Extensions of Fibonacci lattice rules

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Report TW 545, August 2009

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Abstract

We study the trigonometric degree of pairs of embedded cubature rules for the approximation of two-dimensional integrals, where the basic cubature rule is a Fibonacci lattice rule. The embedded cubature rule is constructed by simply doubling the points which results in adding a shifted version of the basic Fibonacci rule. An explicit expression is derived for the trigonometric degree of this particular extension of the Fibonacci rule based on the index of the Fibonacci number.

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Extensions of Fibonacci Lattice Rules

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Dedicated to Ian Sloan's 70th birthday.

1 Introduction

We consider the approximation of integrals

$$I[f] := \int_{[0,1)^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x}$$

by weighted sums of function values. In practice one wants more than one such approximation to obtain information on the accuracy of the approximation. In order to approximate an integral together with an error estimate, one often uses two approximations $Q_1[f]$ and $Q_2[f]$. Then $|Q_1[f] - Q_2[f]|$ can be used as an approximation of the error of the less precise rule. (In practice it is often used as an estimate of the

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error of the most expensive rule, while hoping that this is the most precise. Practical robust error estimates are based on more than one such combinations.)

Given a cubature formula

$$Q_1[f] = \sum_{j=1}^{N} w_j f(\mathbf{x}_j)$$

for the approximation of an integral $I[f]$, we are interested in a cubature formula

$$Q_2[f] = \sum_{j=1}^{N} \bar{w}_j f(\mathbf{x}_j) + \sum_{j=N+1}^{N+M} \bar{w}_j f(\mathbf{x}_j)$$

that reuses the function evaluations of Q_1 and is "better". The quality criterion used in this paper is the *trigonometric degree*, and the cubature formulas will be lattice rules. For other criteria, see [4].

A cubature formula of trigonometric degree d integrates correctly trigonometric polynomials of degree d. Specifically in s dimensions, it integrates $\exp(2\pi \mathrm{i}\, \mathbf{r} \cdot \mathbf{x})$ correctly for all $\mathbf{r} = (r_1, r_2, \ldots, r_s) \in \mathbf{Z}^s$ that satisfy $|\mathbf{r}| := \sum_{k=1}^{s} |r_k| \leq d$.

Embedded pairs of quadrature and cubature formulas of algebraic degree were already studied a long time ago (see, e.g., [6, 2, 3]). We are unaware of an attempt to do this for the trigonometric case. This paper describes a first attempt, limited to the 2-dimensional case. More examples of this type might eventually lead to similar theoretical insights as in the algebraic case.

In the following section we present the necessary background and notation in s dimensions. In §3 we will present the well known class of 2-dimensional Fibonacci lattice rules and known results on their trigonometric degree. In §4 we will investigate a particular extension of these rules to obtain an embedded pair and in §5 we compare their quality with what is theoretically the best possible result.

2 A Short Course on Lattice Rules

For a thorough introduction on lattice rules, we refer to [9]. Results on the trigonometric degree of lattice rules up to that date were mainly published in the Russian literature and summarized in [1].

Definition 1. A multiple integration lattice Λ in $\mathbf{R}^s$ is a subset of $\mathbf{R}^s$ which is discrete and closed under addition and subtraction and which contains $\mathbf{Z}^s$ as a subset.

A lattice rule for approximating an integral over $[0,1)^s$ is a cubature formula where the N points are the points of a multiple integration lattice Λ that lie in $[0,1)^s$ and all points have the same weight $1/N$.

Definition 2. The dual of the lattice Λ is $\Lambda^{\perp} := \{\mathbf{r} \in \mathbf{R}^s : \mathbf{r} \cdot \mathbf{x} \in \mathbf{Z}, \forall \mathbf{x} \in \Lambda\}$.

A lattice Λ can be specified by an $s \times s$ matrix M known as a *generator matrix*, whose rows generate the lattice. This means that all elements of Λ are of the form

$\mathbf{x} = \lambda M$, where $\lambda \in \mathbf{Z}^s$. The dual lattice $\Lambda^\perp$ then has generator matrix $B = (M^{-1})^T$. Since Λ is an integration lattice, its dual $\Lambda^\perp$ is an integer lattice and is generated by an integer-valued matrix B.

The dual of a multiple integration lattice plays an important role in the error representation and is the main tool to prove our results. Assume that f can be expanded into an absolutely convergent multiple Fourier series

$$f(\mathbf{x}) = \sum_{\mathbf{r}\in\mathbf{Z}^s} a(\mathbf{r})\, \mathrm{e}^{2\pi \mathrm{i}\, \mathbf{r}\cdot\mathbf{x}} \;\text{ with }\; a(\mathbf{r}) = \int_{[0,1)^s} \mathrm{e}^{-2\pi \mathrm{i}\, \mathbf{r}\cdot\mathbf{x}} f(\mathbf{x})\, \mathrm{d}\mathbf{x},$$

then the error is given by the next theorem. Note that this assumption limits the functions to be 1-periodic in each dimension.

Theorem 1. *[10] Let Λ be a multiple integration lattice. Then the corresponding lattice rule Q has an error*

$$Q[f] - I[f] = \sum_{\mathbf{r}\in\Lambda^\perp\setminus\mathbf{0}} a(\mathbf{r}).$$

The trigonometric degree of a lattice rule can be determined from the dual lattice:

$$d(Q) := \min_{\mathbf{r}\in\Lambda^\perp\setminus\mathbf{0}} |\mathbf{r}| - 1\ .$$

There is a diamond shaped region (a crosspolytope to be precise) with no points of the dual lattice except the origin inside, and some points on its boundary. The 1-norm of points on the boundary is $d+1$.

Central symmetry (of the integration region and the points in the cubature formulas) plays an important role in the algebraic case. If this symmetry is present, then the cubature formula integrates the odd polynomials exactly automatically. The lower bound for the number of points required is also fundamentally different for the even and the odd degrees. The role that central symmetry plays in the algebraic case is played by *shift symmetry* in the trigonometric case [5].

Definition 3. A cubature formula Q for an integral I on $[0,1)^s$ is shift symmetric if, whenever $(x_1^{(j)}, \ldots, x_s^{(j)})$ is a point of the formula, then so is $(x_1^{(j)} + \frac{1}{2}, \ldots, x_s^{(j)} + \frac{1}{2})$, with both points having the same weight.

The point $(x_1^{(j)} + \frac{1}{2}, \ldots, x_s^{(j)} + \frac{1}{2})$ in the above theorem should actually be interpreted modulo 1, i.e., wrapped around the edges of the unit cube. However, since we are assuming periodic functions we loosen the notation and do not write the traditional fractional braces around the points to denote the modulo 1.

From Definition 3 it follows that N is even for a shift symmetric cubature formula. Furthermore, because a shift symmetric cubature formulas is automatically exact for all trigonometric monomials of odd degree, such a cubature formula has an odd trigonometric degree, see [5]. Finally note that a lattice rule is shift symmetric if and only if $(\frac{1}{2}, \ldots, \frac{1}{2})$ is a point of the lattice.

3 Fibonacci Lattice Rules

We will restrict our investigations in this paper to two dimensions, starting from a well known family of lattice rules. Let F_k be the kth Fibonacci number, defined by $F_0 := 0$, $F_1 := 1$ and $F_k := F_{k-1} + F_{k-2}$ for $1 < k \in \mathbf{N}$. Consider the following lattice rules:

$$\mathscr{L}_k[f] = \frac{1}{F_k} \sum_{j=0}^{F_k - 1} f\left(\frac{j}{F_k}, \frac{jF_{k-1}}{F_k}\right) \qquad (1)$$

and

$$\mathscr{L}'_k[f] = \frac{1}{F_k} \sum_{j=0}^{F_k - 1} f\left(\frac{j}{F_k}, \frac{jF_{k-2}}{F_k}\right).$$

Lattice rules of this form are called Fibonacci lattice rules. The two rules given above are *geometrically equivalent*, meaning that one point set can be changed into the other one by symmetry operations of the unit cube [9]. In this case a reflection on the second coordinate axis maps $\mathscr{L}_k$ to $\mathscr{L}'_k$ since $jF_{k-1} \equiv -jF_{k-2} \pmod{F_k}$. Geometrically equivalent lattice rules have the same trigonometric degree, as is obvious from their dual lattices.

The kth Fibonacci number F_k is even if and only if k is a multiple of 3. This can be observed by looking at the Fibonacci sequence modulo 2: $(F_k \bmod 2)_{k \geq 1} = (1,1,0,\,1,1,0,\,1,1,0,\ldots)$, i.e., the 2nd Pisano period is 3. Only in these cases are the Fibonacci lattice rules shift symmetric. Indeed, the point of $\mathscr{L}_k$ generated by $j = F_k/2$ is $\left(\frac{1}{2}, \frac{F_{k-1}}{2}\right)$ and this then maps to $\left(\frac{1}{2}, \frac{1}{2}\right)$.

The trigonometric degree of Fibonacci lattice rules is known explicitly.

Theorem 2. *[1] If $k = 2m+1$ and $m \geq 2$ then the Fibonacci lattice rule has trigonometric degree $F_{m+2} - 1$. If $k = 2m$ then the Fibonacci lattice rule has trigonometric degree $2F_m - 1$.*

Shift symmetric lattice rules have an odd trigonometric degree [5]. The converse is not always true. A rule of odd trigonometric degree is not necessarily shift symmetric. The family of Fibonacci lattice rules has many examples of this.

The proof in [1] of the above theorem is based on the dual lattice. We sketch it here because we will use the same technique in §4 and because it reveals a structure in the lattice. Starting from the evident generator matrix, e.g., for $k = 2m+1$

$$M = \begin{pmatrix} \frac{1}{F_{2m+1}} & \frac{F_{2m}}{F_{2m+1}} \\ 0 & 1 \end{pmatrix}$$

it follows that $B = (M^{-1})^T = \begin{pmatrix} F_{2m+1} & 0 \\ -F_{2m} & 1 \end{pmatrix}$ is a generator matrix for the dual lattice. When U is any unimodular matrix then UB is also a generator matrix. (A unimodular matrix is a square integer matrix with determinant $+1$ or -1.) Using the unimodular matrix $U = \begin{pmatrix} F_m & F_{m+1} \\ -F_{m-1} & -F_m \end{pmatrix}$, it is shown that

$$B = UA \text{ with } A = \begin{pmatrix} F_m & (-1)^{m+1}F_{m+1} \\ F_{m+1} & (-1)^m F_m \end{pmatrix}. \tag{2}$$

Hence, A is also a generator matrix of this dual lattice. Then, it is proven that no nonzero combination of the rows of this matrix leads to a point with 1-norm smaller than the claimed degree.

Observe that the generator matrix A (2) of the dual lattice is an orthogonal matrix (modulo $F_k = F_{2m+1} = F_m^2 + F_{m+1}^2$). In other words, both generating vectors are orthogonal and have the same length (in 2-norm). So the Fibonacci lattice for odd k has a square unit cell, i.e., this lattice corresponds to a rotated regular grid. This fact was obtained in a different way, and explicitly recognized by Niederreiter and Sloan [8].

4 An Extension of Fibonacci Lattice Rules

We are interested in lattice rules that extend Fibonacci lattice rules so that we obtain an embedded pair of lattice rules. The aim is to obtain a pair that requires less function evaluations than two rules of the corresponding degrees. We are partially successful in that respect.

Consider the lattice rule

$$Q_k[f] = \frac{1}{2F_k} \sum_{j=0}^{2F_k-1} f\left(\frac{j}{2F_k}, \frac{jF_{k-1}}{2F_k}\right). \tag{3}$$

This lattice is shift symmetric if F_{k-1} is odd. Indeed for $j = F_k$ the point $\left(\frac{1}{2}, \frac{F_{k-1}}{2}\right)$ is generated. Whenever F_{k-1} is odd, this maps to $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Next we will show that (3) can be seen as a Fibonacci lattice $\mathscr{L}_k$ plus a shifted version of the same Fibonacci lattice. The above rule can be split in two sums, separating even and odd values of j:

$$\begin{aligned} Q_k[f] &= \frac{1}{2F_k} \sum_{j=0}^{F_k-1} f\left(\frac{j}{F_k}, \frac{jF_{k-1}}{F_k}\right) + \frac{1}{2F_k} \sum_{j=0}^{F_k-1} f\left(\frac{2j+1}{2F_k}, \frac{(2j+1)F_{k-1}}{2F_k}\right) \\ &= \frac{1}{2F_k} \sum_{j=0}^{F_k-1} f\left(\frac{j}{F_k}, \frac{jF_{k-1}}{F_k}\right) + \frac{1}{2F_k} \sum_{j=0}^{F_k-1} f\left(\frac{j}{F_k} + \frac{1}{2F_k}, \frac{jF_{k-1}}{F_k} + \frac{F_{k-1}}{2F_k}\right). \end{aligned}$$

Thus the lattice rule (3) is given by the original rule (1) plus the original rule shifted by $(1, F_{k-1})/(2F_k)$. (The weights are adjusted in the obvious way.) We will now show a formula for its trigonometric degree.

Theorem 3. *The lattice rule given by* (3) *for* $k = 6m + \alpha$, *with* $\alpha = 0, 1, \ldots, 5$, *has trigonometric degree* $F_{3m+\alpha-1} + F_{3m+1} - 1$.

Proof. The proof technique we use here was sketched at the end of §3.

The cubature formula (3) with $k = 6m + \alpha$ is a lattice rule with generator matrix of the corresponding lattice:

$$M = \begin{pmatrix} \frac{1}{2F_{6m+\alpha}} & \frac{F_{6m+\alpha-1}}{2F_{6m+\alpha}} \\ 0 & 1 \end{pmatrix}.$$

We will instead investigate the minor modification of this matrix

$$M = \begin{pmatrix} \frac{1}{2F_{6m}+\alpha} & \frac{(-1)^m F_{6m+\alpha-1}}{2F_{6m+\alpha}} \\ 0 & (-1)^m \end{pmatrix}.$$

If m is even, this is the same. If m is odd, we investigate a geometrically equivalent lattice. As mentioned before, geometrically equivalent lattice rules have the same trigonometric degree.

A generator matrix of the corresponding dual lattice is

$$B = (M^{-1})^T = \begin{pmatrix} 2F_{6m+\alpha} & 0 \\ -F_{6m+\alpha-1} & (-1)^m \end{pmatrix}.$$

Observe that the matrix

$$C = \begin{pmatrix} F_{3m+1} & -2F_{3m} \\ -F_{3m}/2 & F_{3m-1} \end{pmatrix}$$

is an integer unimodular matrix. Indeed, we observed in §3 that F_{3m} is an even number and, making use of Cassini's identity, see (5) below, it follows that

$$\det(C) = F_{3m+1}F_{3m-1} - F_{3m}^2 = (-1)^{3m} = (-1)^m.$$

We will first show that

$$A = \begin{pmatrix} 2F_{3m+\alpha} & 2F_{3m} \\ -F_{3m+\alpha-1} & F_{3m+1} \end{pmatrix} \tag{4}$$

is another generator matrix for the dual lattice. This follows from $CA = B$. It requires some Fibonacci magic to show this. The relevant relations for the sequel are [7]

$$\begin{aligned} F_k F_{n+1} + F_{k-1} F_n &= F_{n+k}, \\ F_{n+1} F_{n-1} - F_n^2 &= (-1)^n. \end{aligned} \tag{5}$$

It follows that

$$\begin{aligned} CA &= \begin{pmatrix} 2F_{3m+1}F_{3m+\alpha} + 2F_{3m}F_{3m+\alpha-1} & 2F_{3m+1}F_{3m} - 2F_{3m}F_{3m+1} \\ -F_{3m}F_{3m+\alpha} - F_{3m-1}F_{3m+\alpha-1} & -F_{3m}^2 + F_{3m-1}F_{3m+1} \end{pmatrix} \\ &= \begin{pmatrix} 2F_{3m+1}F_{3m+\alpha} + 2F_{3m}F_{3m+\alpha-1} & 0 \\ -F_{3m}F_{3m+\alpha} - F_{3m-1}F_{3m+\alpha-1} & (-1)^{3m} \end{pmatrix} = B. \end{aligned}$$

The points on the dual lattice are generated as

$$(i,\, j)A = (i2F_{3m+\alpha} - jF_{3m+\alpha-1},\; i2F_{3m} + jF_{3m+1})$$

for all $i, j \in \mathbf{Z}$. The corresponding lattice rule has trigonometric degree $F_{3m+\alpha-1} + F_{3m+1} - 1$ if all points of the dual lattice, except the point $(0,0)$, lie outside or on the boundary of the diamond shaped region:

$$|i2F_{3m+\alpha} - jF_{3m+\alpha-1}| + |i2F_{3m} + jF_{3m+1}| \geq F_{3m+\alpha-1} + F_{3m+1}. \tag{6}$$

(Some points fall exactly on the boundary, e.g., when $i = 0$ and $j = 1$.) We prove this inequality for different cases of (i, j). For brevity we name parts of the inequality as follows

$$\begin{aligned} \beta(\alpha) &:= |i2F_{3m+\alpha} - jF_{3m+\alpha-1}|, \\ \delta &:= |i2F_{3m} + jF_{3m+1}|, \\ \gamma(\alpha) &:= F_{3m+\alpha-1} + F_{3m+1}, \end{aligned}$$

so that we have to prove for all $\alpha = 0, 1, \ldots, 5$: $\beta(\alpha) + \delta \geq \gamma(\alpha)$. This is a tedious exercise in proving the bound for all different cases. For reference the reader is referred to Fig. 1 on different occasions during the proof, which depicts the possible integer values that the (i, j) can take in generating the dual lattice points.

1. If $i = 0$ then $\beta(\alpha) + \delta = |j|\,(F_{3m+\alpha-1} + F_{3m+1}) \geq F_{3m+\alpha-1} + F_{3m+1}$ since we must have $|j| \geq 1$.
2. If $j = 0$ then $\beta(\alpha) + \delta = |i|\,(2F_{3m+\alpha} + 2F_{3m}) \geq F_{3m+\alpha-1} + F_{3m+1}$ since we must have $|i| \geq 1$ and $2F_{3m} \geq F_{3m} + F_{3m-1} = F_{3m+1}$.
3. If $ij < 0$ (i.e., opposite signs) then $\beta(\alpha) = |i|\,2F_{3m+\alpha} + |j|\,F_{3m+\alpha-1} \geq F_{3m+\alpha-1} + F_{3m+1}$ since $2F_{3m+\alpha} \geq F_{3m+\alpha} + F_{3m+\alpha-1} = F_{3m+\alpha+1} \geq F_{3m+1}$ for all $\alpha \geq 0$.

   We have now checked (and crossed out) the 2nd and 4th quadrant, marked with $ij \leq 0$ on Fig. 1, to fulfill (6).

4. For $ij > 0$ (i.e., same signs) we first look at

   $$\begin{aligned} \beta(\alpha) &= |i2F_{3m+\alpha} - jF_{3m+\alpha-1}| \\ &= |(2i - j)F_{3m+\alpha-1} + 2iF_{3m+\alpha-2}|. \end{aligned}$$

   Again here we consider separate cases based on the signs of the two terms. But note first that $\delta = |i|\,2F_{3m} + |j|\,F_{3m+1} \geq 2F_{3m} + F_{3m+1}$, so we always trivially have the F_{3m+1} term from $\gamma(\alpha)$ and we can close a case quickly if $\beta(\alpha) \geq F_{3m+\alpha-1}$. Also note that we can get extra terms of F_{3m} and F_{3m+1} by having larger bounds on respectively $|i|$ and $|j|$.

   a. If $2i - j = 0$ then $\beta(\alpha) = 2|i|\,F_{3m+\alpha-2} \geq 2F_{3m+\alpha-2} \geq F_{3m+\alpha-1}$.
   b. If $(2i - j)\,i > 0$ (i.e., same signs) then $\beta(\alpha) = |2i - j|\,F_{3m+\alpha-1} + 2|i|\,F_{3m+\alpha-2} \geq F_{3m+\alpha-1}$.

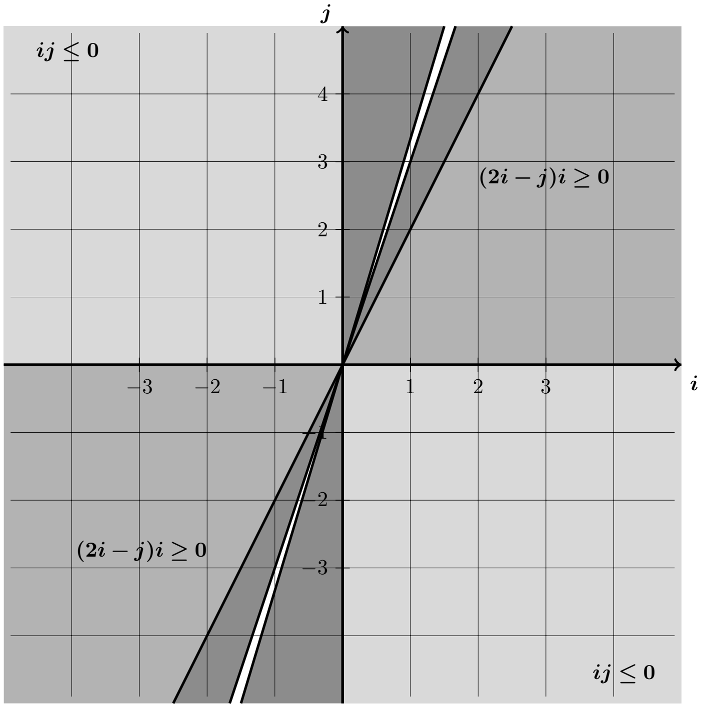

Fig. 1 The different regions to consider for (i, j) when taking linear combinations of the rows of the dual matrix A given by (4).

At this time we have checked the largest parts of the 1st and 3rd quadrant, marked with $(2i - j)i \geq 0$ on Fig. 1, to fulfill (6) as well.

c. If $(2i - j)\,i < 0$ (i.e., opposite signs) then we have more work. First observe that from $(2i - j)\,i < 0$ we can conclude that $|j| > 2$, i.e., $|j| \geq 3$. (This is also visible on the figure.) We can thus refine our estimate for δ in this case to

$$\begin{aligned}\delta &= |i 2F_{3m} + jF_{3m+1}| \\ &= |i|\,2F_{3m} + |j|F_{3m+1} \\ &\geq 2F_{3m} + 3F_{3m+1} \\ &= 2F_{3m+2} + F_{3m+1} \\ &= F_{3m+2} + F_{3m+3} \\ &= F_{3m+4}.\end{aligned}$$

(The different expressions become useful in the following.)
Now we consider the cases for the different values of α since we can easily obtain the required result by filling in the values, except for the case $\alpha = 5$ where more work is needed.

i. If $\alpha = 0, 1, 2, 3, 4$ then we get the following results:

α	$\gamma(\alpha) = F_{3m+\alpha-1} + F_{3m+1}$
0	$F_{3m-1} + F_{3m+1}$
1	$F_{3m} + F_{3m+1} = F_{3m+2}$
2	$F_{3m+1} + F_{3m+1} = 2F_{3m+1}$
3	$F_{3m+2} + F_{3m+1} = F_{3m+3}$
4	$F_{3m+3} + F_{3m+1} = F_{3m+2} + 2F_{3m+1}$

We note that for all of these values of α we have that $\delta \geq \gamma(\alpha)$ (using the refined estimate for δ for all conditions we have set).

ii. If $\alpha = 5$ then

$$\begin{aligned} \beta(5) &= |i2F_{3m+5} - jF_{3m+4}|, \\ \delta &= F_{3m+4}, \\ \gamma(5) &= F_{3m+4} + F_{3m+1}, \end{aligned}$$

where we thankfully use the refined value of δ to only have to show $\beta(5) \geq F_{3m+1}$. We can rewrite $\beta(5)$ as follows

$$\begin{aligned} \beta(5) &= |i2F_{3m+5} - jF_{3m+4}| \\ &= |(10i - 3j)F_{3m+1} + 2(3i - j)F_{3m}|. \end{aligned}$$

Set $a = 10i - 3j$ and $b = 3i - j$ and consider the different sign settings.

A. If $a = 0$ then we need integer solutions of $i = \frac{3}{10}j$. This means $|j| \geq 10$ (and $|i| \geq 3$) and we can make a new refinement for δ, $\delta \geq 6F_{3m} + 10F_{3m+1}$, which solves this case since we have $(2F_{3m} + 3F_{3m+1}) + F_{3m+1} = F_{3m+4} + F_{3m+1}$.

B. If $b = 0$ then we obtain $j = 3i$ and as such $a = 10i - 9i = i$ from which it follows that $\beta(5) = F_{3m+1}$. I.e., for $i = 1$ and $j = 3$ we get a point on the boundary.

C. If $ab > 0$ then the result is trivial as clearly then $\beta(5) \geq F_{3m+1} + 2F_{3m}$.

We now have checked the darkest marked part on Fig. 1 to fulfill (6) and are left with the narrow white wedge in between $j = \frac{10}{3}i$ and $j = 3i$.

D. If $ab < 0$ then we either have

$$\begin{cases} 10i - 3j < 0 \\ 6i - 2j > 0 \end{cases} \qquad \text{or} \qquad \begin{cases} 10i - 3j > 0 \\ 6i - 2j < 0 \end{cases}$$

$$\Leftrightarrow \tfrac{10}{3}i < j < 3i \qquad\qquad \Leftrightarrow 3i < j < \tfrac{10}{3}i.$$

From the inequality on the left side follows that both i and j should be negative, while the inequality on the right side makes it that both i and j are positive. Combined we can write:

$$3|i| < |j| < \tfrac{10}{3}|i|.$$

This means $|j| \geq 4$ (and is easily checked on the figure). Refining the estimate of δ we get

$$\begin{aligned}\delta &= |i| 2F_{3m} + |j| F_{3m+1} \\ &\geq 2F_{3m} + 4F_{3m+1} \\ &= (2F_{3m} + 3F_{3m+1}) + F_{3m+1} \\ &= F_{3m+4} + F_{3m+1},\end{aligned}$$

and thus $\delta \geq \gamma(5)$.

This completes our proof. □

The result of the theorem is put together in Table 1. It is clear from the table that the cases for $\alpha = 1, 2, 3$ are not interesting. The extended rules have the same degree as the original rules, and the number of additional points is the same as for the original rule. In other words, using the embedded rules costs the same as using two rules. In the next section we will show that the remaining cases are however interesting.

Table 1 Trigonometric degrees of Q_k for the different cases of Theorem 3 compared to the degree of the basic rule $\mathscr{L}_k$.

k mod 6	k mod 2	k mod 3	$d(\mathscr{L}_k)$	$d(Q_k)$	comparison
0	0	0	$2F_{3m} - 1$	$2F_{3m} + F_{3m-3} - 1$	higher *
1	1	1	$F_{3m+2} - 1$	$F_{3m+2} - 1$	equal
2	0	2	$2F_{3m+1} - 1$	$2F_{3m+1} - 1$	equal
3	1	0	$F_{3m+3} - 1$	$F_{3m+3} - 1$	equal
4	0	1	$2F_{3m+2} - 1$	$2F_{3m+2} + F_{3m-1} - 1$	higher **
5	1	2	$F_{3m+4} - 1$	$F_{3m+4} + F_{3m+1} - 1$	higher

Note that the line marked with a star (*) includes $k = 6$, i.e., $m = 1$ and $\alpha = 0$. In this exceptional case $d(\mathscr{L}_k) = d(Q_k)$ since then $F_{3m-3} = F_0 = 0$. For all other values of $k \equiv 0 \pmod{6}$ the rule Q_k does have a higher degree than the basic rule $\mathscr{L}_k$. For the double starred line (**) there is no such problem if one allows negative indices for the Fibonacci numbers (see, e.g., [7]). For $k = 4$, i.e., $m = 0$ and $\alpha = 4$, we then have $d(Q_4) = F_3 + F_1 - 1 = 2$, while $d(\mathscr{L}_4) = 2F_2 - 1 = 1$.

Initially, we recognised the five different cases, and we attempted proofs for each α using a generator matrix that suited each case best. There are indeed other useful generator matrices of the dual lattice. We can, e.g., rewrite the matrix A used in the previous proof:

$$A = \begin{pmatrix} 2F_{3m+\alpha} & 2F_{3m} \\ -F_{3m+\alpha-1} & F_{3m+1} \end{pmatrix} = \begin{pmatrix} F_{3m+2+\alpha} - F_{3m-1+\alpha} & F_{3m+2} - F_{3m-1} \\ -F_{3m+\alpha-1} & F_{3m+1} \end{pmatrix}.$$

This can be transformed as follows

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} A = \begin{pmatrix} F_{3m+2+\alpha} & F_{3m-2} \\ -F_{3m+\alpha-1} & F_{3m+1} \end{pmatrix}.$$

For specific values of α this generator matrix is more convenient.

Of special interest is the case $\alpha = 5$. It causes extra difficulties in the general proof given above. If treated separately, that part of the proof becomes easier. The difficult part is to show that the generator matrix B can be transformed to

$$A = \begin{pmatrix} F_{3m+1} & F_{3m+\alpha-1} \\ -F_{3m+\alpha-1} & F_{3m+1} \end{pmatrix}$$

by a unimodular matrix if $\alpha = 5$. With hindsight, this matrix was guessed after the general proof was established. Once this is established, proving the degree of the lattice rule is straightforward. This case is special because now there are four points (instead of two) of the dual lattice, one in each quadrant, lying on the boundary. This lattice has a square unit cell. This special structure is immediately evident from the matrix given above, but not from the matrix used in the general proof.

5 Final Remarks

Not all pairs of embedded cubature rules are interesting from a practical point of view. Let N_d^{opt} denote the number of points used by an optimal rule of degree d, i.e., one with the lowest number of points. If we have two embedded rules of respectively degree d_1 and d_2 with N the total number of points, and $d_1 < d_2$, then a measure for its quality is

$$\gamma := \frac{N}{N_{d_1}^{\mathrm{opt}} + N_{d_2}^{\mathrm{opt}}}.$$

Obviously, we prefer rules with $\gamma < 1$.

Lower bounds on the number of points needed to achieve a given trigonometric degree are known (see, e.g., [1, 5, 4]). In two dimensions the lower bound for a degree d rule is given by

$$N \geq N_d^{\mathrm{opt}} := \begin{cases} 2m^2 + 2m + 1, & \text{for } d = 2m, \\ 2(m+1)^2, & \text{for } d = 2m+1. \end{cases}$$

Furthermore, lattice rules are known that attain this lower bound (see, e.g., [1, 5]).

In Table 2 it can be seen that for the cases $k = 6m$, $6m+4$ and $6m+5$ investigated above, we have that $\gamma < 1$. The data in Table 2 was computationally verified completely up to $k = 54$.

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Table 2 Comparison for the trigonometric degrees of the embedded rule Q_k based on the Fibonacci rule $\mathscr{L}_k$ with the lower bounds for the same degrees. Here $\alpha = k \bmod 6$, $d_1 = d(\mathscr{L}_k)$, $N = 2F_k$ and $d_2 = d(Q_k)$. For $k = 6$ the value of γ should be ignored since there $d_1 = d_2$.

k	α	F_k	d_1	$N_{d_1}^{\text{opt}}$	N	d_2	$N_{d_2}^{\text{opt}}$	$N_{d_1}^{\text{opt}} + N_{d_2}^{\text{opt}}$	γ
4	4	3	1	2	6	2	5	7	0.857
5	5	5	2	5	10	3	8	13	0.769
6	0	8	3	8	16	3	8	16	1.000
10	4	55	9	50	110	10	61	111	0.991
11	5	89	12	85	178	15	128	213	0.836
12	0	144	15	128	288	17	162	290	0.993
16	4	987	41	882	1974	46	1105	1987	0.993
17	5	1597	54	1513	3194	67	2312	3825	0.835
18	0	2584	67	2312	5168	75	2888	5200	0.994
22	4	17711	177	15842	35422	198	19801	35643	0.994
23	5	28657	232	27145	57314	287	41472	68617	0.835
24	0	46368	287	41472	92736	321	51842	93314	0.994
28	4	317811	753	284258	635622	842	355325	639583	0.994
29	5	514229	986	487085	1028458	1219	744200	1231285	0.835
30	0	832040	1219	744200	1664080	1363	930248	1674448	0.994
34	4	5702887	3193	5100818	11405774	3570	6376021	11476839	0.994
35	5	9227465	4180	8740381	18454930	5167	13354112	22094493	0.835
36	0	14930352	5167	13354112	29860704	5777	16692642	30046754	0.994
40	4	102334155	13529	91530450	204668310	15126	114413065	205943515	0.994
41	5	165580141	17710	156839761	331160282	21891	239629832	396469593	0.835
42	0	267914296	21891	239629832	535828592	24475	299537288	539167120	0.994
46	4	1836311903	57313	1642447298	3672623806	64078	2053059121	3695506419	0.994
47	5	2971215073	75024	2814375313	5942430146	92735	4299982848	7114358161	0.835
48	0	4807526976	92735	4299982848	9615053952	103681	5374978562	9674961410	0.994
52	4	32951280099	242785	29472520898	65902560198	271442	36840651125	66313172023	0.994
53	5	53316291173	317810	50501915861	106632582346	392835	77160061448	127661977309	0.835
54	0	86267571272	392835	77160061448	172535142544	439203	96450076808	173610138256	0.994

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