# The smoothed spectral abscissa for robust stability optimization

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*Report TW 505, September 2007*

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**Abstract**

This note concerns the stability optimization of (parameterized) matrices $A(x)$, a problem typically arising in the design of fixed-order or fixed-structured feedback controllers. It is well known that the minimization of the spectral abscissa function $\alpha(A)$ gives rise to very difficult optimization problems, since $\alpha(A)$ is not everywhere differentiable, and even not everywhere Lipschitz. In this note we therefore propose a new stability measure, namely the *smoothed spectral abscissa*, which is based on the inversion of a relaxed $H_2$-type cost function. A regularization parameter $\epsilon$ allows to tune the degree of smoothness, and for $\epsilon$ approaching zero, the smoothed spectral abscissa $\alpha_\epsilon(A)$ converges towards the nonsmooth spectral abscissa from above, so that $\alpha_\epsilon(A) \leq 0$ guarantees asymptotic stability. Evaluation of the smoothed spectral abscissa and its derivatives w.r.t. the matrix parameters can be performed at the cost of solving a primal-dual Lyapunov equation pair, allowing for an efficient integration into a derivative based optimization framework. Two optimization problems are considered: the minimization in function of the parameters $x$ of the smoothed spectral abscissa $\alpha_\epsilon(A(x))$ for a fixed value of $\epsilon$, and the maximization of $\epsilon$ such that $\alpha_\epsilon(A(x)) \leq 0$ is still satisfied. The latter problem can be nicely interpreted as a $H_2$-norm minimization problem, and its solution additionally implies an upper bound on the corresponding $H_\infty$-norm, or a lower bound on the distance to instability. In both cases additional equality and inequality constraints on the variables can be naturally taken into account in the optimization problem.

**Keywords :** robust stability, Lyapunov equation, eigenvalue optimization, pseudospectra.
**AMS(MOS) Classification :** Primary : 49M20, 93D09, 93D15, 93D20, 93B35
Secondary : 93B36, 93B52..

# THE SMOOTHED SPECTRAL ABSCISSA FOR ROBUST STABILITY OPTIMIZATION *

JORIS VANBIERVLIET†, BART VANDEREYCKEN†, WIM MICHIELS† ‡, STEFAN VANDEWALLE†, AND MORITZ DIEHL§

**Abstract.** This note concerns the stability optimization of (parameterized) matrices $A(x)$, a problem typically arising in the design of fixed-order or fixed-structured feedback controllers. It is well known that the minimization of the spectral abscissa function $\alpha(A)$ gives rise to very difficult optimization problems, since $\alpha(A)$ is not everywhere differentiable, and even not everywhere Lipschitz. In this note we therefore propose a new stability measure, namely the *smoothed spectral abscissa*, which is based on the inversion of a relaxed $H_2$-type cost function. A regularization parameter $\epsilon$ allows to tune the degree of smoothness. For $\epsilon$ approaching zero, the smoothed spectral abscissa $\alpha_\epsilon(A)$ converges towards the nonsmooth spectral abscissa from above, so that $\alpha_\epsilon(A) \leq 0$ guarantees asymptotic stability. Evaluation of the smoothed spectral abscissa and its derivatives w.r.t. the matrix parameters can be performed at the cost of solving a primal-dual Lyapunov equation pair, allowing for an efficient integration into a derivative based optimization framework. Two optimization problems are considered: the minimization in function of the parameters $x$ of the smoothed spectral abscissa $\alpha_\epsilon(A(x))$ for a fixed value of $\epsilon$, and the maximization of $\epsilon$ such that $\alpha_\epsilon(A(x)) \leq 0$ is still satisfied. The latter problem can be interpreted as a $H_2$-norm minimization problem, and its solution additionally implies an upper bound on the corresponding $H_\infty$-norm, or a lower bound on the distance to instability. In both cases additional equality and inequality constraints on the variables can be naturally taken into account in the optimization problem.

**AMS subject classifications.** 93D09, 65K10, 49M20

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## 1. Introduction.

Stability optimization of linear and nonlinear continuous-time dynamic systems is both a highly relevant and a difficult task. The optimization parameters often stem from a feedback controller, which can be used to optimize either a performance criterion or the asymptotic stability around a certain steady state. When also robustness against perturbations of the system must be taken into account, the resulting optimization problem becomes even more challenging.

Assuming an adequate parameterization of the desired feedback controller, the problem of finding a suitable steady state along with a stabilizing feedback controller can essentially be transformed into a nonlinear programming problem. By collecting all optimization variables in a vector $x$, we can summarize the described stability optimization problem as

$$\min_x \ \Phi_{\mathbf{stab}}(A(x)), \quad \text{s.t.} \quad g(x) = 0,\ h(x) \leq 0, \tag{1.1}$$

where $A(x)$ is the system matrix depending smoothly on $x$ and the function $\Phi_{\mathbf{stab}}(\cdot)$ shall express our desire to optimize stability, under the given constraints. In the field of linear output feedback control, the closed-loop system matrix $A(x)$ will typically

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be of the form $A + BKC$, with $A$ the open-loop system matrix, $B$ and $C$ the input and output matrices, and $K$ containing the controller parameters $x$ to be optimized.

The most straightforward choice for the objective function $\Phi_{\mathbf{stab}}$ is related to the eigenvalues of $A$, namely the spectral abscissa $\alpha(A)$. This value is defined as the real part of the rightmost eigenvalue of the spectrum $\Lambda(A) = \{z \in \mathbb{C} \mid \det(zI - A) = 0\}$, that is, $\alpha(A) := \sup\{\Re(z) \mid z \in \Lambda(A)\}$.

The spectral abscissa is in general a non-Lipschitz and nonconvex function of $A$ [9, 10] and therefore typically a very hard function to optimize. Still, recent developments have led to algorithms that are able to tackle such nonsmooth objective functions [5, 8, 17, 18]. The extension to infinite-dimensional systems has been made in [20]. Nonetheless, the spectral abscissa is also known to perform quite poorly in terms of robustness against parameter uncertainties. A tiny perturbation or disturbance to a parameter of a system that was optimized in the spectral abscissa can possibly lead to instability. For this reason, more robust approaches have been proposed. Amongst those, the most prominent are $H_\infty$-optimization [1, 2, 4, 15, 16] and, closely related, the minimization of the pseudospectral abscissa $\alpha_\epsilon^{\text{ps}}$ [6, 19]. This latter is defined as the maximal real part of the $\epsilon$-pseudospectrum of $A$:

$$\alpha_\epsilon^{\text{ps}}(A) = \sup\{\Re(z) \mid z \in \Lambda(X) \text{ and } \|X - A\| \leq \epsilon\}.$$

As these optimisation formulations are connected to maximizing the distance to instability of the system under consideration, they inherently take the effect of perturbations into account in the stability measure. However, their objective functions still suffer from nonsmoothness and associated high computational costs in optimization.

**2. The smoothed spectral abscissa.** In this section we introduce a new stability measure, namely the smoothed spectral abscissa. It alleviates the problem of nonsmoothness, and has at the same time certain beneficial robustness properties. At its basis is the following well-known stability criterion.

LEMMA 2.1. *For any matrix norm $\|\cdot\|$ the matrix $A \in \mathbb{R}^{n\times n}$ is Hurwitz stable if and only if the integral $\int_0^\infty \|\exp(At)\|^2 dt$ is finite.*

Inspired by this observation, we introduce the matrix function $f : \mathbb{R}^{n\times n} \times \mathbb{R} \cup \{\infty\} \to \mathbb{R} \cup \{\infty\}$, that uses the Frobenius norm $\|M\|_{\text{F}}^2 := \text{trace}\,(M^{\text{T}}M)$ and takes as its arguments, next to the matrix $A$, also a real-valued relaxation parameter $s$

$$f(A, s) := \int_0^\infty \|V\mathrm{e}^{(A-sI)t}U\|_{\text{F}}^2 dt. \tag{2.1}$$

Here, the matrices $U$ and $V$ are to be seen as respective input and output weighting matrices, with $(A, U)$ controllable and $(V, A)$ observable. It is easy to see that $f(A, s)$ is nothing else than the squared weighted and relaxed $H_2$-norm of a system with transfer function $\mathbf{H}_s(z) = V[zI - (A - sI)]^{-1}U$, i.e.,

$$f(A, s) = \|\mathbf{H}_s\|_{\mathcal{H}_2}^2. \tag{2.2}$$

We conclude with the following properties for $f(A, s)$.

LEMMA 2.2. $\forall A \in \mathbb{R}^{n\times n} : \{f(A, s) \mid s > \alpha(A)\} = \mathbb{R}_0^+$.

*Proof.* If $s > \alpha(A)$, the matrix $A - sI$ is stable and by Lemma 2.1 $f(A, s)$ is finite. Additionally, $f(A, s)$ tends to infinity and zero for $s \to \alpha(A)$ and $s \to \infty$ resp. □

LEMMA 2.3. $\forall\ s > \alpha(A) : \partial f(A, s)/\partial s < 0$ and $\partial^2 f(A, s)/\partial s^2 > 0$.

*Proof.* This can be verified by differentiating the integral in equation (2.1) to $s$ once and twice respectively. □

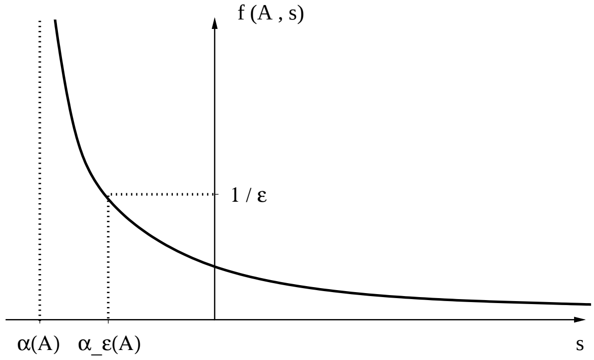



FIG. 2.1. *Typical behaviour of the function $f(A,s)$ in function of $s$. The smoothed spectral abscissa $\alpha_\epsilon(A)$ is defined as the abscissa of the point where this function reaches $\epsilon^{-1}$.*

These last two properties allow us to introduce the implicit function of the relation $f(A,s) = \epsilon^{-1}$ w.r.t. the relaxation argument $s$, as it is well defined on the whole domain, that is, for any $\epsilon > 0$ and any matrix $A \in \mathbb{R}^{n\times n}$. We will call this function the "smoothed spectral abscissa", analogously to the smoothed spectral radius for discrete time systems [11].

DEFINITION 2.4. *The smoothed spectral abscissa is the map $\alpha\colon \mathbb{R}_0^+ \times \mathbb{R}^{n\times n} \mapsto \mathbb{R}$: $(\epsilon, A) \mapsto \alpha_\epsilon(A)$ that uniquely solves*

$$f(A, \alpha_\epsilon(A)) = \epsilon^{-1}. \tag{2.3}$$

Because $f(A,s)$ is analytic in both its arguments for any $s > \alpha(A)$, it follows from the implicit function theorem that $\alpha_\epsilon(A)$ is analytic on its whole domain $\epsilon > 0$, $A \in \mathbb{R}^{n\times n}$. Moreover, it has the following additional properties.

THEOREM 2.5. *$\alpha_\epsilon(A)$ is an increasing function of $\epsilon$, that is, $\partial\alpha_\epsilon(A)/\partial\epsilon > 0$.*

*Proof.* Differentiating (2.3) on both sides w.r.t. $\epsilon$, we obtain

$$\frac{\mathrm{d}f(A,\alpha_\epsilon(A))}{\mathrm{d}\epsilon} = \frac{\partial f(A,s)}{\partial s}\frac{\partial \alpha_\epsilon(A)}{\partial \epsilon} = -\epsilon^{-2} < 0,$$

from which the proposition holds by Lemma 2.3. ☐

THEOREM 2.6. *$\forall\, \epsilon > 0$ : $\alpha_\epsilon(A) > \alpha(A)$ and $\lim_{\epsilon\to 0} \alpha_\epsilon(A) = \alpha(A)$.*

*Proof.* These two properties follow from the fact that $f(A,s)$ is finite and descending for $s > \alpha(A)$, but tends to infinity as $s$ approaches $\alpha(A)$. ☐

Note that this theorem implies that a nonpositive smoothed spectral abscissa ensures an asymptotically stable system.

The above definition and properties are illustrated in Figure 2.1.

**3. Computing the smoothed spectral abscissa and its derivatives.** Having defined the smoothed spectral abscissa, we now a look at its computation. As explained in the previous section, this involves solving the nonlinear equation $f(A,s) = \epsilon^{-1}$ for $s$. Therefore, we first give some properties of the function $f(A,s)$ regarding its evaluation and its derivatives.

LEMMA 3.1. *For all $s > \alpha(A)$, there exist symmetric $n \times n$ matrices $P$ and $Q$ such that*

$$\begin{aligned} f(A,s) &= \operatorname{trace}(VPV^{\mathrm{T}}) = \operatorname{trace}(U^{\mathrm{T}}QU), &(3.1a)\\ \frac{\partial f(A,s)}{\partial s} &= -2\operatorname{trace}(QP) = -2\operatorname{trace}(PQ), &(3.1b)\\ \frac{\partial f(A,s)}{\partial A} &= 2QP, &(3.1c)\end{aligned}$$

*where $P$ and $Q$ satisfy the primal-dual Lyapunov equation pair*

$$\begin{aligned} 0 &= L(P, A, U, s), &(3.2a)\\ 0 &= L(Q, A^{\mathrm{T}}, V^{\mathrm{T}}, s), &(3.2b)\end{aligned}$$

*with $L$ is defined as*

$$L(P, A, U, s) := (A - sI)P + P(A - sI)^{\mathrm{T}} + UU^{\mathrm{T}}.$$

*Proof.* The first part follows immediately by writing out the Frobenius norm in (2.1)

$$f(A, s) = \operatorname{trace}\left(V \int_0^\infty \mathrm{e}^{(A-sI)t} UU^{\mathrm{T}} \mathrm{e}^{(A-sI)^{\mathrm{T}}t} dt\ V^{\mathrm{T}}\right),$$

and by the well-known fact that, since $A - sI$ is stable, the above integral can be identified as the trace of $P$, the solution of equation (3.2a) (see, for instance, [13,21]). Note that solving the dual Lyapunov equation (3.2b) computes a matrix $Q$ that solves the dual integral

$$Q = \int_0^\infty \mathrm{e}^{(A-sI)^{\mathrm{T}}t}\, V^{\mathrm{T}}V\, \mathrm{e}^{(A-sI)t} dt.$$

Since $A$ is fixed in the partial derivative $\frac{\partial f(A,s)}{\partial s}$, we can regard $f$ as a function of $P$, where $P$ depends on $s$ through the Lyapunov relation $L(P(s), A, U, s)$. We propose to use an adjoint differentiation technique, rather than directly computing

$$\frac{\partial f(A,s)}{\partial s} = \frac{\mathrm{d}}{\mathrm{d}s} \operatorname{trace}(VP(s)V^{\mathrm{T}}) = \operatorname{trace}\left(V \frac{\mathrm{d}P}{\mathrm{d}s} V^{\mathrm{T}}\right),$$

with $\frac{\mathrm{d}P}{\mathrm{d}s}$ the solution of the Lyapunov equation $(A - sI)\frac{\mathrm{d}P}{\mathrm{d}s} + \frac{\mathrm{d}P}{\mathrm{d}s}(A - sI)^{\mathrm{T}} - 2P = 0$. Vectorizing the matrix $P$ in a $n^2 \times 1$ vector $p = \operatorname{vec}(P)$, we instead write

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial p}\frac{\partial p}{\partial s} = -\frac{\partial f}{\partial p}\left(\frac{\partial \ell}{\partial p}\right)^{-1}\frac{\partial \ell}{\partial s} \tag{3.3}$$

where $\ell := \operatorname{vec}(L(P, A, U, s))$ represents the vectorized primal Lyapunov formula (3.2a). Making use of the fact that $\operatorname{vec}\left(MXN^T\right) = (N \otimes M)\operatorname{vec}(X)$ [12], we can make $\ell$ explicit in $p$, obtaining the following $n^2 \times n^2$ linear system.

$$\ell(p, A, U, s) = \frac{\partial \ell}{\partial p} p + \operatorname{vec}(UU^{\mathrm{T}}) = 0,$$

with $\frac{\partial \ell}{\partial p} = (A - sI) \otimes I + I \otimes (A - sI)$ and $\otimes$ denoting the (left) Kronecker product. Similarly for the dual Lyapunov equation, we get

$$\ell(q, A^{\mathrm{T}}, V^{\mathrm{T}}, s) = \frac{\partial \ell}{\partial q} q + \operatorname{vec}(V^{\mathrm{T}} V) = 0, \quad \text{with} \quad \frac{\partial \ell}{\partial q} = (A - sI)^{\mathrm{T}} \otimes I + I \otimes (A - sI)^{\mathrm{T}}.$$

It is easily verified that $\frac{\partial \ell}{\partial q} = \frac{\partial \ell}{\partial p}^{\mathrm{T}}$. Replacing $\frac{\partial \ell}{\partial q}$ in $\ell(q, A^{\mathrm{T}}, V^{\mathrm{T}}, s)$, and using in addition the fact that $\operatorname{vec}(V^{\mathrm{T}} V)$ equals $\frac{\partial f}{\partial p}^{\mathrm{T}}$, we find

$$\frac{\partial \ell}{\partial p}^{\mathrm{T}} q + \frac{\partial f}{\partial p}^{\mathrm{T}} = 0 \quad \Leftrightarrow \quad q^{\mathrm{T}} = -\frac{\partial f}{\partial p} \left( \frac{\partial \ell}{\partial p} \right)^{-1}.$$

Substituting into (3.3), along with $\frac{\partial \ell}{\partial s} = -2p$, gives

$$\frac{\partial f}{\partial s} = q^{\mathrm{T}}(-2p) = -2 \operatorname{vec}(Q)^{\mathrm{T}} \operatorname{vec}(P) = -2 \operatorname{trace}(QP).$$

For the third part of the proof, i.e. the proof of the expression for the derivative w.r.t. $A$, we can use a similar adjoint differentiation technique. Hereto, we again let $f$ depend on the vectorized matrix $p = \operatorname{vec}(P)$, which now depends on $a = \operatorname{vec}(A)$ according to the relation $\ell(p(a), a, s) = 0$. Using the previous results, we obtain

$$\operatorname{vec}^{\mathrm{T}} \left( \frac{\partial f}{\partial A} \right) = \frac{\partial f}{\partial a} = \frac{\partial f}{\partial p} \frac{\partial p}{\partial a} = -\frac{\partial f}{\partial p} \left( \frac{\partial \ell}{\partial p} \right)^{-1} \frac{\partial \ell}{\partial a} = q^{\mathrm{T}} \frac{\partial \ell}{\partial a}. \tag{3.4}$$

To find $\frac{\partial \ell}{\partial a}$, we first have to make $\ell$ explicit in $a$, which yields

$$\ell(P, a, U, s) = \frac{\partial \ell}{\partial a} a + \operatorname{vec}(U U^{\mathrm{T}}) = 0 \quad \text{with} \quad \frac{\partial \ell}{\partial a} = (P \otimes I) + (I \otimes P)\, \Pi, \tag{3.5}$$

for $\Pi$ the symmetric permutation matrix that satisfies $\operatorname{vec}(A^{\mathrm{T}}) = \Pi \operatorname{vec}(A)$. Filling in in (3.4) gives

$$\operatorname{vec}^{\mathrm{T}} \left( \frac{\partial F}{\partial A} \right) = \left( \frac{\partial \ell}{\partial a}^{\mathrm{T}} q \right)^{\mathrm{T}} = \left[ \operatorname{vec}(QP) + \Pi^{\mathrm{T}} \operatorname{vec}(PQ) \right]^{\mathrm{T}} = 2 \operatorname{vec}^{\mathrm{T}}(QP) = \operatorname{vec}^{\mathrm{T}}(2QP).$$

By comparison of both sides, we finally obtain that

$$\frac{\partial f}{\partial A} = 2QP,$$

which concludes the proof. ◻

The fact that $f(A, s)$ and its derivative w.r.t. $s$ can be evaluated easily enables us to use e.g. Newton's method for solving the nonlinear equation $f(A, s) = \epsilon^{-1}$ and thus efficiently compute the smoothed spectral abscissa $\alpha_\epsilon(A)$.

As we will want to use derivative-based optimization methods later on to exploit the smoothness of the smoothed spectral abscissa, we need to be able to compute the derivative of $\alpha_\epsilon(A)$ w.r.t. $A$. Fortunately, the cost of this is almost nothing. Indeed, the same ingredients that were needed for the evaluation of $\alpha_\epsilon$, namely the solutions $P$ and $Q$ of one primal-dual Lyapunov equation pair, give us direct access to the derivative of $\alpha_\epsilon(A)$ w.r.t. $A$, as expressed in the following theorem.

THEOREM 3.2. *For fixed $\epsilon$, the derivative of the smoothed spectral abscissa $\alpha_\epsilon(A)$ w.r.t. $A$ equals*

$$\frac{\partial \alpha_\epsilon(A)}{\partial A} = \frac{QP}{\mathrm{trace}\,(QP)},$$

*where $P$ and $Q$ satisfy the Lyapunov equation pair* (3.2a)-(3.2b) *for $s = \alpha_\epsilon(A)$.*

*Proof.* Differentiating the implicit equation $f(A,s) = \epsilon^{-1}$ w.r.t. $A$, and using the chain rule, we obtain

$$\frac{\partial \alpha_\epsilon(A)}{\partial A} = -\left(\frac{\partial f(A,s)}{\partial s}\right)^{-1}\left(\frac{\partial f(A,s)}{\partial A}\right).$$

Recalling (3.1b) and (3.1c) of Lemma 3.1, the result follows directly. □

REMARK 1. *A direct approach to compute this derivative would require the solution of $m+1$ Lyapunov equations with different 'right-hand sides', where $m$ is the number of parameters upon which $A$ depends.*

**4. Robust Stability Optimization.** When it comes to algorithmic optimization, a first major advantage of the criterion of the smoothed spectral abscissa is that it is differentiable everywhere, and that its derivatives can be computed efficiently. This allows us to use derivative based methods without any restriction. Additionally, due to its differentiable dependence on $A$ and its connection with the $H_2$-norm, it is a more robust measure for stability than the spectral abscissa. We will present two smooth formulations of the stability optimization problem (1.1).

The first variant is to simply choose a fixed $\epsilon > 0$ and then solve

$$\min_x \; \alpha_\epsilon(A(x)), \quad \text{s.t.} \quad g(x) = 0,\; h(x) \leq 0. \tag{4.1}$$

Here, $\alpha_\epsilon(A(x))$ is indirectly dependent on $x$, as it is implicitly defined as the solution of the relation $f(A(x),s) = \epsilon^{-1}$ w.r.t. $s$. By regarding the implicit relation of $\alpha_\epsilon$ as a constraint, we can formulate the problem alternatively as

$$\min_x \; s, \quad \text{s.t.} \quad f(A(x),s) = \epsilon^{-1} \quad \text{and} \quad g(x) = 0,\; h(x) \leq 0. \tag{4.2}$$

Should this problem not result in a negative optimum for the chosen $\epsilon$, then one can try again with a smaller $\epsilon$.

In the minimization of the smoothed spectral radius as in formulation (4.1), the choice of $\epsilon$ is somewhat arbitrary. As indicated by Theorem 2.6, $\alpha_\epsilon(A)$ becomes smoother – and thus presumably a more robust measure for stability – with increasing values for $\epsilon > 0$. Thus, we might alternatively search for the largest $\epsilon$ so that the stability certificate $\alpha_\epsilon(A) \leq 0$ still holds. This leads to the optimization problem

$$\max_{x,\epsilon} \; \epsilon \quad \text{s.t.} \quad \alpha_\epsilon(A(x)) \leq 0 \quad \text{and} \quad g(x) = 0,\; h(x) \leq 0. \tag{4.3}$$

Note that this problem requires a stable solution in order to have a feasible starting point. Such points can be easily found by solving the first optimization problem with $\epsilon$ small enough.

Since $\alpha_\epsilon(A)$ is a continously growing function of $\epsilon$, the constraint in problem (4.3) will always be active in the optimum $(x^*, \epsilon^*)$. Hence, it is easy to see that the solution of the first problem (4.1), with $\epsilon$ fixed to $\epsilon^*$, will be exactly zero, and that this minimum is attained in the same parameters as for problem (4.3), i.e.,

$$x^* = \arg\min_x \alpha_{\epsilon^*}(A(x)) \qquad \text{and} \qquad \alpha_{\epsilon^*}(A(x^*)) = 0.$$

By taking the inverse of the objective function of problem (4.3) and by incorporating the stability constraint, we can reformulate this problem as the mimization of the function $f(A(x), 0)$, subject to the constraints, and additionally restricting $x$ to values for which $A(x)$ is Hurwitz. By equation (2.2), this is essentially equivalent to a (squared) $H_2$-norm optimization of a system with transfer function $\mathbf{H}(x)(z) := V(zI - A(x))^{-1}U$. This is expressed in the following theorem.

THEOREM 4.1. *Any solution $x^*$ that solves problem* (4.3) *also satisfies the problem of minimizing the $H_2$-norm over all feasible $x$, that is,*

$$x^* = \arg\min_x \|\mathbf{H}(x)\|_{\mathcal{H}_2}, \quad s.t. \quad g(x) = 0,\ h(x) \le 0,$$

*and its solution $\|\mathbf{H}(x^*)\|_{\mathcal{H}_2}$ equals $\sqrt{1/\epsilon^*}$.*

REMARK 2. *Solving problem* (4.3) *with the restriction $\alpha_\epsilon < s$ (with $s < 0$) would minimize the $H_2$-norm of a system with the shifted transfer function $\mathbf{H}_s$.*

**5. Relation with the $H_\infty$-norm.** Inserting equation (2.2) in the definition of the smoothed spectral abscissa $\alpha_\epsilon$, and using the fact that $f$ is a decreasing function of $s$ (Lemma 2.3), we can derive the following equivalence

$$\alpha_\epsilon(A) < s \quad \Leftrightarrow \quad f(A, s) \ = \ \|\mathbf{H}_s\|^2_{\mathcal{H}_2} < \epsilon^{-1}. \tag{5.1}$$

Hence, we can view $\alpha_\epsilon(A)$ as the maximal shift $s$ for which the squared $H_2$-norm of the shifted system, with transfer function $\mathbf{H}_s$, stays below a bound equal to $\epsilon^{-1}$ (see also Figure 2.1).

By studying the following standard property regarding the *pseudo*-spectral abscissa [6], which we denote by $\alpha^{\rm ps}_\epsilon$,

$$\alpha^{\rm ps}_\epsilon(A) < 0 \quad \Leftrightarrow \quad \max_{\Re(z)=0} \|(zI - A)^{-1}\|_2 \ = \ \|\mathbf{H}\|_{\mathcal{H}_\infty} < \epsilon^{-1}, \tag{5.2}$$

we can give an analoguous interpretation to $\alpha^{\rm ps}_\epsilon(A)$ as we did for the smoothed spectral abscissa. Indeed, applying (5.2) to the shifted matrix $A - sI$, we obtain

$$\alpha^{\rm ps}_\epsilon(A - sI) < 0 \quad \Leftrightarrow \quad \alpha^{\rm ps}_\epsilon(A) < s \quad \Leftrightarrow \quad \|\mathbf{H}_s\|_{\mathcal{H}_\infty} < \epsilon^{-1}. \tag{5.3}$$

Thus, $\alpha^{\rm ps}_\epsilon(A)$ and $\alpha_\epsilon(A)$ are both relaxations of the spectral abscissa in the sense that they are both induced by placing a bound on a norm ($\mathcal{H}_\infty$ and $\mathcal{H}_2$ respectively) that goes to infinity when approaching the spectral abscissa. This analogy enables us to relate these two robust stability measures.

THEOREM 5.1 (Relation to pseudo-spectral abscissa). *For a fixed $\epsilon > 0$ and $U, V = I$, the following holds*

$$\|\mathbf{H}_s\|_{\mathcal{H}_\infty} \quad < \quad 2\|\mathbf{H}_s\|^2_{\mathcal{H}_2} \tag{5.4a}$$

$$\alpha^{\rm ps}_{\epsilon/2}(A) \quad < \quad \alpha_\epsilon(A). \tag{5.4b}$$

*Proof.* The first inequality is based on [3], where $2\lambda_{max}(Q^2)^{\frac{1}{2}}$ is established to be an upper bound on the $H_\infty$-norm of an unweighted system with transfer function $\mathbf{H}_s$,

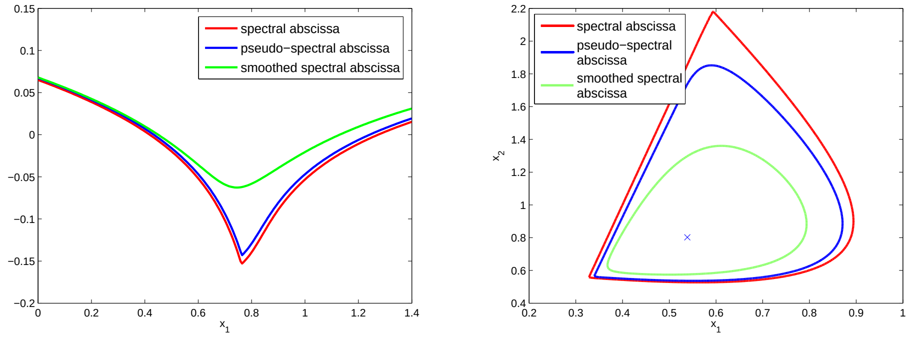



FIG. 6.1. *Evolution w.r.t.* $x_1$ *(left) and stability regions (right) of the spectral abscissa* $\alpha$ *and its pseudo and smoothed relaxations* $\alpha_\epsilon^{\text{ps}}$ *and* $\alpha_{2\epsilon}$ *for the matrix* $A + BK$ *of Example 1 with* $\epsilon = 0.002$.

where $Q$ satisfies (3.2b). Since $Q$ is a positive definite matrix, we can deduce from this the following.

$$\|\mathbf{H}_s\|_{\mathcal{H}_\infty} \;\leq\; 2\lambda_{\max}(Q) \;<\; 2\,\mathrm{trace}\,(Q)\,.$$

This proves (5.4a) directly by Lemma 3.1(a) and (2.2). Suppose then, by (5.1), that for $s = \alpha_\epsilon(A)$ it is true that $\|\mathbf{H}_s\|^2_{\mathcal{H}_2} = \epsilon^{-1}$. Using (5.4a) in connection with (5.3), assertion (5.4b) follows. $\square$

This property has an important implication in terms of robust optimization. It shows that the squared $H_2$-norm constitutes an upper bound to the $H_\infty$-norm, which is directly related to the distance to instability of a system. By minimizing the first norm, one could expect that the second norm should also go down.

On top of this rather intuitive result, (5.4b) also provides guarantees that come with a negative smoothed spectral abscissa. Indeed, if $\alpha_\epsilon < 0$ for some $x$, we are not only sure that the system with system matrix $A(x)$ will be a stable one, but also that this system will have a $H_\infty$-norm smaller than $2/\epsilon$. In other words, we are certain that the distance to instability of the system will be at least $\epsilon/2$.

## 6. Numerical Examples.

We will now put theory into practice by treating two control examples using the smoothed spectral abscissa as the stability criterion. In both examples we assume $U = I$ and $V = I$.

EXAMPLE 1 (Optimal state feedback control).

Consider a linear state feedback controlled system, with a closed-loop system matrix $A + BK$, and where

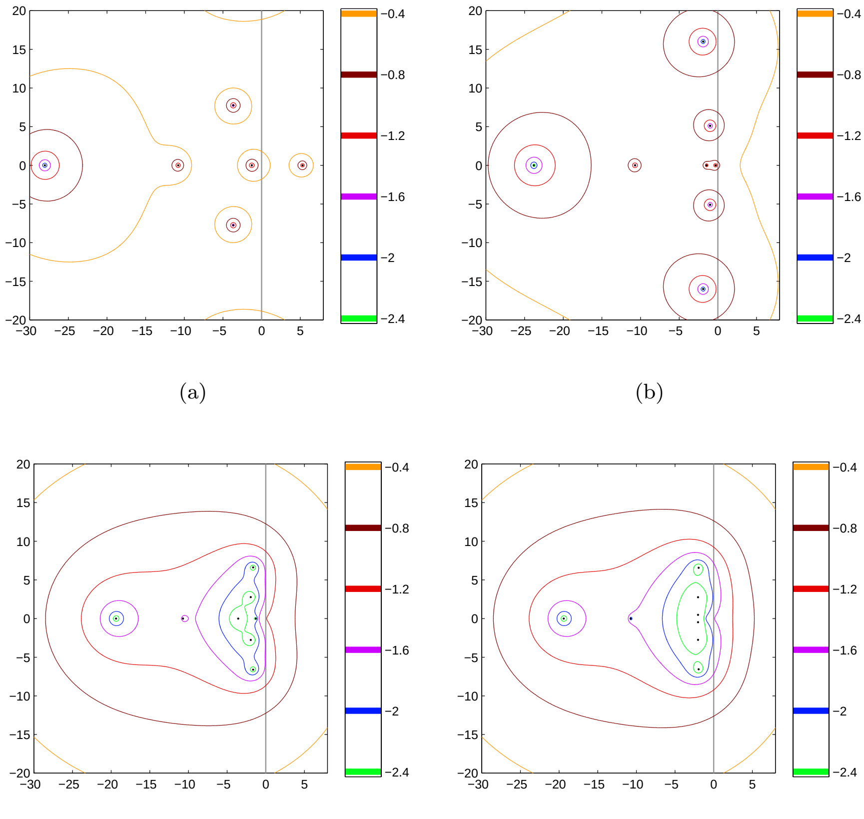
$$A = \begin{bmatrix} 0.1 & -0.03 & 0.2 \\ 0.2 & 0.05 & 0.01 \\ -0.06 & 0.2 & 0.07 \end{bmatrix}, \quad B = \frac{1}{2}\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \quad K^{\mathrm{T}} = \begin{bmatrix} x_1 \\ x_2 \\ 1.4 \end{bmatrix}. \tag{6.1}$$

Figure 6.1 shows the smoothed spectral abscissa in comparison with the pseudo-spectral abscissa and spectral abscissa. In the left frame, $x_2$ is held fixed. We clearly see the smoothness of $\alpha_\epsilon$ in contrast with the nonsmoothness of the pseudo-spectral and spectral abscissa for varying $x_1$. In the right frame, both $x_1$ and $x_2$ are free, and the stability region for the spectral abscissa $\alpha(A(x))$ and robust stability regions for the pseudo and smoothed spectral abscissae $\alpha_\epsilon^{\text{ps}}$ and $\alpha_{2\epsilon}$ are drawn, with the robustness parameter $\epsilon = 2 \cdot 10^{-3}$. Here it is confirmed that the stability region for

| $\epsilon$ | $\min_x \alpha_{2\epsilon}$ | $\alpha_\epsilon^{\mathrm{ps}}$ | $\alpha$ | $\Vert\mathbf{H}_s\Vert_{\mathcal{H}_2}$ | $\Vert\mathbf{H}_s\Vert_{\mathcal{H}_\infty}$ |
|---|---|---|---|---|---|
| $10^{-1}$ | 36.2420 | 5.6539 | 5.2996 | $\infty$ | $\infty$ |
| $10^{-3}$ | -0.0270 | -0.2855 | -0.2889 | 22.0976 | 13.9514 |
| $10^{-5}$ | -1.2432 | -1.2838 | -1.2842 | 54.1900 | 38.0640 |
| $10^{-7}$ | -1.8948 | -1.9328 | -1.9328 | 108.3870 | 89.4209 |

TABLE 6.1

*Solutions to the minimization of the smoothed spectral abscissa of the turbo generator model for designated $\epsilon$-values, in comparison with other stability measures evaluated in these minimizers.*



(a) (b) (c) (d)

FIG. 6.2. *Pseudospectra of the turbo generator model in the minimizers of the smoothed spectral abscissa $\alpha_\epsilon(A(x))$ (problem (4.1)) for the four $\epsilon$-values as used in Table 6.1.*

the smoothed spectral abscissa is contained within the one for the pseudo-spectral abscissa, as stated in Theorem 5.1.

EXAMPLE 2 (Turbo generator model).

Next, we treat Problem 10 of Leibfritz's control problem database [14], which models a turbo generator. This example was also used in [7] for robust stability optimization using the pseudo-spectral abscissa in combination with the gradient sampling algorithm.

We perform the minimization of the smoothed spectral abscissa with a range of four $\epsilon$-values as smoothing parameter, starting with a rather large value, $\epsilon = 2 \cdot 10^{-1}$,

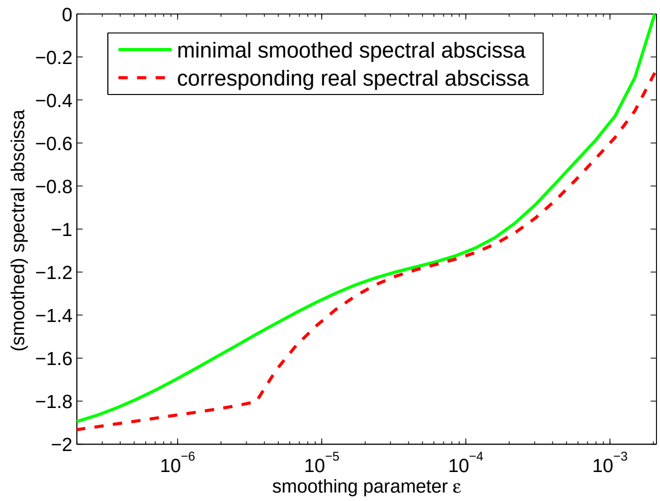



FIG. 6.3. *The minimal smoothed spectral abscissa $\alpha_\epsilon$ for $\epsilon$ ranging from $2 \cdot 10^{-7}$ to $2 \cdot 10^{-3}$, compared to the corresponding spectral abscissa $\alpha$.*

and ending with a very small one, namely $\epsilon = 2 \cdot 10^{-7}$. In the resulting minimizers, the values of several stability measures are given in Table 6.1. The corresponding pseudospectra are plotted in Figure 6.2. For the largest value of $\epsilon$, both the smoothed and spectral abscissa are positive and the minimizer is not stabilising, as confirmed by the spectrum plotted in Figure 6.2(a).

For $\epsilon = 2 \cdot 10^{-3}$, the minimal smoothed spectral abscissa is just below zero and thus yields a stable system. Since the minimum is so close to zero for this $\epsilon$, we can expect it to be close to the maximal $\epsilon$ for which a stabilizing solution can be found. Solving the optimisation problem (4.3) indeed results in an optimal value that is only slightly higher, namely $2.048 \cdot 10^{-3}$. According to Theorem 4.1, the latter also corresponds to a minimal $H_2$-norm of the systems transfer function of approximately 22.1.

In Figure 6.2(b), we see that the eigenvalues are indeed all in the left half complex plane and the $10^{-1.2}$-pseudospectrum is just contained in the left half complex plain, implying a $H_\infty < 15.85$.

Further decrease of $\epsilon$ results in smaller and smaller smoothed and spectral abscissae, but both the $H_2$- and $H_\infty$-norm deteriorate. This is also seen in the two remaining plots of Figure 6.2, where the $10^{-1.6}$- and $10^{-2}$-pseudospectrum are on the verge of leaving the left half complex plane for respective $\epsilon = 2 \cdot 10^{-5}$ and $\epsilon = 2 \cdot 10^{-7}$. Notice the vertical alignment of the spectrum for this last value of $\epsilon$. This configuration is very resemblant to that of a typical minimized spectral abscissa spectrum.

In Figure 6.3, we plot the minimal smoothed spectral abscissa in function of $\epsilon$, where $\epsilon$ is restricted to values that yield negative solutions. It is seen that the minimal smoothed spectral abscissa, although decreasing when $\epsilon$ gets smaller, always lies above the spectral abscissa evaluated in these minimizers. For this interval of stabilizing $\epsilon$, we thus have a range of stable solutions that compromise between a minimized spectral abscissa (fast response time) on the on hand, and a minimized $H_2$-norm (robustness against noise) on the other hand. This is in line with the pseudo-spectral abscissa, where the trade-off chooses between a minimal spectral abscissa and a minimal $H_\infty$-norm.

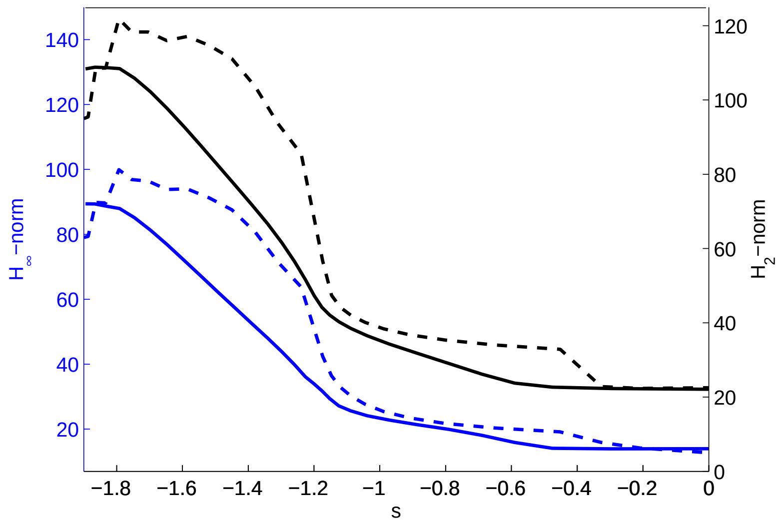



FIG. 6.4. *$H_2$- (in black) and $H_\infty$-norm (in blue) for the range of minimizers obtained by the minimization of the smoothed spectral abscissa (full lines) and of the pseudo-spectral abscissa (dashed lines).*

To compare these two approaches, we define $x_1^*(s)$ and $x_2^*(s)$ as the respective minimizers in function of $s < 0$ of the smoothed and pseudo-spectral abscissa, with smoothing epsilons $\epsilon_1(s)$ and $\epsilon_2(s)$, such that the attained minima equal $s$. Thus,

$$\alpha_{\epsilon_1(s)}(A(x_1^*(s))) = \min_x \alpha_{\epsilon_1(s)}(A(x)) = s,$$
$$\alpha_{\epsilon_2(s)}(A(x_2^*(s))) = \min_x \alpha^{\mathrm{ps}}_{\epsilon_2(s)}(A(x)) = s.$$

According to the remark following Theorem 4.1, $x_1^*(s)$ and $x_2^*(s)$ minimize the $H_2$-norm and $H_\infty$-norm of a shifted system with transfer function $\mathbf{H}_s$.

In Figure 6.4, we show the values as a function of $s$ of the norms $\|zI - A(x_1^*(s))\|_{\mathcal{H}_2}$ and $\|zI - A(x_2^*(s))\|_{\mathcal{H}_2}$. In other words, we compare the $H_2$-norms of the *unshifted* transfer function, evaluated in the smoothed spectral abscissa minimizers $x_1^*$ on the one hand, and the pseudo-spectral abscissa minimizers $x_2^*$ on the other hand. We see that for most $s$ the $H_2$-norms of the smoothed spectral abscissa minimizers are smaller than those of the pseudo-spectral abscissa minimizers, except for $s$ very close to the minimal spectral abscissa. For $s$ approaching zero however, the difference between the two $H_2$-norms becomes very small. This implies that the optimal $H_\infty$-minimizer, i.e. for $s = 0$, is accompagnied with an $H_2$-norm that is only slightly suboptimal.

We can make the same comparison for the $H_\infty$-norm. In Figure 6.4, we also plot $\|zI - A(x_i^*(s))\|_{\mathcal{H}_\infty}$ for $i = 1, 2$. We again observe that the smoothed spectral abscissa minimizers give the best results in a major part of the interval $[\min_x \alpha(A(x))\ 0]$, but this time they are outperformed by the pseudo-spectral abscissa minimizers as $s$ comes closer to zero. This is to be expected since for $s = 0$ the latter yields the optimal $H_\infty$-norm for the unshifted system. Yet again, the gap between the optimal $H_\infty$-norm and the $H_\infty$-norm in the optimal $H_2$-minimizer is very small.

**7. Conclusions.** A smooth relaxation of the nonsmooth spectral abscissa function was introduced as an alternative stability measure, with the advantage that derivative based optimization techniques can readily be used for its optimization. Formulae regarding the efficient computation and derivative evaluation of the smoothed spectral abscissa were derived based on the solution of a primal-dual Lyapunov equation pair.

Besides its direct minimization, which can be used to find stabilizing controllers, a second optimization formulation was shown to be applicable to solve fixed-order $H_2$-optimization problems. Moreover, a guaranteed bound on the distance to instability was established by relating the results to the $H_\infty$-norm. The robust stabilisation by use of these two optimization problems involving the smoothed spectral abscissa was illustrated in numerical examples.

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