On the existence of para-orthogonal rational functions on the unit circle

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Abstract

Similar as in the classical case of polynomials, as is known, para-orthogonal rational functions on the unit circle can be used to obtain quadrature formulas of Szegő-type to approximate some integrals. In the present paper we carry out a thorough discussion of the existence of such rational functions in terms of the underlying Borel measure on the unit circle

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Summary: Similar as in the classical case of polynomials, as is known, para-orthogonal rational functions on the unit circle can be used to obtain quadrature formulas of Szegő-type to approximate some integrals. In the present paper we carry out a thorough discussion of the existence of such rational functions in terms of the underlying Borel measure on the unit circle.

1 Introduction

This paper is related to former work of Bultheel, González-Vera, Hendriksen, and Njåstad on para-orthogonal rational functions on the unit circle (see, e.g., [1, Section 15], [2], [3, Chapter 5], and [4]). In particular, from [2] it is known that the zeros of such a kind of para-orthogonal functions can be used to obtain the nodes of rational Szegő formulas, i.e. quadrature formulas on the unit circle integration exactly a (maximal) class of rational functions. They are the rational equivalent of the more classical Szegő quadrature formulas that are exact in a (maximal) space of Laurent polynomials. The primary Szegő quadrature formulas can be traced in the work of Grenander and Szegő (see, e.g., [11] and [15]). For details on these quadrature formulas, one may also consult [12]. For more information on para-orthogonal polynomials on the unit circle we refer to [5], [6], [7], [10], [13, Section 2.2], [14], and [16] as well.

So far, in all the studies of para-orthogonal functions on the unit circle (even in the classical case of polynomials) the associated inner product spaces were supposed to be chosen such that these particular functions existed. However, this fundamental question can and should be asked. The investigations below give a complete answer. In fact, we carry out a thorough discussion of the existence of para-orthogonal rational functions in terms of the underlying Borel measure on the unit circle that is used to define the associated inner product space. Necessary and sufficient conditions for the existence of the para-orthogonal rational functions are obtained. Particularly, we shall show that the location of the poles of the involved rational functions does not have an influence. Roughly speaking, the situation for para-orthogonal rational functions is similar to the case of orthogonal rational functions, but not exactly the same.

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Key words and phrases: Para-orthogonal rational functions, orthogonal rational functions, reproducing kernels, Borel measures on the unit circle.

A brief synopsis is as follows. For the reader's convenience we recall in Section 2 some notations and some basics on para-orthogonal (resp., orthogonal) rational functions on the unit circle. We give the corresponding definitions and point out some elementary facts that we will make use later on. Subsequently, in Section 3 we turn to the main results of the paper. In the process, we verify some auxiliary results on reproducing kernels of rational function spaces and molecular Borel measures on the unit circle that have some value of their own.

2 Preliminaries

Let $\mathbb{N}$ be the set of all positive integers and let $\mathbb{N}_0 := \{0, 1, \ldots\}$. We define the extended sets $\hat{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ and $\hat{\mathbb{N}}_0 := \mathbb{N}_0 \cup \{\infty\}$. Further, let $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$ and $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ be the unit disk and the unit circle of the complex plane $\mathbb{C}$. The extended complex plane $\mathbb{C} \cup \{\infty\}$ will be denoted by $\hat{\mathbb{C}}$. By convention we set $\frac{1}{0} := \infty$. We also use the Kronecker delta defined as $\delta_{jk} := 1$ if $j = k$ and $\delta_{jk} := 0$ if $j \neq k$.

For the choice of the poles of the rational functions in question, we consider a situation that is somewhat more general than in [1]–[4], where they are restricted to be all outside $\mathbb{D} \cup \mathbb{T}$. Here the poles are given by a sequence $(\alpha_j)_{j=1}^{\infty}$ of complex numbers that satisfy only the restriction $\overline{\alpha_j}\alpha_k \neq 1$ for all $j, k \in \mathbb{N}$. The notation $\mathcal{T}_1$ stands for the set of all such sequences. Obviously, if $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$, then $\alpha_j \notin \mathbb{T}$ for all $j \in \mathbb{N}$.

Let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$. In the following, $\pi_{\alpha,0}$ stands for the constant function on $\hat{\mathbb{C}}$ with value 1, $\mathcal{R}_{\alpha,0}$ denotes the set of all constant (complex-valued) functions defined on $\hat{\mathbb{C}}$, and $\mathbb{P}_{\alpha,0}$ as well as $\mathbb{Z}_{\alpha,0}$ are empty sets. If $n \in \mathbb{N}$, then $\pi_{\alpha,n} : \mathbb{C} \to \mathbb{C}$ is defined by

$$\pi_{\alpha,n}(u) := \prod_{j=1}^{n} (1 - \overline{\alpha_j} u),$$

$\mathcal{R}_{\alpha,n}$ denotes the set of all rational functions f that admit a representation

$$f = \frac{p}{\pi_{\alpha,n}}$$

with some (complex) polynomial p of degree not greater than n, and finally we set

$$\mathbb{P}_{\alpha,n} := \bigcup_{j=1}^{n} \left\{ \tfrac{1}{\overline{\alpha_j}} \right\} \quad \text{and} \quad \mathbb{Z}_{\alpha,n} := \bigcup_{j=1}^{n} \{\alpha_j\}.$$

Henceforth, let $n \in \mathbb{N}_0$ and let μ be a measure defined on the σ-algebra $\mathfrak{B}_{\mathbb{T}}$ of all Borel subsets of $\mathbb{T}$. As in [1]–[4], the space $\mathcal{R}_{\alpha,n}$ of rational functions will be equipped with the (complex-valued) inner product

$$\langle f, g \rangle_{\mu} := \int_{\mathbb{T}} f(z) \overline{g(z)} \, \mu(\mathrm{d}z), \quad \forall\, f, g \in \mathcal{R}_{\alpha,n}.$$

For each $j \in \mathbb{N}$, the notation b_{α_j} stands for the *elementary Blaschke factor* corresponding to α_j, i.e.

$$b_{\alpha_j}(u) := \begin{cases} \eta_j \dfrac{\alpha_j - u}{1 - \overline{\alpha_j} u} & \text{if } u \in \mathbb{C} \setminus \left\{ \frac{1}{\overline{\alpha_j}} \right\} \\ \dfrac{1}{|\alpha_j|} & \text{if } u = \infty \,, \end{cases}$$

where for technical reasons

$$\eta_j := \begin{cases} -1 & \text{if } \alpha_j = 0 \\ \dfrac{\overline{\alpha_j}}{|\alpha_j|} & \text{if } \alpha_j \neq 0 \,. \end{cases}$$

Furthermore, let

$$B_{\alpha,0} := \pi_{\alpha,0} \quad \text{and} \quad B_{\alpha,n} := \prod_{j=1}^{n} b_{\alpha_j}, \quad n \in \mathbb{N}.$$

We define the *adjoint rational function* $f^{[\alpha,n]}$ of some function f belonging to $\mathcal{R}_{\alpha,n}$ as suggested in [3] (but with an other name and notation there), i.e. $f^{[\alpha,n]}$ denotes the rational function which is uniquely determined via

$$f^{[\alpha,n]}(v) = B_{\alpha,n}(v) \overline{f\left(\tfrac{1}{\overline{v}}\right)}, \quad v \in \mathbb{C} \setminus (\mathbb{P}_{\alpha,n} \cup \mathbb{Z}_{\alpha,n} \cup \{0\}).$$

In the particular situation that $n = 0$ or $\alpha_j = 0$ for all $j = 1, 2, \ldots, n$ it follows

$$f^{[\alpha,n]}(v) = v^n \overline{f\left(\tfrac{1}{\overline{v}}\right)}, \quad v \in \mathbb{C} \setminus \{0\},$$

and we will prefer to write $f^{[n]}$ instead of $f^{[\alpha,n]}$ (in Corollary 3.10). Some information on further interrelations between the adjoint rational function $f^{[\alpha,n]}$ and the underlying rational function f can be found in [3, Section 2.2] for the special circumstance that $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{D}$. However, an analog argumentation implies that for all $f, g \in \mathcal{R}_{\alpha,n}$ the properties below are satisfied in the current situation too:

(R1) $f^{[\alpha,n]} \in \mathcal{R}_{\alpha,n}$, $(f^{[\alpha,n]})^{[\alpha,n]} = f$.

(R2) $f^{[\alpha,n]}(\alpha_n) = 0 \Longleftrightarrow f \in \mathcal{R}_{\alpha,n-1}, \quad n \neq 0$.

(R3) $\langle f, g \rangle_\mu = \langle g^{[\alpha,n]}, f^{[\alpha,n]} \rangle_\mu$.

Recall that the measure μ is said to be *nondegenerate of order* n if the Toeplitz matrix

$$\mathbf{T}_n^{(\mu)} := \left(c_{j-k}^{(\mu)} \right)_{j,k=0}^{n}$$

is nonsingular, where $c_{-\ell}^{(\mu)} = \overline{c_\ell^{(\mu)}}$ and

$$c_\ell^{(\mu)} := \int_{\mathbb{T}} \overline{z^\ell} \, \mu(\mathrm{d}z), \quad \ell \in \mathbb{N}_0.$$

We will write $\mathcal{M}$ for the set of all measures defined on $\mathfrak{B}_{\mathbb{T}}$ and $\mathcal{M}_n$ for the set of all measures defined on $\mathfrak{B}_{\mathbb{T}}$ which are nondegenerate of order n. Moreover, we set

$$\mathcal{M}_\infty := \bigcap_{m=0}^{\infty} \mathcal{M}_m.$$

On the other hand, $\mathcal{M}_n^{\mathrm{mol}}$ stands for the set of all measures defined on $\mathfrak{B}_{\mathbb{T}}$ which are molecular of order at most n when $n \in \mathbb{N}$ and, by convention, $\mathcal{M}_0^{\mathrm{mol}}$ denotes the singleton consisting of the zero measure belonging to $\mathcal{M}$. Recall that, for $n \in \mathbb{N}$, a measure $\mu \in \mathcal{M}$ is called *molecular of order at most* n if there is a sequence $(z_j)_{j=1}^n$ of n points belonging to $\mathbb{T}$ such that $\mu(\mathbb{T} \setminus \{z_1, z_2, \ldots, z_n\}) = 0$. It is well-known that the relation

$$\mathcal{M}_n^{\mathrm{mol}} = \mathcal{M} \setminus \mathcal{M}_n \tag{2.1}$$

holds. As a slight modification of that classical result (cf. [9, Theorem 5.6 and Proposition 6.4]), if $f_0, f_1, \ldots, f_n$ is a basis of $\mathcal{R}_{\alpha,n}$, then

$$\det\Big(\big\langle f_j, f_k\big\rangle_\mu\Big)_{j,k=0}^{n} = 0 \quad \iff \quad \mu \in \mathcal{M}_n^{\mathrm{mol}}. \tag{2.2}$$

Let $\mu \in \mathcal{M}_n$. In view of (2.1) and (2.2) it is not hard to accept that $(\mathcal{R}_{\alpha,n}, \langle\cdot,\cdot\rangle_\mu)$ is a finite-dimensional Hilbert space. In analogy to [3, Theorem 2.2.2], it has a reproducing kernel $K_n^{(\alpha,\mu)}$. If $f_0, f_1, \ldots, f_n$ is a basis of $\mathcal{R}_{\alpha,n}$, then the kernel can be represented via

$$K_n^{(\alpha,\mu)}(w,v) = \big(\Upsilon_n(w)\big)^* \left(\Big(\big\langle f_j, f_k\big\rangle_\mu\Big)_{j,k=0}^{n}\right)^{-1} \Upsilon_n(v), \quad \forall\, v,w \in \hat{\mathbb{C}} \setminus \mathbb{P}_{\alpha,n},$$

where $\Upsilon_n := \big(f_0\ f_1\ \cdots\ f_n\big)^\top$. For each $w \in \hat{\mathbb{C}} \setminus \mathbb{P}_{\alpha,n}$, we also use the setting

$$K_{n,w}^{(\alpha,\mu)}(v) := K_n^{(\alpha,\mu)}(w,v), \quad \forall\, v \in \hat{\mathbb{C}} \setminus \mathbb{P}_{\alpha,n}.$$

As in [3], a sequence $(\varphi_k)_{k=0}^{\tau}$ of rational functions (with $\tau \in \hat{\mathbb{N}}_0$) is called an *orthonormal system corresponding to* $(\alpha_j)_{j=1}^\infty$ *and* μ if the two conditions below are satisfied:

(O1) $\varphi_k \in \mathcal{R}_{\alpha,k},\ k = 0, 1, \ldots, \tau$.

(O2) $\big\langle \varphi_j, \varphi_k\big\rangle_\mu = \delta_{jk},\ j,k = 0, 1, \ldots, \tau$.

If $\alpha_j = 0$ for all $j = 1, 2, \ldots, \tau$ (or if $\tau = 0$), then an orthonormal system $(\varphi_k)_{k=0}^{\tau}$ corresponding to $(\alpha_j)_{j=1}^\infty$ and μ simply consists of polynomials. In that particular case we will also refer to $(\varphi_k)_{k=0}^{\tau}$ as *orthonormal polynomial system corresponding to* μ.

Taking (2.1) and (2.2) into account, some elementary considerations imply the following (see also [3, Section 2.2]):

Proposition 2.1 *Let $\tau \in \hat{\mathbb{N}}_0$ and $\mu \in \mathcal{M}$, then there exists an orthonormal system $(\varphi_k)_{k=0}^{\tau}$ corresponding to $(\alpha_j)_{j=1}^\infty$ and μ if and only if $\mu \in \mathcal{M}_\tau$.*

Finally, we arrive at the notation for the para-orthogonal rational functions. A sequence $(\chi_j)_{j=1}^{\tau}$ of rational functions (with $\tau \in \hat{\mathbb{N}}$) is called a *para-orthogonal system corresponding to* $(\alpha_j)_{j=1}^\infty$ *and* μ if, for each element, the following three conditions hold:

(P1) $\chi_j \in \mathcal{R}_{\alpha,j}^{q\times q}$.

(P2) $\langle \chi_j, B_{\alpha,0}\rangle_\mu \neq 0$ and $\langle \chi_j, B_{\alpha,j}\rangle_\mu \neq 0$.

(P3) If $g \in \mathcal{R}_{\alpha,j}$ such that $g(\alpha_j) = 0$ and $g^{[\alpha,j]}(\alpha_j) = 0$, then $\langle \chi_j, g\rangle_\mu = 0$.

Similar as in the case of orthonormal systems, if $\alpha_j = 0$ for all $j = 1, 2, \dots, \tau$, then we will also refer to $(\chi_j)_{j=1}^{\tau}$ as *para-orthogonal polynomial system corresponding to* μ.

The terminology above is guided by [2] (see also [1, Section 15], [3, Chapter 5], and [12]). The phrase para-orthogonal therein is chosen because of deficiencies in orthogonality properties of these sequences. In view of (P3) and the difference with the definition implemented in [2] we note that (R2) implies the relation

$$\mathcal{R}'_{\alpha,j} = \{g \in \mathcal{R}_{\alpha,j-1} : g(\alpha_j) = 0\}, \tag{2.3}$$

where we use from now on the setting

$$\mathcal{R}'_{\alpha,j} := \{g \in \mathcal{R}_{\alpha,j} : g(\alpha_j) = g^{[\alpha,j]}(\alpha_j) = 0\}, \quad j = 1, 2, \dots, \tau.$$

Moreover, the para-orthogonal systems of rational functions in [2] include additionally the index $j = 0$ (involving some constant function). For that what follows, this seems to be a bit nonnatural and unwieldy, since it would require a separate (trivial, but unesthetic) treatment of the case $j = 0$ each time. Hence, the systems here begin with $j = 1$.

As is generally known, there is a relation between orthonormal and para-orthogonal systems of rational functions. In fact (cf. [2, Theorem 2]), this is given by the following:

Theorem 2.2 *Let $\tau \in \hat{\mathbb{N}}$ and $\mu \in \mathcal{M}_\tau$. Furthermore, let $(\varphi_k)_{k=0}^{\tau}$ be an orthonormal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ. Then a sequence $(\chi_j)_{j=1}^{\tau}$ of rational functions is a para-orthogonal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ if and only if, for each element, there are two nonzero complex numbers a_j and b_j such that*

$$\chi_j = a_j\varphi_j + b_j\varphi_j^{[\alpha,j]}.$$

As a consequence of Theorem 2.2 and the Christoffel–Darboux formulas for orthonormal systems of rational functions (cf. [3, Theorem 3.1.3]) we get then:

Corollary 2.3 *Let $\tau \in \hat{\mathbb{N}}$, $\mu \in \mathcal{M}_\tau$, and let $(\varphi_k)_{k=0}^{\tau}$ be an orthonormal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ. Furthermore, let $v_j \in \hat{\mathbb{C}} \setminus \mathbb{P}_{\alpha,j}$ for $j = 1, 2, \dots, \tau$ be such that $\varphi_j(v_j) \neq 0$ and $\varphi_j^{[\alpha,j]}(v_j) \neq 0$ and let*

$$\chi_j := \left(1 - \overline{b_{\alpha_j}(v_j)}b_{\alpha_j}\right)K_{j-1,v_j}^{(\alpha,\mu)}.$$

Then $(\chi_j)_{j=1}^{\tau}$ is a para-orthogonal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ.

Remark 2.4 The location of the poles and zeros of orthogonal rational functions (cf. [3, Corollary 3.2.2]) yields that, in any case, one can choose $v_j \in \mathbb{T}$ for $j = 1, 2, \dots, \tau$ in Corollary 2.3 to catch a para-orthogonal system $(\chi_j)_{j=1}^{\tau}$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ.

3 The main results

In view of Theorem 2.2 and Proposition 2.1 it is clear that (for $\tau \in \hat{\mathbb{N}}$) there is a para-orthogonal system $(\chi_j)_{j=1}^{\tau}$ corresponding to some $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ and $\mu \in \mathcal{M}$, when the underlying measure μ belongs to $\mathcal{M}_\tau$. In this section we study now as to which extent this property is necessary and sufficient. In particular, we will see that, for $\tau \in \mathbb{N}$, actually the condition $\mu \in \mathcal{M}_{\tau-1}$ is essential.

At first, we prove some auxiliary results concerning the reproducing kernel $K_n^{(\alpha,\mu)}$. Here and in the following, we assume $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ and $n \in \mathbb{N}_0$.

Lemma 3.1 *Let $\mu \in \mathcal{M}$ be such that there exist $n+1$ functions $f_0, f_1, \dots, f_n \in \mathcal{R}_{\alpha,n}$ which are point evaluation kernels, i.e., for each $f \in \mathcal{R}_{\alpha,n}$, the reproducing properties*

$$\left\langle f, f_k \right\rangle_\mu = f(w_k), \quad k = 0, 1, \dots, n,$$

hold, where $w_0, w_1, \dots, w_n$ are pairwise distinct points belonging to $\hat{\mathbb{C}} \setminus \mathbb{P}_{\alpha,n}$. Then the measure μ belongs to $\mathcal{M}_n$ and $f_k = K_{n,w_k}^{(\alpha,\mu)}$ for each $k = 0, 1, \dots, n$.

Proof: First of all, we prove by contraposition that

$$\det\Big(\left\langle B_{\alpha,j}, B_{\alpha,\ell} \right\rangle_\mu\Big)_{j,\ell=0}^{n} \neq 0. \tag{3.1}$$

Suppose that (3.1) does not hold. Thus, suppose that there is a nonzero complex vector $\mathbf{u} = \begin{pmatrix} u_0 \; u_1 \; \cdots \; u_n \end{pmatrix} \in \mathbb{C}^{1\times n+1}$ such that

$$\mathbf{u}\left(\left\langle B_{\alpha,j}, B_{\alpha,\ell} \right\rangle_\mu\right)_{j,\ell=0}^{n} = \begin{pmatrix} 0 \; 0 \; \cdots \; 0 \end{pmatrix}.$$

If we put

$$f := \sum_{\ell=0}^{n} u_\ell B_{\alpha,\ell},$$

then $f \in \mathcal{R}_{\alpha,n}$. Furthermore, from $f_k \in \mathcal{R}_{\alpha,n}$ we can infer that there is a complex vector $\mathbf{c}_k = \begin{pmatrix} c_{0;k} \; c_{1;k} \; \cdots \; c_{n;k} \end{pmatrix} \in \mathbb{C}^{1\times n+1}$ such that

$$f_k = \sum_{\ell=0}^{n} c_{\ell;k} B_{\alpha,\ell}, \quad k = 0, 1, \dots, n.$$

Hence, in view of the reproducing property we get

$$f(w_k) = \left\langle f, f_k \right\rangle_\mu = \mathbf{u}\left(\left\langle B_{\alpha,j}, B_{\alpha,\ell} \right\rangle_\mu\right)_{j,\ell=0}^{n} \mathbf{c}_k^* = 0, \quad k = 0, 1, \dots, n.$$

Since $w_0, w_1, \dots, w_n \in \hat{\mathbb{C}} \setminus \mathbb{P}_{\alpha,n}$ are some pairwise distinct points and since $f \in \mathcal{R}_{\alpha,n}$, from the fundamental theorem of algebra one can conclude that f must be the zero function, i.e. $u_\ell = 0$ for $\ell = 0, 1, \dots, n$. This is a contradiction to the choice of $\mathbf{u}$. Therefore, the assumption was wrong and (3.1) follows. By using (3.1) in combination with (2.1) and (2.2) we conclude that the measure μ belongs to $\mathcal{M}_n$. The remaining part of the assertion is then a consequence of the uniqueness of reproducing kernels in finite-dimensional Hilbert spaces (cf. [8, Theorem 10]). □

Lemma 3.2 *If $\mu \in \mathcal{M}_n$, then*

$$\big(K_{n,w}^{(\alpha,\mu)}\big)^{[\alpha,n]}(v) = \big(K_{n,v}^{(\alpha,\mu)}\big)^{[\alpha,n]}(w), \quad \forall\, v, w \in \hat{\mathbb{C}} \setminus \mathbb{P}_{\alpha,n}.$$

In particular $K_{n,z}^{(\alpha,\mu)} = \overline{B_{\alpha,n}(z)}\big(K_{n,z}^{(\alpha,\mu)}\big)^{[\alpha,n]}$ holds for each $z \in \mathbb{T}$.

Proof: Let $v, w \in \hat{\mathbb{C}} \setminus \mathbb{P}_{\alpha,n}$. In view of the reproducing property of the kernel $K_n^{(\alpha,\mu)}$, (R1), and (R3) it follows that

$$\begin{aligned}\big(K_{n,w}^{(\alpha,\mu)}\big)^{[\alpha,n]}(v) &= \Big\langle \big(K_{n,w}^{(\alpha,\mu)}\big)^{[\alpha,n]}, K_{n,v}^{(\alpha,\mu)} \Big\rangle_\mu \\ &= \Big\langle \big(K_{n,v}^{(\alpha,\mu)}\big)^{[\alpha,n]}, K_{n,w}^{(\alpha,\mu)} \Big\rangle_\mu = \big(K_{n,v}^{(\alpha,\mu)}\big)^{[\alpha,n]}(w).\end{aligned}$$

In particular, taking into account that $|B_{\alpha,n}(z)| = 1$ for $z \in \mathbb{T}$, we get

$$\begin{aligned}K_{n,z}^{(\alpha,\mu)}(v) &= |B_{\alpha,n}(z)|^2 \overline{K_n^{(\alpha,\mu)}(v,z)} = \overline{B_{\alpha,n}(z)} B_{\alpha,n}(z) \overline{K_{n,v}^{(\alpha,\mu)}\big(\tfrac{1}{\overline{z}}\big)} \\ &= \overline{B_{\alpha,n}(z)}\big(K_{n,v}^{(\alpha,\mu)}\big)^{[\alpha,n]}(z) = \overline{B_{\alpha,n}(z)}\big(K_{n,z}^{(\alpha,\mu)}\big)^{[\alpha,n]}(v),\end{aligned}$$

i.e. for each $z \in \mathbb{T}$ the identity $K_{n,z}^{(\alpha,\mu)} = \overline{B_{\alpha,n}(z)}\big(K_{n,z}^{(\alpha,\mu)}\big)^{[\alpha,n]}$. □

Based on Lemma 3.1 and Lemma 3.2 we are now able to prove that (for $\tau \in \mathbb{N}$) there exists a para-orthogonal system $(\chi_j)_{j=1}^{\tau}$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ, when the underlying measure μ belongs just to $\mathcal{M}_{\tau-1}$. The next lemma relates this with Corollary 2.3 and Remark 2.4. In fact, we obtain:

Lemma 3.3 *Let $\tau \in \mathbb{N}$ and $\mu \in \mathcal{M}_{\tau-1}$. If $z \in \mathbb{T}$, then the following are equivalent:*

(i) $\big\langle B_{\alpha,\tau}, K_{\tau-1,z}^{(\alpha,\mu)} \big\rangle_\mu = B_{\alpha,\tau}(z)$.

(ii) $\big\langle (1 - \overline{b_{\alpha_\tau}(z)} b_{\alpha_\tau}) K_{\tau-1,z}^{(\alpha,\mu)}, B_{\alpha,0} \big\rangle_\mu = 0$.

(iii) $\big\langle (1 - \overline{b_{\alpha_\tau}(z)} b_{\alpha_\tau}) K_{\tau-1,z}^{(\alpha,\mu)}, B_{\alpha,\tau} \big\rangle_\mu = 0$.

Moreover, there is only a finite set Δ of at most τ pairwise distinct points belonging to $\mathbb{T}$ such that (i) *is satisfied. In particular, if $z_\tau \in \mathbb{T} \setminus \Delta$ and if we set*

$$\chi_j := \big(1 - \overline{b_{\alpha_j}(z_j)} b_{\alpha_j}\big) K_{j-1,z_j}^{(\alpha,\mu)}, \quad j = 1, 2, \ldots, \tau,$$

(wherein $z_j \in \mathbb{T}$ can be arbitrarily chosen for $j = 1, 2, \ldots, \tau - 1$), then the system $(\chi_j)_{j=1}^{\tau}$ is a para-orthogonal corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ.

Proof: Let $z \in \mathbb{T}$. In view of the definition of $B_{\alpha,\tau}$ and the reproducing property of the kernel $K_{\tau-1}^{(\alpha,\mu)}$ we get the equality

$$\begin{aligned}\Big\langle (1-\overline{b_{\alpha_\tau}(z)}b_{\alpha_\tau})K_{\tau-1,z}^{(\alpha,\mu)}, B_{\alpha,\tau}\Big\rangle_\mu &= \Big\langle K_{\tau-1,z}^{(\alpha,\mu)}, B_{\alpha,\tau}\Big\rangle_\mu - \overline{b_{\alpha_\tau}(z)}\Big\langle K_{\tau-1,z}^{(\alpha,\mu)}, B_{\alpha,\tau-1}\Big\rangle_\mu \\ &= \overline{\Big\langle B_{\alpha,\tau}, K_{\tau-1,z}^{(\alpha,\mu)}\Big\rangle_\mu} - \overline{B_{\alpha,\tau}(z)}. \end{aligned} \tag{3.2}$$

By virtue of $B_{\alpha,0}(z)=1$ and using Lemma 3.2, $|B_{\alpha,\tau}(z)|=1$, (R1), and (R3) we obtain

$$\begin{aligned}\Big\langle (1-\overline{b_{\alpha_\tau}(z)}b_{\alpha_\tau})K_{\tau-1,z}^{(\alpha,\mu)}, B_{\alpha,0}\Big\rangle_\mu &= \Big\langle K_{\tau-1,z}^{(\alpha,\mu)}, B_{\alpha,0}\Big\rangle_\mu - \overline{b_{\alpha_\tau}(z)}\Big\langle b_{\alpha_\tau}K_{\tau-1,z}^{(\alpha,\mu)}, B_{\alpha,0}\Big\rangle_\mu \\ &= 1-\overline{B_{\alpha,\tau}(z)}\Big\langle b_{\alpha_\tau}\big(K_{\tau-1,z}^{(\alpha,\mu)}\big)^{[\alpha,\tau-1]}, B_{\alpha,0}\Big\rangle_\mu \\ &= \overline{B_{\alpha,\tau}(z)}\Big(B_{\alpha,\tau}(z) - \Big\langle B_{\alpha,\tau}, K_{\tau-1,z}^{(\alpha,\mu)}\Big\rangle_\mu\Big). \end{aligned} \tag{3.3}$$

¿From (3.2) and (3.3) the equivalence of the statements (i), (ii), and (iii) now follows. Moreover, Corollary 2.3 yields in combination with Remark 2.4 the remaining part of the assertion in the case of $\mu \in \mathcal{M}_\tau$.
We assume now that $\mu \in \mathcal{M}_{\tau-1} \setminus \mathcal{M}_\tau$. Furthermore, we suppose that (at least)

$$\Big\langle B_{\alpha,\tau}, K_{\tau-1,z_k}^{(\alpha,\mu)}\Big\rangle_\mu = B_{\alpha,\tau}(z_k) \tag{3.4}$$

for $k=0,1,\ldots,\tau$, where $z_0,z_1,\ldots,z_\tau$ are some pairwise distinct points belonging to $\mathbb{T}$. Let $f \in \mathcal{R}_{\alpha,\tau}$. Hence, there is an $h \in \mathcal{R}_{\alpha,\tau-1}$ and a $c \in \mathbb{C}$ such that $f = cB_{\alpha,\tau}+h$. Therefore, from (3.4) and the reproducing property of $K_{\tau-1}^{(\alpha,\mu)}$ it follows

$$\begin{aligned}\Big\langle f, K_{\tau-1,z_k}^{(\alpha,\mu)}\Big\rangle_\mu &= c\Big\langle B_{\alpha,\tau}, K_{\tau-1,z_k}^{(\alpha,\mu)}\Big\rangle_\mu + \Big\langle h, K_{\tau-1,z_k}^{(\alpha,\mu)}\Big\rangle_\mu \\ &= cB_{\alpha,\tau}(z_k)+h(z_k) \;=\; f(z_k)\end{aligned}$$

for $k=0,1,\ldots,\tau$. Accordingly, Lemma 3.1 implies $\mu \in \mathcal{M}_\tau$. Consequently, there are at most τ pairwise distinct points $z_0,z_1,\ldots,z_{\tau-1} \in \mathbb{T}$ such that (3.4) holds for each $k=0,1,\ldots,\tau-1$. In other words, there is a finite set Δ of at most τ elements such that $\langle B_{\alpha,\tau}, K_{\tau-1,z}^{(\alpha,\mu)}\rangle_\mu \neq B_{\alpha,\tau}(z)$ for each $z \in \mathbb{T}\setminus\Delta$. Let $z_\tau \in \mathbb{T}\setminus\Delta$. Thus, by virtue of (3.2) and (3.3) one can conclude that, by setting

$$\chi_\tau := \big(1-\overline{b_{\alpha_\tau}(z_\tau)}b_{\alpha_\tau}\big)K_{\tau-1,z_\tau}^{(\alpha,\mu)},$$

we have $\langle \chi_\tau, B_{\alpha,\tau}\rangle_\mu \neq 0$ and $\langle \chi_\tau, B_{\alpha,0}\rangle_\mu \neq 0$. In particular, for the special case $\tau=1$ it follows that $(\chi_j)_{j=1}^{\tau}$ forms a para-orthogonal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ.
Now let $\tau \geq 2$. In addition, let $z_j \in \mathbb{T}$ and let

$$\chi_j := \big(1-\overline{b_{\alpha_j}(z_j)}b_{\alpha_j}\big)K_{j-1,z_j}^{(\alpha,\mu)}, \quad j=1,2,\ldots,\tau-1.$$

Taking $\mu \in \mathcal{M}_{\tau-1}$ into account, then from Corollary 2.3 we can infer that $(\chi_j)_{j=1}^{\tau-1}$ is a para-orthogonal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ. Suppose that $g \in \mathcal{R}'_{\alpha,\tau}$. By (2.3) the function g admits a representation

$$g = \frac{p_\tau}{\pi_{\alpha,\tau-1}} p$$

with some polynomial p of degree not greater than $\tau - 2$ and $p_\tau(v) := \alpha_\tau - v$ for $v \in \mathbb{C}$. If we put $q_\tau(v) := \overline{\eta_\tau}(1 - \overline{\alpha_\tau} v)$ for $v \in \mathbb{C}$, then the reproducing property of $K_{\tau-1}^{(\alpha,\mu)}$ and the definition of b_{α_τ} lead to

$$\begin{aligned}
\left\langle \chi_\tau, g \right\rangle_\mu &= \left\langle K_{\tau-1,z_\tau}^{(\alpha,\mu)}, \frac{p_\tau}{\pi_{\alpha,\tau-1}} p \right\rangle_\mu - \overline{b_{\alpha_\tau}(z_\tau)} \left\langle b_{\alpha_\tau} K_{\tau-1,z_\tau}^{(\alpha,\mu)}, \frac{p_\tau}{\pi_{\alpha,\tau-1}} p \right\rangle_\mu \\
&= \overline{\left(\frac{p_\tau(z_\tau)}{\pi_{\alpha,\tau-1}(z_\tau)} p(z_\tau) \right)} - \overline{b_{\alpha_\tau}(z_\tau)} \left\langle K_{\tau-1,z_\tau}^{(\alpha,\mu)}, \frac{q_\tau}{\pi_{\alpha,\tau-1}} p \right\rangle_\mu \\
&= \overline{g(z_\tau)} - \overline{b_{\alpha_\tau}(z_\tau)} \overline{\left(\frac{q_\tau(z_\tau)}{\pi_{\alpha,\tau-1}(z_\tau)} p(z_\tau) \right)} = 0.
\end{aligned}$$

To that $(\chi_j)_{j=1}^{\tau}$ forms indeed a para-orthogonal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ in the case $\tau \geq 2$ as well. □

To demonstrate the converse statement, i.e. that the existence of a para-orthogonal system $(\chi_j)_{j=1}^{\tau}$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ implies $\mu \in \mathcal{M}_{\tau-1}$, we play on the following lemma saying that if $\mu \in \mathcal{M}_\tau^{\mathrm{mol}}$, then orthogonality on $\mathcal{R}_{\alpha,\tau+1}$ is equivalent with orthogonality on $\mathcal{R}'_{\alpha,\tau+1} = \{g \in \mathcal{R}_{\alpha,\tau+1} : g(\alpha_{\tau+1}) = g^{[\alpha,\tau+1]}(\alpha_{\tau+1}) = 0\}$.

Lemma 3.4 *Let $\tau \in \mathbb{N}$ and $\mu \in \mathcal{M}_\tau^{\mathrm{mol}}$. Further, let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ and $\chi \in \mathcal{R}_{\alpha,\tau+1}$. Then $\left\langle \chi, f \right\rangle_\mu = 0$ for each $f \in \mathcal{R}_{\alpha,\tau+1}$ if and only if $\left\langle \chi, g \right\rangle_\mu = 0$ for all $g \in \mathcal{R}'_{\alpha,\tau+1}$.*

Proof: If $\left\langle \chi, f \right\rangle_\mu = 0$ holds for each $f \in \mathcal{R}_{\alpha,\tau+1}$, then clearly

$$\left\langle \chi, g \right\rangle_\mu = 0, \quad \forall\, g \in \mathcal{R}'_{\alpha,\tau+1}. \tag{3.5}$$

Conversely, assume that (3.5) holds. Since $\mu \in \mathcal{M}_\tau^{\mathrm{mol}}$, there are pairwise distinct points $z_1, z_2, \ldots, z_\tau \in \mathbb{T}$ and some nonnegative numbers $d_1, d_2, \ldots, d_\tau$ such that

$$\mu = \sum_{j=1}^{\tau} d_j \varepsilon_{z_j}, \tag{3.6}$$

where ε_{z_j} denotes the Dirac measure defined on $\mathfrak{B}_{\mathbb{T}}$ with unit mass located at z_j for $j = 1, 2, \ldots, \tau$. Therefore, using some standard facts from integration theory we obtain

$$\left\langle \chi, f \right\rangle_\mu = \int_{\mathbb{T}} \chi(z) \overline{f(z)}\, \mu(\mathrm{d}z) = \sum_{j=1}^{\tau} d_j \chi(z_j) \overline{f(z_j)} \tag{3.7}$$

for all $f \in \mathcal{R}_{\alpha,\tau+1}$. Furthermore, for $k = 1, 2, \ldots, \tau$, let

$$g_k := \frac{\pi_{\alpha,\tau}(z_k)\, p_{\tau+1}}{(\alpha_{\tau+1} - z_k)\, \pi_{\alpha,\tau}} h_k, \tag{3.8}$$

where $p_{\tau+1}(v) := \alpha_{\tau+1} - v$ for $v \in \mathbb{C}$ and where

$$h_k(v) := \begin{cases} 1 & \text{if } \tau = 1 \\ \displaystyle\prod_{\substack{j=1 \\ j \neq k}}^{\tau} \frac{z_j - v}{z_j - z_k} & \text{if } \tau > 1 \end{cases} \tag{3.9}$$

for $v \in \mathbb{C}$. Since (3.8), (3.9), and (2.3) imply that $g_k \in \mathcal{R}'_{\alpha,\tau+1}$ with $g_k(z_j) = \delta_{kj}$ for $j = 1, 2, \ldots, \tau$, from (3.7) and (3.5) we conlude that

$$d_k \chi(z_k) = d_k \chi(z_k) \overline{g_k(z_k)} = \sum_{j=1}^{\tau} d_j \chi(z_j) \overline{g_k(z_j)} = \big\langle \chi, g_k \big\rangle_\mu = 0.$$

This yields in combination with (3.7) the equality $\big\langle \chi, f \big\rangle_\mu = 0$ for all $f \in \mathcal{R}_{\alpha,\tau+1}$. □

Summing up the auxiliary results above we get the following characterization concerning the existence of para-orthogonal systems of rational functions:

Theorem 3.5 *Let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ and $\tau \in \mathbb{N}$. The following statements are equivalent:*

(i) $\mu \in \mathcal{M}_{\tau-1}$.

(ii) *There exists a sequence $(z_j)_{j=1}^{\tau}$ of points belonging to $\mathbb{T}$ such that, by setting*

$$\chi_j := \big(1 - \overline{b_{\alpha_j}(z_j)} b_{\alpha_j}\big) K_{j-1,z_j}^{(\alpha,\mu)}, \tag{3.10}$$

a para-orthogonal system $(\chi_j)_{j=1}^{\tau}$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ is given.

(iii) *There is a para-orthogonal system $(\chi_j)_{j=1}^{\tau}$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ such that, for each $j = 1, 2, \ldots, \tau$ and each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j}$, with some $z_j \in \mathbb{T}$ the equality*

$$\chi_j(v) = \frac{1 - \overline{\alpha_{j-1}} v}{1 - \overline{\alpha_j} v} \Big(\varphi_{j-1}^{[\alpha,j-1]}(v) - \overline{\Big(\frac{b_{\alpha_{j-1}}(z_j) \varphi_{j-1}(z_j)}{\varphi_{j-1}^{[\alpha,j-1]}(z_j)} \Big)} b_{\alpha_{j-1}}(v) \varphi_{j-1}(v) \Big) \tag{3.11}$$

holds, where $(\varphi_k)_{k=0}^{\tau-1}$ is an orthonormal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ.

(iv) *There is a para-orthogonal system $(\chi_j)_{j=1}^{\tau}$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ.*

(v) *There is a $\chi_\tau \in \mathcal{R}_{\alpha,\tau}$ such that $\big\langle \chi_\tau, f \big\rangle_\mu \neq 0$ holds for some $f \in \mathcal{R}_{\alpha,\tau}$ and that $\big\langle \chi_\tau, g \big\rangle_\mu = 0$ holds for each $g \in \mathcal{R}'_{\alpha,\tau} = \{g \in \mathcal{R}_{\alpha,\tau} : g(\alpha_\tau) = g^{[\alpha,\tau]}(\alpha_\tau) = 0\}$.*

Proof: By Lemma 3.3, (i) leads to (ii). Moreover, the equivalence of (ii) and (iii) is a consequence of the Christoffel–Darboux formulas for orthonormal systems of rational functions (cf. [3, Theorem 3.1.3]). In addition, (ii) clearly implies (iv). Taking (P1)–(P3) into account, one can see that (iv) yields immediately (v) as well. We suppose now (v). In the particular case of $\tau = 1$, from $\left\langle \chi_1, f \right\rangle_\mu \neq 0$ for some $f \in \mathcal{R}_{\alpha,1}$ we find that $\mu \in \mathcal{M} \setminus \mathcal{M}_0^{\mathrm{mol}}$. If $\tau > 1$, then an application of Lemma 3.4 shows that $\mu \in \mathcal{M} \setminus \mathcal{M}_{\tau-1}^{\mathrm{mol}}$. Consequently, by (2.1) we infer that (v) entails (i). □

Since $z_j \in \mathbb{T}$, we observe (cf. [3, Corollary 3.1.4]) that in (3.11),

$$\overline{\left(\frac{b_{\alpha_{j-1}}(z_j)\varphi_{j-1}(z_j)}{\varphi_{j-1}^{[\alpha,j-1]}(z_j)} \right)} \in \mathbb{T}, \quad j = 1, 2, \ldots, \tau.$$

With the background of Theorem 3.5 one can immediately conclude that in the case of infinite sequences of rational functions the situation is as follows:

Corollary 3.6 *Let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$. The following statements are equivalent:*

(i) $\mu \in \mathcal{M}_\infty$.

(ii) *There is a sequence $(z_j)_{j=1}^{\infty}$ of points belonging to $\mathbb{T}$ so that the system $(\chi_j)_{j=1}^{\infty}$ with χ_j given by* (3.10) *is para-orthogonal corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ.*

(iii) *There is a para-orthogonal system $(\chi_j)_{j=1}^{\infty}$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ such that, for each $j \in \mathbb{N}$ and each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j}$, with some $z_j \in \mathbb{T}$ the equality* (3.11) *holds, where $(\varphi_k)_{k=0}^{\infty}$ is an orthonormal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ.*

(iv) *There is a para-orthogonal system $(\chi_j)_{j=1}^{\infty}$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ.*

(v) *For each $j \in \mathbb{N}$, there is a $\chi_j \in \mathcal{R}_{\alpha,j}$ such that $\left\langle \chi_j, f \right\rangle_\mu \neq 0$ with some $f \in \mathcal{R}_{\alpha,j}$ and that $\left\langle \chi_j, g \right\rangle_\mu = 0$ if $g \in \mathcal{R}'_{\alpha,j} = \{g \in \mathcal{R}_{\alpha,j} : g(\alpha_j) = g^{[\alpha,j]}(\alpha_j) = 0\}$.*

Since the condition (i) in Theorem 3.5 (resp., (i) in Corollary 3.6) does not depend on the concrete choice of the underlying sequence $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ (i.e. the location of the poles of the rational functions involved does not have an influence), it follows:

Corollary 3.7 *Let $\tau \in \hat{\mathbb{N}}$. The following statements are equivalent:*

(i) *For each sequence $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$, there is a para-orthogonal system $(\chi_j)_{j=1}^{\tau}$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ.*

(ii) *There exists a para-orthogonal system $(\chi_j)_{j=1}^{\tau}$ corresponding to some sequence $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ and μ.*

(iii) *There exists a para-orthogonal polynomial system $(\chi_j)_{j=1}^{\tau}$ corresponding to μ.*

Corollary 2.3 and Remark 2.4 (see also Lemma 3.3) suggest that some converse is also fulfilled. Namely, if (ii) (resp., (iii)) of Theorem 3.5 holds for every choice of points $z_1, z_2, \ldots, z_\tau \in \mathbb{T}$, then $\mu \in \mathcal{M}_\tau$. In the final part of this paper we want to verify that statement. We start by the following result concerning molecular Borel measures on $\mathbb{T}$, that can be seen as an addendum to Lemma 3.3.

Lemma 3.8 *Let $\tau \in \mathbb{N}$ and let $\mu \in \mathcal{M}_\tau^{\mathrm{mol}} \cap \mathcal{M}_{\tau-1}$. Then there exist exactly τ pairwise distinct points $z_1, z_2, \ldots, z_\tau$ belonging to $\mathbb{T}$ such that*

$$\left\langle B_{\alpha,\tau}, K_{\tau-1,z_j}^{(\alpha,\mu)} \right\rangle_\mu = B_{\alpha,\tau}(z_j), \quad j = 1, 2, \ldots, \tau.$$

Moreover, the measure μ admits the representation (3.6) *with these points $z_1, z_2, \ldots, z_\tau$ and $d_j = \frac{1}{K_{\tau-1,z_j}^{(\alpha,\mu)}(z_j)}$ for $j = 1, 2, \ldots, \tau$. In particular,*

$$\sum_{\substack{j=1\\ j\neq k}}^{\tau} B_{\alpha,\tau}(z_j) \frac{K_{\tau-1,z_j}^{(\alpha,\mu)}(z_k)}{K_{\tau-1,z_j}^{(\alpha,\mu)}(z_j)} = 0 \quad \text{and} \quad \sum_{\substack{j=1\\ j\neq k}}^{\tau} b_{\alpha_\tau}(z_j) \frac{K_{\tau-1,z_k}^{(\alpha,\mu)}(z_j)}{K_{\tau-1,z_j}^{(\alpha,\mu)}(z_j)} = 0, \quad k = 1, 2, \ldots, \tau.$$

Proof: The choice of μ implies in combination with (2.1) that $\mu \in \mathcal{M}_\tau^{\mathrm{mol}} \setminus \mathcal{M}_{\tau-1}^{\mathrm{mol}}$. Thus, the measure μ admits the representation (3.6) with some pairwise distinct points $z_1, z_2, \ldots, z_\tau \in \mathbb{T}$ and some positive numbers $d_1, d_2, \ldots, d_\tau$. Let $k = 1, 2, \ldots, \tau$. Furthermore, let the polynomial h_k of degree $\tau - 1$ be given by (3.9) and let

$$f_k := \frac{\pi_{\alpha,\tau-1}(z_k) h_k}{\pi_{\alpha,\tau-1}}.$$

Since $f_k \in \mathcal{R}_{\alpha,\tau-1}$ so that $f_k(z_j) = \delta_{kj}$ for $j = 1, 2, \ldots, \tau$, in view of the reproducing property of $K_{\tau-1}^{(\alpha,\mu)}$, (3.6), and some facts from integration theory (cf. (3.7)), we get

$$1 = \left\langle f_k, K_{\tau-1,z_k}^{(\alpha,\mu)} \right\rangle_\mu = \sum_{j=1}^{\tau} d_j f_k(z_j) \overline{K_{\tau-1,z_k}^{(\alpha,\mu)}(z_j)} = d_k \overline{K_{\tau-1,z_k}^{(\alpha,\mu)}(z_k)} = d_k K_{\tau-1,z_k}^{(\alpha,\mu)}(z_k),$$

or in other words

$$d_k = \frac{1}{K_{\tau-1,z_k}^{(\alpha,\mu)}(z_k)}. \tag{3.12}$$

Because of Lemma 3.3 we already know that there is a set Δ of at most τ pairwise distinct points z belonging to $\mathbb{T}$ such that the identity

$$\left\langle B_{\alpha,\tau}, K_{\tau-1,z}^{(\alpha,\mu)} \right\rangle_\mu = B_{\alpha,\tau}(z) \tag{3.13}$$

is satisfied. Let $z \in \mathbb{T}$. By virtue of (3.6) it follows

$$\left\langle B_{\alpha,\tau}, K_{\tau-1,z}^{(\alpha,\mu)} \right\rangle_\mu = \sum_{j=1}^{\tau} d_j B_{\alpha,\tau}(z_j) \overline{K_{\tau-1,z}^{(\alpha,\mu)}(z_j)}.$$

Consequently, we see that (3.13) is equivalent to

$$\sum_{j=1}^{\tau} d_j B_{\alpha,\tau}(z_j) K^{(\alpha,\mu)}_{\tau-1,z_j}(z) = B_{\alpha,\tau}(z). \tag{3.14}$$

Moreover, by using Lemma 3.2 and (R1) we obtain

$$\begin{aligned}\left\langle B_{\alpha,\tau}, K^{(\alpha,\mu)}_{\tau-1,z}\right\rangle_{\mu} &= \sum_{j=1}^{\tau} d_j B_{\alpha,\tau}(z_j) B_{\alpha,\tau-1}(z)\overline{\left(K^{(\alpha,\mu)}_{\tau-1,z}\right)^{[\alpha,\tau-1]}(z_j)}\\ &= B_{\alpha,\tau-1}(z)\sum_{j=1}^{\tau} d_j B_{\alpha,\tau}(z_j)\overline{B_{\alpha,\tau-1}(z_j)} K^{(\alpha,\mu)}_{\tau-1,z}(z_j)\\ &= B_{\alpha,\tau-1}(z)\sum_{j=1}^{\tau} d_j b_{\alpha_\tau}(z_j) K^{(\alpha,\mu)}_{\tau-1,z}(z_j).\end{aligned}$$

Thus, (3.13) is equivalent to

$$\sum_{j=1}^{\tau} d_j b_{\alpha_\tau}(z_j) K^{(\alpha,\mu)}_{\tau-1,z}(z_j) = b_{\alpha_\tau}(z) \tag{3.15}$$

as well. Taking into account that the function

$$f := \sum_{j=1}^{\tau} \overline{b_{\alpha_\tau}(z_j)} f_j$$

belongs to $\mathcal{R}_{\alpha,\tau-1}$, whereby the relation $f(z_j) = \overline{b_{\alpha_\tau}(z_j)}$ holds for $j = 1, 2, \ldots, \tau$, the reproducing property of $K^{(\alpha,\mu)}_{\tau-1}$ and (3.6) provide us

$$b_{\alpha_\tau}(z_k) = \left\langle K^{(\alpha,\mu)}_{\tau-1,z_k}, f\right\rangle_{\mu} = \sum_{j=1}^{\tau} d_j K^{(\alpha,\mu)}_{\tau-1,z_k}(z_j)\overline{f(z_j)} = \sum_{j=1}^{\tau} d_j b_{\alpha_\tau}(z_j) K^{(\alpha,\mu)}_{\tau-1,z_k}(z_j).$$

Therefore, the equality in (3.15) holds when $z = z_k$. Since the identities (3.13) and (3.15) are equivalent and since the set Δ has at most τ elements, one can conclude

$$\Delta = \{z_1, z_2, \ldots, z_\tau\}.$$

Furthermore, by using (3.12), the equivalence of (3.13) and (3.14) results in

$$\sum_{\substack{j=1\\ j\neq k}}^{\tau} B_{\alpha,\tau}(z_j)\frac{K^{(\alpha,\mu)}_{\tau-1,z_j}(z_k)}{K^{(\alpha,\mu)}_{\tau-1,z_j}(z_j)} = \sum_{j=1}^{\tau} d_j B_{\alpha,\tau}(z_j) K^{(\alpha,\mu)}_{\tau-1,z_j}(z_k) - B_{\alpha,\tau}(z_k) = 0$$

and the equivalence of (3.13) and (3.15) gives rise to

$$\sum_{\substack{j=1\\ j\neq k}}^{\tau} b_{\alpha_\tau}(z_j)\frac{K^{(\alpha,\mu)}_{\tau-1,z_k}(z_j)}{K^{(\alpha,\mu)}_{\tau-1,z_j}(z_j)} = \sum_{j=1}^{\tau} d_j b_{\alpha_\tau}(z_j) K^{(\alpha,\mu)}_{\tau-1,z_k}(z_j) - b_{\alpha_\tau}(z_k) = 0.$$

□

Proposition 3.9 *Let $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ and $\tau \in \hat{\mathbb{N}}$. The following statements are equivalent:*

(i) $\mu \in \mathcal{M}_\tau$.

(ii) *For every choice of points $z_1, z_2, \ldots, z_\tau \in \mathbb{T}$, a para-orthogonal system $(\chi_j)_{j=1}^{\tau}$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ is given by* (3.10).

(iii) *For every choice of points $z_1, z_2, \ldots, z_\tau \in \mathbb{T}$, there is a para-orthogonal system $(\chi_j)_{j=1}^{\tau}$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ such that, for each $j = 1, 2, \ldots, \tau$ and each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j}$, the equality* (3.11) *holds, where $(\varphi_k)_{k=0}^{\tau-1}$ is an orthonormal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ.*

Proof: In view of Corollary 2.3 and Remark 2.4 we see that (i) implies (ii). Moreover, the equivalence of (ii) and (iii) is a consequence of the Christoffel–Darboux formulas for orthonormal systems of rational functions (cf. [3, Theorem 3.1.3]). Now we suppose (ii). Consider at first $\tau \in \mathbb{N}$. For that, we prove by contraposition that (i) follows. Assume that $\mu \in \mathcal{M} \setminus \mathcal{M}_\tau$. By virtue of (ii), Theorem 3.5, and (2.1) we get $\mu \in \mathcal{M}_\tau^{\mathrm{mol}} \cap \mathcal{M}_{\tau-1}$. Thus, a combination of Lemma 3.8 and Lemma 3.3 shows that there is sequence $(z_j)_{j=1}^{\tau}$ of points belonging to $\mathbb{T}$ such that (3.10) does not provide a para-orthogonal system $(\chi_j)_{j=1}^{\tau}$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ. This is a contradiction to (ii). Consequently, (ii) results in (i) in the case $\tau \in \mathbb{N}$. Based on this, the case $\tau = \infty$ is obvious. □

Note that, in the particular case $\tau = \infty$, the statements in Proposition 3.9 and Corollary 3.6 actually coincide. (In Proposition 3.9 and below, $\tau - 1$ means ∞ when $\tau = \infty$.)

Similar to Corollary 3.7, from Proposition 3.9 one can conclude that:

Corollary 3.10 *Let $\tau \in \hat{\mathbb{N}}$. The following statements are equivalent:*

(i) *For each $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ and each $z_1, z_2, \ldots, z_\tau \in \mathbb{T}$, a para-orthogonal system $(\chi_j)_{j=1}^{\tau}$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ is given by* (3.10).

(ii) *For every choice of points $z_1, z_2, \ldots, z_\tau \in \mathbb{T}$, a para-orthogonal system $(\chi_j)_{j=1}^{\tau}$ corresponding to some $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ and μ is given by* (3.10).

(iii) *For every choice of points $z_1, z_2, \ldots, z_\tau \in \mathbb{T}$, a para-orthogonal polynomial system $(\chi_j)_{j=1}^{\tau}$ corresponding to μ is given by*

$$\chi_j(v) := (1 - \overline{z_j} v) \begin{pmatrix} v^0 & v^1 & \cdots & v^{j-1} \end{pmatrix} \left(\mathbf{T}_{j-1}^{(\mu)} \right)^{-1} \begin{pmatrix} \overline{z_j}^{\,0} \\ \overline{z_j}^{\,1} \\ \vdots \\ \overline{z_j}^{\,j-1} \end{pmatrix}, \quad v \in \mathbb{C}.$$

(iv) *For each $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ and each $z_1, z_2, \ldots, z_\tau \in \mathbb{T}$, there is a para-orthogonal system $(\chi_j)_{j=1}^{\tau}$ corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ such that, for all $j = 1, 2, \ldots, \tau$ and all $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j}$, the equality* (3.11) *holds, where $(\varphi_k)_{k=0}^{\tau-1}$ is an orthonormal system corresponding to $(\alpha_j)_{j=1}^{\infty}$ and μ.*

(v) *For every choice of points* $z_1, z_2, \ldots, z_\tau \in \mathbb{T}$*, there is a para-orthogonal system* $(\chi_j)_{j=1}^{\tau}$ *corresponding to some* $(\alpha_j)_{j=1}^{\infty} \in \mathcal{T}_1$ *and* μ *so that, for all* $j = 1, 2, \ldots, \tau$ *and all* $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,j}$*, the equality* (3.11) *holds, where* $(\varphi_k)_{k=0}^{\tau-1}$ *is an orthonormal system corresponding to* $(\alpha_j)_{j=1}^{\infty}$ *and* μ*.*

(vi) *For every choice of points* $z_1, z_2, \ldots, z_\tau \in \mathbb{T}$*, there is a para-orthogonal polynomial system* $(\chi_j)_{j=1}^{\tau}$ *corresponding to* μ *such that the equality*

$$\chi_j(v) = \varphi_{j-1}^{[j-1]}(v) - \overline{\left(\frac{z_j \varphi_{j-1}(z_j)}{\varphi_{j-1}^{[j-1]}(z_j)}\right)} v \varphi_{j-1}(v), \quad v \in \mathbb{C},$$

holds, where $(\varphi_k)_{k=0}^{\tau-1}$ *is an orthonormal polynomial system corresponding to* μ*.*

Remark 3.11 Because of Lemma 3.8, Proposition 3.9, and (2.1) it follows that the set Δ in Lemma 3.3 is either empty (in the case of $\mu \in \mathcal{M}_\tau$) or consists of exactly τ elements (when $\mu \in \mathcal{M}_{\tau-1} \setminus \mathcal{M}_\tau$). Moreover, one can see that the location of the poles of the involved rational functions does not have an influence of the shape of Δ.

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