Orthogonal rational functions with complex poles: The Favard theorem

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Orthogonal rational functions with complex poles: The Favard theorem*

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Abstract

Let $\{\varphi_n\}$ be a sequence of rational functions with arbitrary complex poles, generated by a certain three-term recurrence relation. In this paper we show that under some mild conditions, the rational functions φ_n form an orthonormal system with respect to a Hermitian positive-definite inner product.

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1 Introduction

In [8] a Favard theorem was given for Laurent polynomials. Later, several Favard theorems were determined for classes of rational functions with restrictions on the poles: first, the restriction that the poles are complex and outside the extended real line (or, using an inverse Cayley Transformation, outside the unit circle), see e.g. [1, 2, 4, 7]; afterwards, the restriction that the poles are all on the extended real line (or on the unit circle), see e.g. [3, 4]. Finally, in [5, Thm. 3.10] a Favard theorem was given for rational functions without restrictions on the poles.

The complete proof of this last Favard type theorem was omitted in [5] because at first it seemed that the outline of the proof would be similar to the proof given in [4, Chapt. 11.9]. However, a detailed study in [6] revealed that Theorem 3.10 could not be proved as in [4, Chapt. 11.9]; hence, Theorem 3.10 is still unproved.

The aim of this paper is to give a complete proof for Theorem 3.10 in [5]. The outline is as follows. We start with an overview of the theoretical preliminaries in Section 2. Next, in Section 3 we give some intermediate results required for the proof of the Favard theorem. Finally, Section 4 contains a complete proof of the Favard theorem formulated in [5, Thm. 3.10].

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2 Preliminaries

The field of complex numbers will be denoted by $\mathbb{C}$ and the Riemann sphere by $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. For the real line we use the symbol $\mathbb{R}$, while the extended real line will be denoted by $\overline{\mathbb{R}}$. If the value $a \in X$ is omitted in the set X, this will be represented by X_a; e.g.

$$\mathbb{C}_0 = \mathbb{C} \setminus \{0\}.$$

Let $c = a + \mathbf{i}b$, where $a, b \in \mathbb{R}$. Then we denote the real part of c by $\Re\{c\} = a$ and the imaginary part by $\Im\{c\} = b$.

Suppose a sequence of poles $\mathcal{A}_n = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset \overline{\mathbb{C}}_0$ is given. Define the factors

$$Z_k(x) = Z_{\alpha_k}(x) = \frac{x}{1 - x/\alpha_k}, \quad k = 1, 2, \ldots$$

and the basis functions

$$b_0(x) \equiv 1, \quad b_k(x) = b_{k-1}(x) Z_k(x), \quad k = 1, 2, \ldots \quad .$$

Then the space of rational functions with poles in $\mathcal{A}_n$ is defined as

$$\mathcal{L}_n = \text{span}\{b_0, b_1, \ldots, b_n\}.$$

We denote with $\mathcal{P}_n$ the space of polynomials of degree less than or equal to n. Let π_n be given by

$$\pi_0(x) \equiv 1, \quad \pi_n(x) = \prod_{k=1}^{n} (1 - x/\alpha_k).$$

Then we may write equivalently

$$\mathcal{L}_n = \left\{ \frac{p_n}{\pi_n} : p_n \in \mathcal{P}_n \right\}.$$

In the remainder, we will use the notation $\pi_{n\setminus j}$, with $n \geq j$, to denote the polynomial

$$\pi_{n\setminus j} = \frac{\pi_n}{\pi_j} \in \mathcal{P}_{n-j}.$$

In the special case in which $j = 0$ or $j = n$ we have that $\pi_{n\setminus 0} = \pi_n$, respectively $\pi_{n\setminus n} = \pi_0 = 1$.

Note that the value $\alpha_\emptyset = 0$ represents a forbidden value for the poles α_k. Since we consider only a countable number of poles α_k, we can always find a point $\alpha_\emptyset \mathbb{C}$ so that $\alpha_k \neq \alpha_\emptyset$ for every $k \geq 1$. A simple transformation can bring this $\alpha_\emptyset$ to any position that we would prefer. Therefore, this forbidden value $\alpha_\emptyset$ is not a real restriction on the sequence of poles, and we may assume it to be fixed by the value zero.

We define the involution operation or substar conjugate of a function $f \in \mathcal{L}_\infty$ as

$$f_*(x) = \overline{f(\overline{x})}.$$

This way we have that $f(x)$ has a pole in $x = \alpha$ iff $f_*(x)$ has a pole in $x = \overline{\alpha}$. With $\mathcal{L}_{n*}$ we then denote the space of rational functions given by $\mathcal{L}_{n*} = \{f_* : f \in \mathcal{L}_n\}$.

Next, let us consider an inner product that is defined by a linear functional M:

$$\langle f, g\rangle = M\{fg_*\}, \quad f, g \in \mathcal{L}_\infty.$$

When $M\{ff_*\} \neq 0$ for all $f \neq 0$ that are in $\mathcal{L}_\infty$, then the functional is called quasi-definite; moreover, when $M\{ff_*\} > 0$ for all $f \neq 0$ that are in $\mathcal{L}_\infty$, it is called positive-definite. Finally, the functional is called Hermitian if for every $f \in \mathcal{L}_\infty \cdot \mathcal{L}_{\infty *}$ it holds that $M\{f_*\} = \overline{M\{f\}}$.

Suppose there exists a sequence of rational functions $\{\varphi_n\}$, with $\varphi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$, so that the φ_n form an orthonormal system with respect to the Hermitian positive-definite linear functional M; i.e. $M\{\varphi_j \varphi_{k*}\} = \delta_{jk}$, where δ_{jk} denotes the Kronecker Delta. Further, assume that these rational functions are of the form $\varphi_n(x) = p_n(x)/\pi_n(x)$. We then call φ_n degenerate (respectively exceptional) iff $p_n(\overline{\alpha}_{n-1}) = 0$ (respectively $p_n(\alpha_{n-1}) = 0$). A zero of p_n at ∞ then means that the degree of p_n is less than n. If φ_n is not degenerate and not exceptional, it is called regular. In [4, Chapt. 11.1] (for all the poles on the extended real line), and [9, Thm. 2.1.1] and [5, Sec. 3] (for arbitrary complex poles), the following three-term recurrence relation has been proved.

Theorem 2.1. *Let $E_0 \in \mathbb{C}_0$, $\alpha_{-1} \in \overline{\mathbb{R}}_0$ and $\alpha_0 \in \overline{\mathbb{C}}_0$ be arbitrary but fixed in advance. Then the sequence of orthonormal rational functions $\{\varphi_n\}$ is regular iff for every $n \geq 1$ there exists a three-term recurrence relation of the form*

$$\varphi_n(x) = E_n Z_n(x) \left[1 + \frac{F_n}{Z_{n-1}(x)}\right] \varphi_{n-1}(x) + C_n \frac{Z_n(x)}{Z_{n-2*}(x)} \varphi_{n-2}(x), \qquad (1)$$

with

$$\begin{aligned} E_n &\neq 0 \\ C_n = -E_n \left[1 + F_n/Z_{n-1}(\overline{\alpha}_{n-1})\right] / \overline{E}_{n-1} &\neq 0. \end{aligned}$$

The initial conditions are $\varphi_{-1}(x) \equiv 0$ and $\varphi_0(x) \equiv \frac{\eta}{\sqrt{M\{1\}}}$, where η is a unimodular constant.

The previous theorem starts from a system of rational functions $\{\varphi_n\}$ for which the φ_n are orthonormal with respect to a Hermitian positive-definite inner product, to prove the existence of a three-term recurrence relation iff the system is regular. In order to derive a Favard type theorem, however, we need to verify whether the statement holds in the opposite direction; i.e., starting from a regular system of rational functions $\{\varphi_n\}$ for which the φ_n are generated by the three-term recurrence relation (1), we have to prove whether there exists a Hermitian positive-definite inner product for which the φ_n form an orthonormal system.

In the next section we give some intermediate results required for the proof of this Favard type theorem. But first we need the following auxiliary results that have been proved in [5, Sec. 3].

Lemma 2.2. *Let $A(\alpha, \beta)$ be given by*

$$A(\alpha, \beta) = \frac{1}{Z_\alpha(x)} - \frac{1}{Z_\beta(x)}.$$

Then the following statements hold:

1. $A(\alpha, \beta) = \frac{1}{Z_\alpha(\beta)}$ *and hence is independent of* x*,*

2. $A(\alpha, \beta) = -A(\beta, \alpha)$*,*

3. $\overline{A(\alpha, \beta)} = A(\overline{\alpha}, \overline{\beta})$*,*

4. $A(\alpha, \beta) - A(\gamma, \beta) = A(\alpha, \gamma)$*,*

5. $A(\alpha, \beta) + A(\alpha, \gamma) = 2A\left(\alpha, \frac{2\beta\gamma}{\beta+\gamma}\right)$*,*

6. $\frac{Z_\beta(x)}{Z_\alpha(x)} = A(\alpha, \beta) Z_\beta(x) + 1$*,*

7. $\frac{b_k(x)}{Z_\alpha(x)} = A(\alpha, \alpha_k) b_k(x) + b_{k-1}(x) \quad \in \begin{cases} \mathcal{L}_k \setminus \mathcal{L}_{k-1}, & \alpha \neq \alpha_k \\ \mathcal{L}_{k-1}, & \alpha = \alpha_k \end{cases}.$

Lemma 2.3. *Suppose* $\varphi_n(x) = \kappa_n b_n(x) + \kappa'_n b_{n-1}(x) + f_{n-2}(x)$*, where* $\kappa_n, \kappa'_n \in \mathbb{C}$*,* $\kappa_n \neq 0$ *and* $f_{n-2} \in \mathcal{L}_{n-2}$*. Then it holds that*

$$E_n = \frac{\kappa_n + \kappa'_n A(\alpha_n, \alpha_{n-1})}{\kappa_{n-1}}.$$

Lemma 2.4. *Let* $a_j(x), b_j(x), c_j(x), d_j(x), A_j, B_j$ *and* C_j*, with* $j = 1, \ldots, 4$*, be given by Table 1. Then it holds that*

$$\frac{a_j(x)}{b_j(x)} \cdot \frac{c_{j*}(x)}{d_{j*}(x)} = A_j a_j(x) + B_j c_{j*}(x) + C_j.$$

If $\alpha = \overline{\gamma}$ *in Table 1, then the equality holds in the sense that the limit of the right hand side for* $(\alpha, \gamma) \to (a, \overline{a})$ *tends to the left hand side with* $\alpha = \overline{\gamma} = a$*.*

j	$a_j(x)$	$b_j(x)$	$c_j(x)$	$d_j(x)$	A_j	B_j	C_j
1	$Z_\alpha(x)$	$Z_\beta(x)$	1	1	$A(\beta, \alpha)$	0	1
2	$Z_\alpha(x)$	1	$Z_\gamma(x)$	1	$\frac{1}{A(\overline{\gamma}, \alpha)}$	$\frac{1}{A(\alpha, \overline{\gamma})}$	0
3	$Z_\alpha(x)$	$Z_\beta(x)$	$Z_\gamma(x)$	1	$\frac{A(\beta, \alpha)}{A(\overline{\gamma}, \alpha)}$	$\frac{A(\beta, \overline{\gamma})}{A(\alpha, \overline{\gamma})}$	0
4	$Z_\alpha(x)$	$Z_\beta(x)$	$Z_\gamma(x)$	$Z_\delta(x)$	$\frac{A(\beta, \alpha) A(\overline{\delta}, \alpha)}{A(\overline{\gamma}, \alpha)}$	$\frac{A(\overline{\delta}, \overline{\gamma}) A(\beta, \overline{\gamma})}{A(\alpha, \overline{\gamma})}$	1

Table 1: Definition of $a_j(x), b_j(x), c_j(x), d_j(x), A_j, B_j$ and C_j for $j = 1, \ldots, 4$, with $\{\alpha, \beta, \gamma, \delta\} \subset \overline{\mathbb{C}}_0$.

3 Intermediate results

In this section we will assume that $\{\varphi_n\}_{n=0}^{\infty}$ is a sequence of rational functions in $\mathcal{L}_\infty$ and that the following assumptions are satisfied:

(A1) $\alpha_{-1} \in \overline{\mathbb{R}}_0$ and $\alpha_n \in \overline{\mathbb{C}}_0$, $n = 0, 1, \ldots$,

(A2) φ_n is generated by the three-term recurrence relation (1),

(A3) $\varphi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$, $n = 1, 2, \ldots$, and $\varphi_0 \in \mathbb{C}_0$,

(A4) $F_n \in \mathbb{C}$ and $E_n \in \mathbb{C}_0$, $n = 1, 2, \ldots$,

(A5)

$$\frac{1}{|E_n|^2} = 2\Re\left\{ \frac{F_n\left[1 + \overline{F}_n A(\overline{\alpha}_{n-1}, \omega_n)\right] + \left|\frac{1+F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})}{\overline{E}_{n-1}}\right|^2 A(\alpha_{n-2}, \omega_n)}{A(\alpha_n, \overline{\alpha}_n)} \right\}, \quad (2)$$

$n = 1, 2, \ldots$, where $\omega_n = \frac{|\alpha_n|^2}{\Re\{\alpha_n\}}$ and $E_0 \in \mathbb{C}_0$,

(A6) $C_n \overline{E}_{n-1} = -E_n\left[1 + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})\right] \in \mathbb{C}_0$, $n = 1, 2, \ldots$.

Let $\mathcal{S}_n$ and $\mathcal{S}_\infty$ be given by

$$\mathcal{S}_n = \text{span}\,\{\varphi_k \varphi_{l*} : 0 \leq k, l \leq n \quad \text{and} \quad k \neq l\}, \quad n > 0,$$

respectively

$$\mathcal{S}_\infty = \text{span}\,\{\mathcal{S}_n : n = 1, 2, \ldots\}.$$

Then we define M on $\mathcal{S}_\infty$ by setting

$$\forall f \in \mathcal{S}_\infty : M\{f\} = 0.$$

Note that it is always possible to define M in such a way, independent of whether the assumptions given by (A1)–(A6) are satisfied.

Next, let $\mathcal{T}_n$ and $\mathcal{T}_\infty$ be given by

$$\mathcal{T}_n = \text{span}\,\{\varphi_k \varphi_{k*} : 0 \leq k \leq n\}, \quad n > 0,$$

respectively

$$\mathcal{T}_\infty = \text{span}\,\{\mathcal{T}_n : n = 1, 2, \ldots\}.$$

Clearly we then have that $\mathcal{S}_n + \mathcal{T}_n = \mathcal{L}_n \cdot \mathcal{L}_{n*}$ for every $n > 0$. Hence, it remains to extend the definition of M to $\mathcal{T}_\infty$ in such a way that $M\{\varphi_k \varphi_{k*}\} = 1$ for every $k \geq 0$. But before we do, note that $\mathcal{S}_{n*} = \{f_* : f \in \mathcal{S}_n\} = \mathcal{S}_n$ and $\mathcal{T}_{n*} = \{f_* : f \in \mathcal{T}_n\} = \mathcal{T}_n$ for every $n > 0$.

3.1 Extension of M to $\mathcal{T}_\infty$

If we want $M\{\varphi_k \varphi_{k*}\} = 1$ for every $k \geq 0$, it is necessary and sufficient that

$$\phi = \sum_{k=0}^{\infty} a_k \varphi_k \varphi_{k*} \in (\mathcal{T}_\infty \cap \mathcal{S}_\infty) \quad \text{iff} \quad \sum_{k=0}^{\infty} a_k = 0. \quad (3)$$

We will now prove by induction that the condition given by (3) holds true under the assumptions given by (A1)–(A6).

Initialisation $n = 1$**.**

Consider the subspace $\mathcal{L}_1 \cdot \mathcal{L}_{1*} = S_1 + T_1$. We then have the following theorems.

Theorem 3.1. *Under the assumptions given by (A1)–(A6) it holds that*

$$\varphi_1\varphi_{1*} - \varphi_0\varphi_{0*} \in \mathcal{S}_1.$$

Proof. From the three-term recurrence relation (1), Lemma 2.2 and Lemma 2.4 it follows that

$$\begin{aligned}
\frac{\varphi_1\varphi_{1*}}{|E_1|^2} &= |1 + F_1 A(\alpha_0, \alpha_1)|^2 Z_1\varphi_0 Z_{1*}\varphi_{0*} + |F_1|^2\varphi_0\varphi_{0*} \\
&\quad + \overline{F}_1 [1 + F_1 A(\alpha_0, \alpha_1)] Z_1\varphi_0\varphi_{0*} + F_1 \left[1 + \overline{F}_1 A(\overline{\alpha}_0, \overline{\alpha}_1)\right] Z_{1*}\varphi_{0*}\varphi_0 \\
&= |F_1|^2\varphi_0\varphi_{0*} + \frac{[1 + F_1 A(\alpha_0, \alpha_1)] \left[1 + \overline{F}_1 A(\overline{\alpha}_0, \alpha_1)\right]}{A(\overline{\alpha}_1, \alpha_1)} Z_1\varphi_0\varphi_{0*} \\
&\quad + \frac{\left[1 + \overline{F}_1 A(\overline{\alpha}_0, \overline{\alpha}_1)\right] [1 + F_1 A(\alpha_0, \overline{\alpha}_1)]}{A(\alpha_1, \overline{\alpha}_1)} Z_{1*}\varphi_{0*}\varphi_0.
\end{aligned}$$

Further, we have that

$$[1 + F_1 A(\alpha_0, \alpha_1)] Z_1\varphi_0\varphi_{0*} = \frac{\varphi_1\varphi_{0*}}{E_1} - F_1\varphi_0\varphi_{0*},$$

and

$$\left[1 + \overline{F}_1 A(\overline{\alpha}_0, \overline{\alpha}_1)\right] Z_{1*}\varphi_{0*}\varphi_0 = \frac{\varphi_{1*}\varphi_0}{\overline{E}_1} - \overline{F}_1\varphi_0\varphi_{0*}.$$

Hence, there exists a function $f_1 \in \mathcal{S}_1$ so that

$$\frac{\varphi_1\varphi_{1*}}{|E_1|^2} = \left[|F_1|^2 - 2\Re\left\{F_1\left(\frac{1 + \overline{F}_1 A(\overline{\alpha}_0, \alpha_1)}{A(\overline{\alpha}_1, \alpha_1)}\right)\right\}\right]\varphi_0\varphi_{0*} + f_1.$$

Note that

$$\begin{aligned}
\frac{|F_1|^2}{2} - F_1\left(\frac{1 + \overline{F}_1 A(\overline{\alpha}_0, \alpha_1)}{A(\overline{\alpha}_1, \alpha_1)}\right) &= F_1\left[\frac{1 + \frac{\overline{F}_1}{2}\left(2A(\overline{\alpha}_0, \alpha_1) - A(\overline{\alpha}_1, \alpha_1)\right)}{A(\alpha_1, \overline{\alpha}_1)}\right] \\
&= F_1\left[\frac{1 + \overline{F}_1 A(\overline{\alpha}_0, \omega_1)}{A(\alpha_1, \overline{\alpha}_1)}\right],
\end{aligned}$$

where $\omega_1 = \frac{|\alpha_1|^2}{\Re\{\alpha_1\}}$. Finally, because

$$2\Re\left\{\frac{A(\alpha_{-1}, \omega_1)}{A(\alpha_1, \overline{\alpha}_1)}\right\} = 0$$

for $\alpha_{-1} \in \mathbb{R}_0$, it follows from assumption (A5) that

$$|F_1|^2 - 2\Re\left\{F_1\left[\frac{1 + \overline{F}_1 A(\alpha_0, \alpha_1)}{A(\overline{\alpha}_1, \alpha_1)}\right]\right\} = \frac{1}{|E_1|^2}.$$

Consequently,

$$\varphi_1\varphi_{1*} - \varphi_0\varphi_{0*} = |E_1|^2 f_1 \in \mathcal{S}_1.$$

□

Theorem 3.2. *Under the assumptions given by (A1)–(A6) it holds that*

$$\varphi_j \varphi_{j*} \notin \mathcal{S}_1 \quad \textit{for} \quad j = 0, 1.$$

Proof. First, consider the case that $\alpha_1 \in \overline{\mathbb{R}}_0$, and suppose that

$$\varphi_1 \varphi_{1*} = \frac{p_1 p_{1*}}{\pi_1^2} \in \mathcal{S}_1 = \text{span}\,\{\varphi_1 \varphi_{0*}, \varphi_{1*}\varphi_0\}.$$

We then have that there exists a constant $c \in \mathbb{C}_0$ so that

$$\frac{c p_1(x) + \overline{c} p_{1*}(x)}{\pi_1(x)} = \frac{p_1(x) p_{1*}(x)}{\pi_1^2(x)}.$$

Or equivalently,

$$\pi_1(x) \left[c p_1(x) + \overline{c} p_{1*}(x) \right] = p_1(x) p_{1*}(x).$$

Taking $x = \alpha_1$, it then follows that $p_1(\alpha_1) = 0$, contradicting our assumption given by (A3). Consequently, $\varphi_1 \varphi_{1*} \notin \mathcal{S}_1$, and from Theorem 3.1 it then follows that $\varphi_0 \varphi_{0*} \notin \mathcal{S}_1$.

Finally, consider the case that $\alpha_1 \notin \overline{\mathbb{R}}$, and suppose that

$$\varphi_0 \varphi_{0*} \in \mathcal{S}_1 = \text{span}\,\{\varphi_1 \varphi_{0*}, \varphi_{1*}\varphi_0\}.$$

We then have that there exists a constant $c \in \mathbb{C}_0$ so that

$$\frac{c p_1(x)}{\pi_1(x)} + \frac{\overline{c} p_{1*}(x)}{\pi_{1*}(x)} = \varphi_0 \varphi_{0*}.$$

Or equivalently,

$$\pi_{1*}(x) c p_1(x) + \pi_1(x) \overline{c} p_{1*}(x) = \pi_1(x) \pi_{1*}(x) \varphi_0 \varphi_{0*}.$$

Taking $x = \alpha_1$ or $x = \overline{\alpha}_1$, it then follows that

$$\pi_{1*}(\alpha_1) c p_1(\alpha_1) = 0,$$

respectively

$$\pi_1(\overline{\alpha}_1) \overline{c} p_{1*}(\overline{\alpha}_1) = 0.$$

Due to our assumption given by (A3) we find that $c = \overline{c} = 0 \notin \mathbb{C}_0$. Hence, $\varphi_0 \varphi_{0*} \notin \mathcal{S}_1$, and from Theorem 3.1 it then follows that $\varphi_1 \varphi_{1*} \notin \mathcal{S}_1$. □

Hence, we now have proved that

$$\phi_1 = a_0 \varphi_0 \varphi_{0*} + a_1 \varphi_1 \varphi_{1*} \in (\mathcal{T}_1 \cap \mathcal{S}_1) \quad \text{iff} \quad a_0 + a_1 = 0.$$

Induction for $n > 1$.

Consider the subspaces $\mathcal{L}_j \cdot \mathcal{L}_{j*} = S_j + T_j$, with $j = n-1, n$, and suppose that for $j = n-1$ it holds that

$$\phi_{n-1} = \sum_{k=0}^{n-1} a_k \varphi_k \varphi_{k*} \in (\mathcal{T}_{n-1} \cap \mathcal{S}_{n-1}) \quad \text{iff} \quad \sum_{k=0}^{n-1} a_k = 0.$$

We then have to prove for $j = n$ that

$$\phi_n = \sum_{k=0}^{n} a_k \varphi_k \varphi_{k*} \in (\mathcal{T}_n \cap \mathcal{S}_n) \quad \text{iff} \quad \sum_{k=0}^{n} a_k = 0.$$

First we need the following lemma.

Lemma 3.3. *There exist functions $g_n, h_n \in \mathcal{S}_n$ so that*

$$\begin{aligned}&[1 + F_n A(\alpha_{n-1}, \alpha_n)]\, Z_n \varphi_{n-1} \varphi_{n-1*} \\ &\quad = A(\overline{\alpha}_{n-2}, \alpha_n) \frac{1 + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})}{\overline{E}_{n-1}} Z_n \varphi_{n-2} \varphi_{n-1*} - F_n \varphi_{n-1} \varphi_{n-1*} + g_n\end{aligned}$$

and

$$\begin{aligned}&A(\overline{\alpha}_{n-2}, \alpha_n) \frac{1 + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})}{\overline{E}_{n-1}} Z_n \varphi_{n-2} \varphi_{n-2*} \\ &= [1 + F_n A(\alpha_{n-1}, \alpha_n)]\, Z_n \varphi_{n-1} \varphi_{n-2*} - \frac{1 + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})}{\overline{E}_{n-1}} \varphi_{n-2} \varphi_{n-2*} + h_n.\end{aligned}$$

Proof. From the three-term recurrence relation (1), Lemma 2.2 and Lemma 2.4 it follows that

$$\begin{aligned}\frac{\varphi_n \varphi_{k*}}{E_n} = Z_n \Bigg\{ &\left(1 + F_n A(\alpha_{n-1}, \alpha_n) + \frac{F_n}{Z_n}\right) \varphi_{n-1} \\ &- \frac{1 + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})}{\overline{E}_{n-1}} \left(A(\overline{\alpha}_{n-2}, \alpha_n) + \frac{1}{Z_n}\right) \varphi_{n-2} \Bigg\} \varphi_{k*}\end{aligned}$$

The equalities are now easily verified by taking $k = n-1$, respectively $k = n-2$. □

Theorem 3.4. *Under the assumptions given by (A1)–(A6) it holds that*

$$\varphi_n \varphi_{n*} - \varphi_{n-1} \varphi_{n-1*} \in \mathcal{S}_n.$$

Proof. First, note that it follows from the three-term recurrence relation (1), Lemma 2.2 and Lemma 2.4 that

$$\frac{\varphi_n \varphi_{n*}}{|E_n|^2} = H_n H_{n*} - H_n G_{n*} - H_{n*} G_n + G_n G_{n*},$$

where

$$H_n = Z_n \left(1 + F_n A(\alpha_{n-1}, \alpha_n) + \frac{F_n}{Z_n}\right) \varphi_{n-1}$$

and

$$G_n = Z_n \frac{1 + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})}{\overline{E}_{n-1}} \left(A(\overline{\alpha}_{n-2}, \alpha_n) + \frac{1}{Z_n} \right) \varphi_{n-2}.$$

Next, let a_n, b_n, c_n and d_n be defined as

$$a_n = A(\overline{\alpha}_{n-2}, \alpha_n) \frac{1 + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})}{\overline{E}_{n-1}} \left(\frac{1 + \overline{F}_n A(\overline{\alpha}_{n-1}, \alpha_n)}{A(\overline{\alpha}_n, \alpha_n)} \right),$$

$$b_n = A(\overline{\alpha}_{n-2}, \overline{\alpha}_n) \frac{1 + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})}{\overline{E}_{n-1}} \left(\frac{1 + \overline{F}_n A(\overline{\alpha}_{n-1}, \overline{\alpha}_n)}{A(\alpha_n, \overline{\alpha}_n)} \right),$$

$$c_n = 2\Re \left\{ \frac{F_n \left[1 + \overline{F}_n A(\overline{\alpha}_{n-1}, \omega_n) \right]}{A(\alpha_n, \overline{\alpha}_n)} \right\},$$

and

$$d_n = 2\Re \left\{ \frac{\left| \frac{1 + \hat{F}_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})}{\overline{E}_{n-1}} \right|^2 A(\alpha_{n-2}, \omega_n)}{A(\alpha_n, \overline{\alpha}_n)} \right\},$$

where $\omega_n = \frac{|\alpha_n|^2}{\Re\{\alpha_n\}}$. Based on Lemma 2.2, 2.4 and 3.3, we find after some computations similar to those in [5, Thm. 3.5] that there exist functions $f_n, g_n \in \mathcal{S}_n$ and $h_{n-1} \in \mathcal{S}_{n-1}$ so that

$$H_n H_{n*} = a_n Z_n \varphi_{n-2} \varphi_{n-1*} + \overline{a}_n Z_{n*} \varphi_{n-2*} \varphi_{n-1} + c_n \varphi_{n-1} \varphi_{n-1*} + f_n,$$

$$G_n G_{n*} = b_n Z_{n*} \varphi_{n-1*} \varphi_{n-2} + \overline{b}_n Z_n \varphi_{n-1} \varphi_{n-2*} + d_n \varphi_{n-2} \varphi_{n-2*} + g_n$$

and

$$H_n G_{n*} + H_{n*} G_n = -\big(a_n Z_n \varphi_{n-2} \varphi_{n-1*} + \overline{a}_n Z_{n*} \varphi_{n-2*} \varphi_{n-1} + b_n Z_{n*} \varphi_{n-1*} \varphi_{n-2} + \overline{b}_n Z_n \varphi_{n-1} \varphi_{n-2*} \big) + h_{n-1}.$$

Consequently, we have that

$$\frac{\varphi_n \varphi_{n*}}{|E_n|^2} = (c_n + d_n) \varphi_{n-1} \varphi_{n-1*} + k_n,$$

where

$$k_n = f_n + g_n - h_{n-1} - d_n (\varphi_{n-1} \varphi_{n-1*} - \varphi_{n-2} \varphi_{n-2*}).$$

From the induction hypotheses it now follows that

$$\varphi_{n-1} \varphi_{n-1*} - \varphi_{n-2} \varphi_{n-2*} \in \mathcal{S}_{n-1} \subseteq \mathcal{S}_n,$$

so that $k_n \in \mathcal{S}_n$. Finally, we have that $c_n + d_n = \frac{1}{|E_n|^2}$ due to our assumption (A5), so that

$$\varphi_n \varphi_{n*} - \varphi_{n-1} \varphi_{n-1*} = |E_n|^2 k_n \in \mathcal{S}_n.$$

□

As a consequence of the previous theorem, we now have the following corollary.

Corollary 3.5. *Under the assumptions given by (A1)–(A6) it holds for* $j = 0, \ldots, n-2$ *that*

$$\varphi_n \varphi_{n*} - \varphi_j \varphi_{j*} \in \mathcal{S}_n.$$

Proof. Note that

$$\varphi_n \varphi_{n*} - \varphi_j \varphi_{j*} = (\varphi_n \varphi_{n*} - \varphi_{n-1} \varphi_{n-1*}) + (\varphi_{n-1} \varphi_{n-1*} - \varphi_j \varphi_{j*}),$$

where it follows from the induction hypotheses that

$$\varphi_{n-1} \varphi_{n-1*} - \varphi_j \varphi_{j*} \in \mathcal{S}_{n-1} \subseteq \mathcal{S}_n.$$

The statement now directly follows from Theorem 3.4. □

It remains to prove that $\varphi_k \varphi_{k*} \notin \mathcal{S}_n$ for $k = 0, \ldots, n$. Therefore we first need the following lemma.

Lemma 3.6. *Under the assumptions given by (A1)–(A6) it holds for every* $g_{n-2} \in \mathcal{L}_{n-2}$ *that*

$$\frac{Z_{n-1*} g_{n-2*} \varphi_n}{Z_n} \in \mathcal{S}_{n-1}.$$

Proof. First, note that there exist coefficients $a_1, a_2, \ldots, a_{n-1}$ so that

$$Z_{n-1}(x) g_{n-2}(x) = \sum_{k=1}^{n-1} a_k b_k(x).$$

From the three-term recurrence relation (1) it now follows that

$$\frac{b_{k*} \varphi_n}{E_n Z_n} = \left[1 + \frac{F_n(x)}{Z_{n-1}}\right] \varphi_{n-1} b_{k*} - \frac{1 + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})}{\overline{E}_{n-1}} \varphi_{n-2} \frac{b_{k*}}{Z_{n-2*}}. \tag{4}$$

By means of the last statement in Lemma 2.2 it is easily verified that the right hand side of (4) is in $\mathcal{S}_{n-1}$ for $k = 1, \ldots, n-2$. While for $k = n-1$ we have that

$$\left[1 + \frac{F_n(x)}{Z_{n-1}}\right] \varphi_{n-1} b_{n-1*} = [1 + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})] \varphi_{n-1} b_{n-1*} + F_n \varphi_{n-1} b_{n-2*}$$

and

$$\varphi_{n-2} \frac{b_{n-1*}}{Z_{n-2*}} = A(\overline{\alpha}_{n-2}, \overline{\alpha}_{n-1}) \varphi_{n-2} b_{n-1*} + \varphi_{n-2} b_{n-2*}.$$

Consequently,

$$\begin{aligned} &\frac{b_{n-1*} \varphi_n}{E_n [1 + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})] Z_n} \\ &\qquad = \left(\varphi_{n-1} - \frac{A(\overline{\alpha}_{n-2}, \overline{\alpha}_{n-1})}{\overline{E}_{n-1}} \varphi_{n-2}\right) b_{n-1*} - \frac{1}{\overline{E}_{n-1}} \varphi_{n-2} b_{n-2*} + k_{n-1}, \end{aligned}$$

where

$$k_{n-1} = \frac{F_n}{1 + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})} \varphi_{n-1} b_{n-2*} \in \mathcal{S}_{n-1}.$$

Suppose that $\varphi_{n-1} = \kappa_{n-1} b_{n-1} + \kappa'_{n-1} b_{n-2} + f_{n-3}$, where $\kappa_{n-1}, \kappa'_{n-1} \in \mathbb{C}$, $\kappa_{n-1} \neq 0$ and $f_{n-3} \in \mathcal{L}_{n-3}$. Then we get that

$$\begin{aligned}
&\left(\varphi_{n-1} - \frac{A(\overline{\alpha}_{n-2}, \overline{\alpha}_{n-1})}{\overline{E}_{n-1}} \varphi_{n-2}\right) b_{n-1*} \\
&\quad = \left(\varphi_{n-1} - \frac{A(\overline{\alpha}_{n-2}, \overline{\alpha}_{n-1})}{\overline{E}_{n-1}} \varphi_{n-2}\right) \frac{1}{\overline{\kappa}_{n-1}} \left(\varphi_{n-1*} - \overline{\kappa}'_{n-1} b_{n-2*} - f_{n-3*}\right) \\
&\qquad = \frac{1}{\overline{\kappa}_{n-1}} \left[\varphi_{n-1}\varphi_{n-1*} - \frac{\overline{\kappa}'_{n-1} A(\overline{\alpha}_{n-1}, \overline{\alpha}_{n-2})}{\overline{E}_{n-1}} \varphi_{n-2} b_{n-2*} - h_{n-1}\right],
\end{aligned}$$

where

$$\begin{aligned}
h_{n-1} = \varphi_{n-1} &\left(\overline{\kappa}'_{n-1} b_{n-2*} + f_{n-3*}\right) \\
&+ \frac{A(\overline{\alpha}_{n-2}, \overline{\alpha}_{n-1})}{\overline{E}_{n-1}} \varphi_{n-2} \left(\varphi_{n-1*} - f_{n-3*}\right) \in \mathcal{S}_{n-1}.
\end{aligned}$$

Hence, with $h'_{n-1} = (\overline{\kappa}_{n-1} k_{n-1} - h_{n-1}) \in \mathcal{S}_{n-1}$ we have that

$$\begin{aligned}
&\frac{b_{n-1*}\varphi_n}{E_n \left[1 + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})\right] Z_n} \\
&\quad = \frac{1}{\overline{\kappa}_{n-1}} \left[\varphi_{n-1}\varphi_{n-1*} - \frac{\overline{\kappa}_{n-1} + \overline{\kappa}'_{n-1} A(\overline{\alpha}_{n-1}, \overline{\alpha}_{n-2})}{\overline{E}_{n-1}} \varphi_{n-2} b_{n-2*} + h'_{n-1}\right].
\end{aligned}$$

Finally, suppose that $\varphi_{n-2} = \kappa_{n-2} b_{n-2} + l_{n-3}$, where $\kappa_{n-2} \neq 0$ and $l_{n-3} \in \mathcal{L}_{n-3}$. Then it follows from Lemma 2.3 that

$$\frac{b_{n-1*}\varphi_n}{E_n \left[1 + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})\right] Z_n} = \frac{1}{\overline{\kappa}_{n-1}} \left[\varphi_{n-1}\varphi_{n-1*} - \varphi_{n-2}\varphi_{n-2*} + g'_{n-1}\right],$$

where $g'_{n-1} = (h'_{n-1} + \varphi_{n-2} l_{n-3*}) \in \mathcal{S}_{n-1}$, and

$$\varphi_{n-1}\varphi_{n-1*} - \varphi_{n-2}\varphi_{n-2*} \in \mathcal{S}_{n-1}$$

as well due to the induction hypotheses. □

Theorem 3.7. *Under the assumptions given by (A1)–(A6) it holds that*

$$\varphi_j \varphi_{j*} \notin \mathcal{S}_n \quad \textit{for} \quad j = 0, \ldots, n.$$

Proof. First, consider the case that $\alpha_n \in \overline{\mathbb{R}}_0$, and suppose that

$$\varphi_n \varphi_{n*} = \frac{p_n p_{n*}}{\pi_n \pi_{n*}} \in \mathcal{S}_n = \left(\mathcal{S}_{n-1} + \varphi_n \cdot \mathcal{L}_{n-1*} + \varphi_{n*} \cdot \mathcal{L}_{n-1}\right).$$

We then have that there exist a polynomial $r_{n-1} \in \mathcal{P}_{n-1}$ and function $h_{n-1} \in \mathcal{S}_{n-1}$ so that

$$h_{n-1}(x) + \frac{p_n(x)r_{n-1}(x) + p_{n*}(x)r_{n-1*}(x)}{\pi_{n-1}(x)\pi_{n*}(x)} = \frac{p_n(x)p_{n*}(x)}{\pi_n(x)\pi_{n*}(x)}.$$

Or equivalently,

$$\pi_n(x)\pi_{n*}(x)h_{n-1}(x) \\ + \pi_{n\backslash(n-1)}(x)\left[p_n(x)r_{n-1}(x) + p_{n*}(x)r_{n-1*}(x)\right] = p_n(x)p_{n*}(x).$$

Taking $x = \alpha_n$, it then follows that $p_n(\alpha_n) = 0$, contradicting our assumption given by (A3). Consequently, $\varphi_n\varphi_{n*} \notin \mathcal{S}_n$, and from Theorem 3.4 and Corollary 3.5 it then follows that $\varphi_j\varphi_{j*} \notin \mathcal{S}_n$ for $j = 0, \ldots, n$.

Finally, consider the case that $\alpha_n \notin \overline{\mathbb{R}}$, and suppose that

$$\varphi_{n-1}\varphi_{n-1*} = \frac{p_{n-1}p_{n-1*}}{\pi_{n-1}\pi_{n-1*}} \in \mathcal{S}_n = \left(\mathcal{S}_{n-1} + \varphi_n \cdot \mathcal{L}_{n-1*} + \varphi_{n*} \cdot \mathcal{L}_{n-1}\right).$$

We then have that there exist a polynomial $r_{n-1} \in \mathcal{P}_{n-1}$ and function $h_{n-1} \in \mathcal{S}_{n-1}$ so that

$$h_{n-1}(x) + \frac{p_n(x)r_{n-1}(x)}{\pi_n(x)\pi_{n-1*}(x)} + \frac{p_{n*}(x)r_{n-1*}(x)}{\pi_{n*}(x)\pi_{n-1}(x)} = \frac{p_{n-1}(x)p_{n-1*}(x)}{\pi_{n-1}(x)\pi_{n-1*}(x)}.$$

Or equivalently,

$$\pi_n(x)\pi_{n*}(x)h_{n-1}(x) + \pi_{n*\backslash(n-1)*}(x)p_n(x)r_{n-1}(x) \\ + \pi_{n\backslash(n-1)}(x)p_{n*}(x)r_{n-1*}(x) = \pi_{n\backslash(n-1)}(x)\pi_{n*\backslash(n-1)*}(x)p_{n-1}(x)p_{n-1*}(x).$$

Taking $x = \alpha_n$ or $x = \overline{\alpha}_n$, it then follows that

$$\pi_{n*\backslash(n-1)*}(\alpha_n)p_n(\alpha_n)r_{n-1}(\alpha_n) = 0,$$

respectively

$$\pi_{n\backslash(n-1)}(\overline{\alpha}_n)p_{n*}(\overline{\alpha}_n)r_{n-1*}(\overline{\alpha}_n) = 0.$$

Consequently, $r_{n-1}(\alpha_n) = r_{n-1*}(\overline{\alpha}_n) = 0$ due to our assumption given by (A3). Hence, there exists a function $g_{n-2} \in \mathcal{L}_{n-2}$ so that

$$\frac{p_n(x)r_{n-1}(x)}{\pi_n(x)\pi_{n-1*}(x)} + \frac{p_{n*}(x)r_{n-1*}(x)}{\pi_{n*}(x)\pi_{n-1}(x)} \\ = \frac{Z_{n-1*}(x)g_{n-2*}(x)\varphi_n(x)}{Z_n(x)} + \frac{Z_{n-1}(x)g_{n-2}(x)\varphi_{n*}(x)}{Z_{n*}(x)}.$$

From Lemma 3.6 it now follows that

$$h_{n-1} + \frac{p_n r_{n-1}}{\pi_n\pi_{n-1*}} + \frac{p_{n*}r_{n-1*}}{\pi_{n*}\pi_{n-1}} \in \mathcal{S}_{n-1},$$

while it follows from the induction hypotheses that $\varphi_{n-1}\varphi_{n-1*} \notin \mathcal{S}_{n-1}$. Hence, $\varphi_{n-1}\varphi_{n-1*} \notin \mathcal{S}_n$, and from Theorem 3.4 and Corollary 3.5 it then follows that $\varphi_j\varphi_{j*} \notin \mathcal{S}_n$ for $j = 0, \ldots, n$. □

Thus, we now have proved the following theorem.

Theorem 3.8. *Under the assumptions given by (A1)–(A6) it holds that*

$$\phi = \sum_{k=0}^{\infty} a_k \varphi_k \varphi_{k*} \in (\mathcal{T}_\infty \cap \mathcal{S}_\infty) \quad \textit{iff} \quad \sum_{k=0}^{\infty} a_k = 0.$$

3.2 An equivalent for assumption (A5)

Note that the denominator of the right hand side in (2) equals zero whenever $\alpha_n \in \overline{\mathbb{R}}_0$. On the other hand, from Lemma 2.4 it follows that the equality in (2) must hold in a limiting sense. Therefore, the numerator of the right hand side in (2) has to equal zero as well whenever $\alpha_n \in \overline{\mathbb{R}}_0$. We will now give an equivalent formulation for assumption (A5) that is more interesting from a numerical point of view.

In [5, Sec. 3] it has been proved that Equation (2) is equivalent with

$$\left|\frac{E_{n-1}}{E_n}\right|^2 = \frac{\left[\Im\{F_n\} - |F_n|^2 \frac{\Im\{\alpha_{n-1}\}}{|\alpha_{n-1}|^2}\right]\left[|E_{n-1}|^2 - 4\frac{\Im\{\alpha_{n-1}\}}{|\alpha_{n-1}|^2}\frac{\Im\{\alpha_{n-2}\}}{|\alpha_{n-2}|^2}\right] + \frac{\Im\{\alpha_{n-2}\}}{|\alpha_{n-2}|^2}}{\frac{\Im\{\alpha_n\}}{|\alpha_n|^2}}. \tag{5}$$

Note that for $\alpha_{n-1} \in \overline{\mathbb{R}}_0$, Equation (5) reduces to

$$\Im\{F_n\} = \frac{\Im\{\alpha_n\}}{|\alpha_n|^2} \cdot \frac{1}{|E_n|^2} - \frac{\Im\{\alpha_{n-2}\}}{|\alpha_{n-2}|^2} \cdot \frac{1}{|E_{n-1}|^2}.$$

While for $\alpha_{n-1} \notin \overline{\mathbb{R}}$, we have that

$$\frac{2\Im\{\alpha_{n-1}\}}{|\alpha_{n-1}|^2} = -\frac{\mathbf{i}}{Z_{n-1}(\overline{\alpha}_{n-1})},$$

so that Equation (5) is equivalent with

$$\Re\{F_n\}^2 + \left(\Im\{F_n\} - \mathbf{i}Z_{n-1}(\overline{\alpha}_{n-1})\right)^2 = \left[\mathbf{i}Z_{n-1}(\overline{\alpha}_{n-1})\right]^2 \frac{|E_{n-1}|^2}{|E_n|^2} \cdot \frac{\Delta_n}{\Delta_{n-1}},$$

where

$$\Delta_n = |E_n|^2 - 4\frac{\Im\{\alpha_n\}}{|\alpha_n|^2} \cdot \frac{\Im\{\alpha_{n-1}\}}{|\alpha_{n-1}|^2}.$$

Clearly, we may assume that $\Delta_n/\Delta_{n-1} \geq 0$; otherwise the collection of (finite) F_n satisfying Equation (5) will be empty. Moreover, if $\Delta_n/\Delta_{n-1} = 0$, it follows that $F_n = -Z_{n-1}(\overline{\alpha}_{n-1}) = -1/A(\alpha_{n-1}, \overline{\alpha}_{n-1})$. Consequently,

$$C_n \overline{E}_{n-1} = E_n \left[1 + F_n A(\alpha_{n-1}, \overline{\alpha}_{n-1})\right] = 0,$$

contradicting assumption (A6). Therefore, we should have that

$$\Delta_n/\Delta_{n-1} > 0. \tag{6}$$

Finally, note that $\Delta_0 = |E_0|^2 > 0$. Thus, suppose that $\Delta_{n-1} > 0$. By induction it then follows from assumption (A4) or (6) that $\Delta_n > 0$, if respectively $\alpha_{n-1} \in \overline{\mathbb{R}}_0$ or $\alpha_{n-1} \notin \overline{\mathbb{R}}$.

4 Favard theorem

We are now able to prove the following Favard type theorem.

Theorem 4.1 (Favard)**.** *Let $\{\varphi_n\}$ be a sequence of rational functions, and assume that the following conditions are satisfied:*

(A1) $\alpha_{-1} \in \overline{\mathbb{R}}_0$ *and* $\alpha_n \in \overline{\mathbb{C}}_0$, $n = 0, 1, \ldots$,

(A2) φ_n *is generated by the three-term recurrence relation* (1)*,*

(A3) $\varphi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$, $n = 1, 2, \ldots$, *and* $\varphi_0 \in \mathbb{C}_0$,

(A4) $F_n \in \mathbb{C}$ *and* $E_n \in \mathbb{C}_0$, $n = 1, 2, \ldots$,

(A5)

$$\Im\{F_n\} = \frac{\Im\{\alpha_n\}}{|\alpha_n|^2} \cdot \frac{1}{|E_n|^2} - \frac{\Im\{\alpha_{n-2}\}}{|\alpha_{n-2}|^2} \cdot \frac{1}{|E_{n-1}|^2},$$

if $\alpha_{n-1} \in \overline{\mathbb{R}}_0$*, respectively*

$$\Re\{F_n\}^2 + (\Im\{F_n\} - \mathbf{i}Z_{n-1}(\overline{\alpha}_{n-1}))^2 = [\mathbf{i}Z_{n-1}(\overline{\alpha}_{n-1})]^2 \frac{|E_{n-1}|^2}{|E_n|^2} \cdot \frac{\Delta_n}{\Delta_{n-1}},$$

if $\alpha_{n-1} \notin \overline{\mathbb{R}}$*, where*

$$\Delta_n = |E_n|^2 - 4\frac{\Im\{\alpha_n\}}{|\alpha_n|^2} \cdot \frac{\Im\{\alpha_{n-1}\}}{|\alpha_{n-1}|^2} > 0,$$

$n = 1, 2, \ldots$, *with* $E_0 \in \mathbb{C}_0$,

(A6) $C_n \overline{E}_{n-1} = -E_n \left[1 + F_n / Z_{n-1}(\overline{\alpha}_{n-1})\right] \in \mathbb{C}_0$, $n = 1, 2, \ldots$.

Then there exists a functional M on $\mathcal{L}_\infty \cdot \mathcal{L}_{\infty *}$ so that

$$\langle f, g \rangle = M\{f g_*\}$$

defines a Hermitian positive-definite inner product on $\mathcal{L}_\infty$ for which the φ_n form an orthonormal system.

Proof. From Theorem 3.8 it follows that the linear functional M can be defined in such a way that the orthonormality relations are satisfied. For every $h \in \mathcal{L}_\infty \cdot \mathcal{L}_{\infty *}$ there exist functions $f = \sum a_i \varphi_i \in \mathcal{L}_\infty$ and $g = \sum b_j \varphi_j \in \mathcal{L}_\infty$ so that $h = f g_*$. Consequently,

$$M\{h_*\} = M\left\{\sum \overline{a}_i \varphi_{i*} \cdot \sum b_j \varphi_j\right\} = \sum \overline{a}_i b_i = \overline{\sum \overline{b}_i a_i} = \overline{M\{h\}}.$$

Finally, the positivity is guaranteed by

$$M\{f f_*\} = \sum |a_i|^2 > 0$$

for every $f \neq 0$. □

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