m|$ ). Since  $x_0 \in (0, \infty)$ , this is a contradiction. ■

**Proposition 8.3.** *Let  $P_{2m-1}(\xi) = 0$ ,  $\xi \neq x_0$ . Then  $P_{2m+1}(\xi)$  and  $P_{2m-3}(\xi)$  have opposite sign.*

PROOF. It follows from (4.1–4.3) that

$$\varphi_{2m+1}(\xi) = -\frac{(\alpha_{m-1} - \xi)(\beta_{m-1} - \xi)b_{2m+1}(\xi)}{(\alpha_m - \xi)(\beta_m - \xi)b_{2m-1}(\xi)} \varphi_{2m-3}(\xi). \quad (8.3)$$

We may write

$$b_{2k+1}(z) = \frac{(Q_{2k+1} + R_{2k+1})(z - x_0)}{z - \alpha_{k+1}}.$$

Then according to Proposition 8.2 we have  $b_{2k+1}(x) < 0$  for  $x \in (0, x_0)$ ,  $b_{2k+1}(x) > 0$  for  $x \in (x_0, \infty)$ , for all  $k$ . Hence we conclude that the signs of the factors on the right-hand side of (8.3) cancel, and the result follows. ■

**Theorem 8.4.** *For every  $p = 1, 2, 3, \dots$ , the sequence  $\{x_p^{(2m+1)}\}_{m=p}^{\infty}$  is decreasing and for  $p = -1, -2, -3, \dots$  the sequence  $\{x_p^{(2m+1)}\}_{m=-p}^{\infty}$  is increasing, i.e.,*

$$\begin{aligned} x_p^{(2m+1)} &< x_p^{(2m-1)}, \quad \text{for } p = 1, 2, \dots, \quad m > p \\ x_p^{(2m+1)} &> x_p^{(2m-1)}, \quad \text{for } p = -1, -2, \dots, \quad m > -p. \end{aligned} \quad (8.4)$$

PROOF. We first consider  $p > 0$ . Recall that  $x_0^{(2m+1)} = x_0$  for all  $m$ . It follows from Propositions 8.1 and 8.3 that  $P_5(x_1^{(3)}) < 0$  since  $P_1(x) > 0$  for  $x > x_0^{(1)}$ , and  $x_1^{(3)} > x_0^{(1)}$ . From (7.7) we have  $x_0^{(3)} < x_1^{(3)} < x_2^{(5)}$ . Since  $\lim_{x \rightarrow \infty} P_5(x) = \infty$  by Proposition 8.3, we have for  $x \in (x_0^{(5)}, x_1^{(5)})$ . We then conclude from  $P_5(x_1^{(3)}) < 0$  that  $x_1^{(3)} \in (x_1^{(5)}, x_2^{(5)})$ . Thus

$$x_1^{(5)} < x_1^{(3)}. \quad (8.5)$$

Since  $x_2^{(5)} > x_2^{(4)} > x_1^{(3)}$  by (7.2), we have  $P_3(x_2^{(5)}) > 0$  by Proposition 8.1, and consequently  $P_7(x_2^{(5)}) < 0$  by Proposition 8.3. From (7.2) we have  $x_1^{(7)} < x_2^{(5)} < x_3^{(7)}$ . Proposition 8.1 implies that  $P_7(x) < 0$  for  $x \in (x_2^{(7)}, x_3^{(7)})$  and  $P_7(x) > 0$  for  $x \in (x_1^{(7)}, x_2^{(7)})$ . Hence  $x_2^{(5)} \in (x_2^{(7)}, x_3^{(7)})$ . Thus

$$x_2^{(7)} < x_2^{(5)}. \quad (8.6)$$

Since  $x_0^{(3)} < x_1^{(5)} < x_1^{(3)}$  (cf. (8.1)) we have  $P_3(x_1^{(5)}) < 0$  by Proposition 8.1, hence  $P_7(x_1^{(5)}) > 0$  by Proposition 8.3. As above we have  $P_7(x) < 0$  for  $x \in (x_2^{(7)}, x_3^{(7)})$  and for  $x \in (x_0^{(7)}, x_1^{(7)})$ . Hence  $x_1^{(5)} \in (x_1^{(7)}, x_2^{(7)})$ . Thus

$$x_1^{(7)} < x_1^{(5)}. \quad (8.7)$$

Arguing in the same way, repeatedly using the results of Propositions 8.1 and 8.3 and the already obtained results, we find by induction that the first line of (8.4) is satisfied for all  $m$  and  $p = 1, 2, \dots, m$ .

Similarly we prove that the second line in (8.4) is satisfied for all  $m$  and  $p = -1, -2, -3, \dots, -m$ .

■

## 9. Support points and convergence of zero sequences

In this section we assume that all the conditions and hence all the results of Section 6, 7, and 8 are satisfied.

**Theorem 9.1.** *Consider an indeterminate rational Stieltjes moment problem where the recurrence coefficients satisfy (7.3) and (7.4). Then for each  $k = \pm 1, \pm 2, \dots$ , the zero sequence  $\{x_k^{(2m)}\}_m$  converges to the zero  $x_k^{(0)}$  of  $\Phi_0$  and the zero sequence  $\{x_k^{(2m+1)}\}_m$  converges to the zero  $x_k^{(\infty)}$  of  $\Phi_\infty$ .*

PROOF. We first consider  $k > 0$ . The sequence  $\{x_k^{(2m+1)}\}_{m=k}$  is decreasing by Theorem 8.4 and is bounded below by  $x_0$ , hence converges to a limit  $y_k$ . It follows from Proposition 7.1 and Proposition 7.3 that every  $x_p^{(\infty)}$  must coincide with at least one of the values  $y_k$  and every  $y_k$  must coincide with a value  $x_p^{(\infty)}$ .

The zeros  $x_k^{(2m+1)}$  are simple zeros of  $\Phi_{2m+1}$  and the zeros  $x_p^{(\infty)}$  are simple zeros of  $\Phi_\infty$ . Consequently each  $x_p^{(\infty)}$  is the limit of exactly one sequence  $\{x_k^{(2m+1)}\}_m$ , i.e., every  $x_p^{(\infty)}$  coincides with

exactly one  $y_k$ , and *vice versa*. By the ordering of the zeros  $x_k^{(2m+1)}$  and the ordering of the zeros  $x_p^{(\infty)}$  (cf. (7.6)) it follows that  $y_k = x_k^{(\infty)}$  for  $k = 1, 2, \dots$

In the same way we find that  $\{x_k^{(2m+1)}\}_{m \geq -k}$  converges to  $x_k^{(\infty)}$  for  $k = -1, -2, \dots$

Finally, Proposition 7.3 implies that the accumulation points of the set  $\cup_{m=1}^{\infty} Z_{2m}$  in  $(0, \infty)$  are exactly the zeros  $x_k^{(0)}$  of  $\Phi_0$ . Since  $x_{k-1}^{(2m+1)} < x_k^{(2m)} < x_k^{(2m+1)}$  for  $k = 1, 2, \dots$ , all the accumulation points of  $\{x_k^{(2m)}\}_m$  lie in  $[x_{k-1}^{(\infty)}, x_k^{(\infty)}]$ . The only zero of  $\Phi_0$  in  $[x_{k-1}^{(\infty)}, x_k^{(\infty)}]$  is  $x_k^{(0)}$ , hence  $\{x_k^{(2m)}\}_m$  converges to  $x_k^{(0)}$ . In the same way we establish that  $\{x_k^{(2m)}\}_m$  converges to  $x_k^{(0)}$  for  $k = -1, -2, \dots$

■

**Remark 9.2.** It follows from Theorem 9.1 that  $\text{supp } \mu_0$  and  $\text{supp } \mu_{\infty}$  consist of infinitely many points to the left of  $x_0$  and to the right of  $x_0$ . Since  $\Phi_0$  and  $\Phi_{\infty}$  are holomorphic in an open set containing  $(0, \infty)$ , we conclude that  $x_k^{(0)}$  and  $x_k^{(\infty)}$  tend to 0 as  $k$  tends to  $-\infty$  and  $x_k^{(0)}$  and  $x_k^{(\infty)}$  tend to  $\infty$  as  $k$  tends to  $\infty$ . In particular, the origin belongs to the support of both  $\mu_0$  and  $\mu_{\infty}$ .

When  $\{\alpha_n\}$  tends monotonically to 0 and  $\{\beta_n\}$  tends monotonically to  $-\infty$ , then the fact that the support accumulates at the origin and at infinity follows without the special assumptions in Theorem 6.3 from a Carleman-type criterion giving sufficient condition for determinacy. See [6, Theorem 6.2], [8, Theorem 16.2].

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