Schur-Nevanlinna-Potapov sequences of rational matrix functions

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Abstract

We study particular sequences of rational matrix-valued functions with poles outside the unit circle. Such kind of sequences are recursively constructed based on a sequence of complex numbers with norm less than one and a sequence of strictly contractive matrices. In fact, such sequences are closely related to the so-called Schur-Nevanlinna-Potapov algorithm for matrix-valued Schur functions but also to orthogonal rational matrix-valued functions. As a main result of paper, we shall see that rational matrix-valued functions belonging to Schur-Nevanlinna-Potapov sequences can be used to parameterize the set of all solutions of an interpolation problem of Nevanlinna-Pick type for matrix-valued Schur functions.

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SCHUR-NEVANLINNA-POTAPOV SEQUENCES OF RATIONAL MATRIX FUNCTIONS

ADHEMAR BULTHEEL* AND ANDREAS LASAROW**

ABSTRACT. We study particular sequences of rational matrix-valued functions with poles outside the unit circle. Such kind of sequences are recursively constructed based on a sequence of complex numbers with norm less than one and a sequence of strictly contractive matrices. In fact, such sequences are closely related to the so-called Schur-Nevanlinna-Potapov algorithm for matrix-valued Schur functions but also to orthogonal rational matrix-valued functions. As a main result of paper, we shall see that rational matrix-valued functions belonging to Schur-Nevanlinna-Potapov sequences can be used to parameterize the set of all solutions of an interpolation problem of Nevanlinna-Pick type for matrix-valued Schur functions.

1. INTRODUCTION

Interpolation problems for certain classes of holomorphic functions in the open unit disk have been investigated for nearly a century, beginning with the fundamental papers of Carathéodory [8], Pick [24], Schur [27], and Nevanlinna [23]. Nowadays, there is an extensive literature on several types of such problems including various generalizations (see, e.g., [1], [20], [13], [2], [15], [5], [12], [30], [6], [4], and the references there). The present paper is another contribution to this topic and deals with an interpolation problem for matrix-valued Schur functions in $\mathbb{D} := \{w \in \mathbb{C} : \ |w| < 1\}$. Throughout the paper, $\mathbb{C}$ will denote the set of all complex numbers, p and q will be positive integers, and $\mathbb{C}^{p\times q}$ will represent the set of all complex $p \times q$ matrices. A matrix-valued function $V : \mathbb{D} \to \mathbb{C}^{p\times q}$ is called a $p \times q$ *Schur function* (in $\mathbb{D}$) if V is holomorphic in $\mathbb{D}$ and if $\mathbf{I}_q - \big(V(w)\big)^* V(w)$ is a non-negative Hermitian matrix for all $w \in \mathbb{D}$, where $\mathbf{I}_q$ stands for the identity matrix in $\mathbb{C}^{q\times q}$ and $\big(V(w)\big)^*$ denotes the adjoint matrix of $V(w)$. The notation $\mathcal{S}_{p\times q}(\mathbb{D})$ will stand for the set of all $p \times q$ Schur functions (in $\mathbb{D}$).

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We shall investigate the following *multiple Nevanlinna-Pick problem* for matrix-valued Schur functions.

Problem (MNP):

Let m be a non-negative integer, let $\beta_0, \beta_1, \dots, \beta_m$ be mutually distinct points in $\mathbb{D}$, let $l_0, l_1, \dots, l_m$ be non-negative integers, and let $\mathbf{V}_{kt}$ be a complex $p \times q$ matrix for all $t \in \{0, 1, \dots, l_k\}$ and all $k \in \{0, 1, \dots, m\}$. Find necessary and sufficient conditions for the existence of a $p \times q$ Schur function V satisfying

$$\frac{1}{t!} V^{(t)}(\beta_k) = \mathbf{V}_{kt}, \quad 0 \le t \le l_k,\ 0 \le k \le m. \tag{1.1}$$

Furthermore, describe the set $\mathcal{S}_\Delta$ of all $p \times q$ Schur functions V fulfilling (1.1).

Problem (MNP) is an interpolation problem where not only values for the function itself, but also for its derivatives up to a certain order are prescribed. Hence, Problem (MNP) can be conceived as a generalization of the Schur coefficient problem which corresponds to the special case $m = 0$ (see [27] and [1]) and of the classical Nevanlinna-Pick problem which is obtained by taking $l_0 = l_1 = \cdots = l_m = 0$ (see [24] and [23]). Furthermore, from the point of view of engineering, applications of interpolation problems of Nevanlinna-Pick type abound in circuit theory, system identification, robust control, and signal processing to mention a few (see, e.g., [14], [10], [28], [16], and [3]).

It is well-known that there exists a $V \in \mathcal{S}_{p\times q}(\mathbb{D})$ satisfying (1.1) if and only if a generalized Schwarz-Pick-Potapov matrix $\mathbf{P}_\Delta$ (resp., $\widetilde{\mathbf{P}}_\Delta$) for Problem (MNP) is non-negative Hermitian (see, e.g., [21, Theorem 2.1]). The main issues of these block matrices and of the Schur-Nevanlinna-Potapov algorithm for matrix-valued Schur functions will be briefly recalled in Section 2. In the scalar case of complex-valued functions (i.e. $p = q = 1$), such algorithm was introduced by Nevanlinna [23] as an extension of the classical algorithm of Schur [27]. For the matrix case we refer to [9] (see also [18]). Starting from a $V \in \mathcal{S}_\Delta$, we shall deduce that the Schur-Nevanlinna-Potapov algorithm for V does not break down after the n-th step, where the integer n is given by Problem (MNP) via

$$n := m + \sum_{k=0}^{m} l_k, \tag{1.2}$$

iff $\mathbf{P}_\Delta$ (resp., $\widetilde{\mathbf{P}}_\Delta$) is positive Hermitian. We call this the non-degenerate case.

As the main result of this note we prove in Section 6 that, for the non-degenerate case, the set of solutions $\mathcal{S}_\Delta$ can be characterized as a linear fractional transformation of the set $\mathcal{S}_{p\times q}(\mathbb{D})$. More precisely, with the data of Problem (MNP), one may associate Schur-Nevanlinna-Potapov sequences $(O_k)_{k=0}^n$, $(Q_k)_{k=0}^n$, $(P_k)_{k=0}^n$, and $(R_k)_{k=0}^n$ whose elements are matrix-valued functions and denoting by $O_n^{[\alpha,n]}$, $Q_n^{[\alpha,n]}$, $P_n^{[\alpha,n]}$, and $R_n^{[\alpha,n]}$ the adjoints of O_n, Q_n, P_n, and R_n as defined in (2.9) below, then a function V belongs to $\mathcal{S}_\Delta$ if and only if there exists a $p \times q$ Schur function S such that

$$V(w) = \Big(O_n^{[\alpha,n]}(w) + b_{\alpha_n}(w) Q_n(w) S(w)\Big)\Big(P_n^{[\alpha,n]}(w) + b_{\alpha_n}(w) R_n(w) S(w)\Big)^{-1} \tag{1.3}$$

or equivalently

$$V(w) = \Big(Q_n^{[\alpha,n]}(w) + b_{\alpha_n}(w) S(w) O_n(w)\Big)^{-1}\Big(R_n^{[\alpha,n]}(w) + b_{\alpha_n}(w) S(w) P_n(w)\Big) \tag{1.4}$$

for all $w \in \mathbb{D}$, where b_{α_n} is the elementary Blaschke factor corresponding to certain point α_n (belonging to $\mathbb{D}$), i.e.

$$b_{\alpha_n}(v) := \begin{cases} v & \text{if } \alpha_n = 0\,, \\ \dfrac{\overline{\alpha_n}}{|\alpha_n|}\dfrac{\alpha_n - v}{1-\overline{\alpha_n}v} & \text{if } \alpha_n \neq 0\,, \end{cases} \tag{1.5}$$

for each $v \in \mathbb{C} \setminus \left\{\frac{1}{\overline{\alpha_n}}\right\}$ (where we use the convention $\frac{1}{0} := \infty$).

Some basics on Schur-Nevanlinna-Potapov sequences of rational matrix functions are explained in Section 3. These sequences are generated during the Schur-Nevanlinna-Potapov algorithm for matrix-valued Schur functions but they are also related to orthogonal rational matrix-valued functions on the unit circle. Like in the case of orthogonal functions (cf. [19] and [22]), we will see that the Christoffel-Darboux formulae are important tools for studying Schur-Nevanlinna-Potapov sequences of rational matrix-valued functions (see Section 4 and the inverse question discussed in Section 5).

Note that for instance the general considerations in [5] include already a description of the solution set $\mathcal{S}_\Delta$ based on an operator-theoretic approach. The essential new feature of this paper is that the functions O_n, Q_n, P_n, and R_n which appear in (1.3) and (1.4) are closely related to the theory of orthogonal rational matrix-valued functions on the unit circle which were introduced in the scalar case by Djrbashian [11] (see also [6] and other papers cited there). But the explicit interplay between Schur-Nevanlinna-Potapov sequences and orthogonal rational matrix-valued functions will be done in a forthcoming work. The results of this paper can be seen as generalizations of two particular cases. For example, Theorem 6.4 below is a matricial version of [7, Theorem 6.3] which formulates the scalar case. But our approach can also be seen as a generalization of the approach used in [17] to solve a Taylor coefficient problem at the point zero for matrix-valued Schur functions. Inspired by these special cases, we study in Section 7 the geometric structure of the sets $\{V(w)\colon\ V \in \mathcal{S}_\Delta\}$ if $w \in \mathbb{D}$ is a fixed point and $\left\{\frac{1}{(l_k+1)!}V^{(l_k+1)}(\beta_k)\,:\ V \in \mathcal{S}_\Delta\right\}$ for some fixed interpolation point β_k with $k \in \{0, 1, \ldots, m\}$ according to Problem (MNP). In particular, we show that these sets are so-called Weyl matrix balls and we determine their parameters based on Theorem 6.4.

2. Preliminaries

Henceforth, we write $\mathbb{N}_0$ and $\mathbb{N}$ to denote the set of all non-negative integers and the set of all positive integers, respectively. Moreover, if $j \in \mathbb{N}_0$ and if $\tau \in \mathbb{N}_0$ or $\tau = \infty$, then $\mathbb{N}_{j,\tau}$ stands for the set of all integers k satisfying $j \leq k \leq \tau$.

With regard to Problem (MNP), we assume that the following data are given: $m \in \mathbb{N}_0$, mutually different points $\beta_0, \beta_1, \ldots, \beta_m \in \mathbb{D}$, numbers $l_0, l_1, \ldots, l_m \in \mathbb{N}_0$, and matrices $\mathbf{V}_{kt} \in \mathbb{C}^{p\times q}$, $t \in \mathbb{N}_{0,l_k}$, $k \in \mathbb{N}_{0,m}$. We denote this data set by Δ, i.e.

$$\Delta := \left\{\left(\beta_k, l_k, (\mathbf{V}_{kt})_{t=0}^{l_k}\right)_{k=0}^{m}\right\}. \tag{2.1}$$

For a given function $V \in \mathcal{S}_{p\times q}(\mathbb{D})$ we define similarly

$$\Delta^{[V]} := \left\{\left(\beta_k, l_k, \left(\tfrac{1}{t!}V^{(t)}(\beta_k)\right)_{t=0}^{l_k}\right)_{k=0}^{m}\right\}.$$

In particular, $V \in \mathcal{S}_\Delta$ if and only if $\Delta^{[V]} = \Delta$. Furthermore, we put the non-negative integer n as in (1.2).

The *generalized Schwarz-Pick-Potapov block matrix* (with respect to the data Δ) of size $(n+1)p \times (n+1)p$ (resp., $(n+1)q \times (n+1)q$) is defined as

$$\mathbf{P}_\Delta := \big(\mathbf{P}_{jk}\big)_{j,k=0}^{m} \quad \Big(\text{resp., } \widetilde{\mathbf{P}}_\Delta := \big(\widetilde{\mathbf{P}}_{jk}\big)_{j,k=0}^{m}\Big),$$

where the complex $(l_j+1)p \times (l_k+1)p$ (resp., $(l_j+1)q \times (l_k+1)q$) matrices

$$\mathbf{P}_{jk} := \big(\mathbf{p}_{st}^{(jk)}\big)_{\substack{s=0,1,\ldots,l_j\\ t=0,1,\ldots,l_k}} \quad \Big(\text{resp., } \widetilde{\mathbf{P}}_{jk} := \big(\widetilde{\mathbf{p}}_{st}^{(jk)}\big)_{\substack{s=0,1,\ldots,l_j\\ t=0,1,\ldots,l_k}}\Big), \quad j,k \in \mathbb{N}_{0,m},$$

are determined by the entries

$$\begin{aligned}\mathbf{p}_{st}^{(jk)} := & \sum_{r=0}^{\min\{s,t\}} \frac{(s+t-r)!}{(s-r)!r!(t-r)!} \frac{\beta_j^{t-r}\overline{\beta_k}^{s-r}}{(1-\beta_j\overline{\beta_k})^{s+t-r+1}} \mathbf{I}_p \\ & - \sum_{\ell=0}^{s}\sum_{h=0}^{t}\sum_{r=0}^{\min\{\ell,h\}} \frac{(h+\ell-r)!}{(\ell-r)!r!(h-r)!} \frac{\beta_j^{h-r}\overline{\beta_k}^{h-r}}{(1-\beta_j\overline{\beta_k})^{h+\ell-r+1}} \mathbf{V}_{j,s-\ell}\mathbf{V}_{k,t-h}^*\end{aligned}$$

$$\begin{aligned}\Big(\text{resp., } \widetilde{\mathbf{p}}_{st}^{(jk)} := & \sum_{r=0}^{\min\{s,t\}} \frac{(s+t-r)!}{(s-r)!r!(t-r)!} \frac{\overline{\beta_j}^{t-r}\beta_k^{s-r}}{(1-\overline{\beta_j}\beta_k)^{s+t-r+1}} \mathbf{I}_q \\ & - \sum_{\ell=0}^{s}\sum_{h=0}^{t}\sum_{r=0}^{\min\{\ell,h\}} \frac{(h+\ell-r)!}{(\ell-r)!r!(h-r)!} \frac{\overline{\beta_j}^{h-r}\beta_k^{h-r}}{(1-\overline{\beta_j}\beta_k)^{h+\ell-r+1}} \mathbf{V}_{j,s-\ell}^*\mathbf{V}_{k,t-h}\Big).\end{aligned}$$

In the sequel, 0 stands also for the zero matrix of appropriate size, and if $\mathbf{A}$, $\mathbf{B}$ are Hermitian matrices of the same size, then $\mathbf{A} \geq \mathbf{B}$ (resp., $\mathbf{A} > \mathbf{B}$) means that $\mathbf{A} - \mathbf{B}$ is a non-negative (resp., positive) Hermitian matrix.

The following criterion to have $\mathcal{S}_\Delta \neq \varnothing$ is well known (see, e.g., [15, Section 5 in Chapter X], [5, Section 1.1 in Chapter 1], or [21, Theorem 2.1]).

Theorem 2.1. *For a given data set Δ as in* (2.1)*, there exists a $V \in \mathcal{S}_{p\times q}(\mathbb{D})$ fulfilling* (1.1) *if and only if $\mathbf{P}_\Delta \geq 0$ (resp., $\widetilde{\mathbf{P}}_\Delta \geq 0$).*

Note that, in view of Theorem 2.1, if $\mathbf{P}_\Delta \geq 0$ (resp., $\widetilde{\mathbf{P}}_\Delta \geq 0$) then [21, Theorem 3.1] implies the rank identity

$$\operatorname{rank} \mathbf{P}_\Delta = \operatorname{rank} \widetilde{\mathbf{P}}_\Delta + (n+1)(p-q).$$

Since the main goal of this paper is to obtain the description of $\mathcal{S}_\Delta$ via (1.3) and (1.4) for the non-degenerate case, we will always assume in the further course

$$\mathbf{P}_\Delta > 0 \quad \Big(\text{resp., } \widetilde{\mathbf{P}}_\Delta > 0\Big).$$

The next aim is to show that $\mathbf{P}_\Delta > 0$ (resp., $\widetilde{\mathbf{P}}_\Delta > 0$) implies that at least $n+1$ steps of the Schur-Nevanlinna-Potapov algorithm [18, Section 5] can be performed for a $V \in \mathcal{S}_\Delta$ without breaking down. For the reader's convenience, we first point out some basics on linear fractional matrix transformations (see, e.g., [26], [13], and [12]) that we shall apply. Like in [12, Section 1.6] we use the following notation. If $\mathbf{A} \in \mathbb{C}^{p\times p}$, $\mathbf{B} \in \mathbb{C}^{p\times q}$, $\mathbf{C} \in \mathbb{C}^{q\times p}$, $\mathbf{D} \in \mathbb{C}^{q\times q}$, and

$$\mathbf{\Theta} := \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \quad \left(\text{resp., } \mathbf{\Xi} := \begin{pmatrix} \mathbf{D} & \mathbf{C} \\ \mathbf{B} & \mathbf{A} \end{pmatrix}\right)$$

and if $\mathbf{X} \in \mathbb{C}^{p\times q}$ is so that $\det(\mathbf{CX}+\mathbf{D})\neq 0$ (resp., $\det(\mathbf{XC}+\mathbf{A})\neq 0$) then we put

$$\begin{gathered}\mathfrak{S}_{\mathbf{\Theta}}(\mathbf{X}) := (\mathbf{AX}+\mathbf{B})(\mathbf{CX}+\mathbf{D})^{-1}\\ \Big(\text{resp., } \mathfrak{T}_{\mathbf{\Xi}}(\mathbf{X}) := (\mathbf{XC}+\mathbf{A})^{-1}(\mathbf{XD}+\mathbf{B})\Big).\end{gathered} \tag{2.2}$$

Recall that a complex $p \times q$ matrix $\mathbf{S}$ is said to be *contractive* (resp., *strictly contractive*) if $\mathbf{I}_q - \mathbf{S}^*\mathbf{S}$ is a non-negative (resp., positive) Hermitian matrix. For a strictly contractive $p\times q$ matrix $\mathbf{S}$, we know from [12, Lemma 3.6.32] that

$$\mathbf{H}_{\mathbf{S}}\mathbf{H}_{-\mathbf{S}} = \mathbf{I}_{p+q} \tag{2.3}$$

and

$$(\mathbf{H}_{\mathbf{S}})^* \begin{pmatrix} \mathbf{I}_p & 0 \\ 0 & -\mathbf{I}_q \end{pmatrix} \mathbf{H}_{\mathbf{S}} = \begin{pmatrix} \mathbf{I}_p & 0 \\ 0 & -\mathbf{I}_q \end{pmatrix}, \tag{2.4}$$

where we use from now on the notation

$$\mathbf{H}_{\mathbf{S}} := \begin{pmatrix} (\mathbf{I}_p - \mathbf{SS}^*)^{-\frac{1}{2}} & \mathbf{S}(\mathbf{I}_q - \mathbf{S}^*\mathbf{S})^{-\frac{1}{2}} \\ \mathbf{S}^*(\mathbf{I}_p - \mathbf{SS}^*)^{-\frac{1}{2}} & (\mathbf{I}_q - \mathbf{S}^*\mathbf{S})^{-\frac{1}{2}} \end{pmatrix} \tag{2.5}$$

with $\mathbf{A}^{\frac{1}{2}}$ denoting the (unique) non-negative Hermitian square root of a non-negative Hermitian matrix $\mathbf{A}$ and $\mathbf{A}^{-\frac{1}{2}} = (\mathbf{A}^{-1})^{\frac{1}{2}} = (\mathbf{A}^{\frac{1}{2}})^{-1}$ its inverse.

Now we explain briefly the *Schur-Nevanlinna-Potapov algorithm* (SNP algorithm for short) for $p \times q$ Schur functions stated in [18, Section 5] (see also [9]). Let $S \in \mathcal{S}_{p\times q}(\mathbb{D})$ and let $\zeta_0, \zeta_1, \zeta_2, \ldots \in \mathbb{D}$. We set $S_0 := S$ and $\mathbf{S}_0 := S_0(\zeta_0)$. If $\mathbf{S}_0$ is a strictly contractive $p\times q$ matrix then for each $w \in \mathbb{D}$ the matrix $\mathbf{I}_q - \mathbf{S}_0^* S_0(w)$ is non-singular (cf. [12, Remark 1.1.2 and Lemma 1.1.13]) and we can define the matrix-valued function (holomorphic in $\mathbb{D}$)

$$S_1 := \frac{1}{b_{\zeta_0}}(\mathbf{I}_p - \mathbf{S}_0\mathbf{S}_0^*)^{-\frac{1}{2}}(S_0 - \mathbf{S}_0)(\mathbf{I}_q - \mathbf{S}_0^* S_0)^{-1}(\mathbf{I}_q - \mathbf{S}_0^*\mathbf{S}_0)^{\frac{1}{2}}$$

and recursively if for $k \in \mathbb{N}_0$ the function S_k is already defined and

$$\mathbf{S}_k := S_k(\zeta_k) \tag{2.6}$$

is a strictly contractive $p \times q$ matrix then

$$S_{k+1} := \frac{1}{b_{\zeta_k}}(\mathbf{I}_p - \mathbf{S}_k\mathbf{S}_k^*)^{-\frac{1}{2}}(S_k - \mathbf{S}_k)(\mathbf{I}_q - \mathbf{S}_k^* S_k)^{-1}(\mathbf{I}_q - \mathbf{S}_k^*\mathbf{S}_k)^{\frac{1}{2}}, \tag{2.7}$$

where b_{ζ_k} denotes the elementary Blaschke factor corresponding to ζ_k (see (1.5)). If $S \in \mathcal{S}_{p\times q}(\mathbb{D})$ and $\zeta_0, \zeta_1, \zeta_2, \ldots \in \mathbb{D}$ such that the SNP algorithm can be carried out at least r times (that is after obtaining S_r and $\mathbf{S}_r$) then $(\mathbf{S}_k)_{k=0}^r$ given by (2.6) is called the sequence of *SNP parameters associated with* the pair $[S, (\zeta_k)_{k=0}^r]$.

Using (2.5), (2.2), and [12, Lemma 1.1.12], the relation (2.7) can be written as

$$b_{\zeta_k}(w)S_{k+1}(w) = \mathfrak{S}_{\mathbf{H}_{-\mathbf{S}_k}}\big(S_k(w)\big), \quad w \in \mathbb{D}.$$

Thus, for each $w \in \mathbb{D}$, (2.3) and [12, Proposition 1.6.2] imply

$$S_k(w) = \mathfrak{S}_{(\mathbf{H}_{-\mathbf{S}_k})^{-1}}\big(b_{\zeta_k}(w)S_{k+1}(w)\big) = \mathfrak{S}_{\widehat{\mathbf{\Phi}}_k(w)}\big(S_{k+1}(w)\big)$$

with

$$\widehat{\mathbf{\Phi}}_k(w) := \mathbf{H}_{\mathbf{S}_k}\begin{pmatrix} b_{\zeta_k}(w)\mathbf{I}_p & 0 \\ 0 & \mathbf{I}_q \end{pmatrix}$$

and, by virtue of [12, Proposition 1.6.3], consequently

$$\begin{aligned} S(w) \equiv S_0(w) &= \mathfrak{S}_{\widehat{\mathbf{\Phi}}_0(w)}\Big(\mathfrak{S}_{\widehat{\mathbf{\Phi}}_1(w)}\Big(\cdots\Big(\mathfrak{S}_{\widehat{\mathbf{\Phi}}_k(w)}\big(S_{k+1}(w)\big)\cdots\Big)\Big)\Big) \\ &= \mathfrak{S}_{\widehat{\mathbf{\Phi}}_0(w)\widehat{\mathbf{\Phi}}_1(w)\cdots\widehat{\mathbf{\Phi}}_k(w)}\big(S_{k+1}(w)\big). \end{aligned} \tag{2.8}$$

The algorithm defines $p \times q$ Schur functions $S_0, S_1, S_2, \ldots$ (cf. (2.7), (2.4), and [12, Theorem 1.6.1]). It breaks down after the r-th step (that means here after obtaining S_r and $\mathbf{S}_r$) if and only if $\mathbf{I}_p - \mathbf{S}_r^*\mathbf{S}_r$ (resp., $\mathbf{I}_q - \mathbf{S}_r\mathbf{S}_r^*$) is a singular matrix. In addition, the considerations in [18, Section 5] show that the feasibility of the SNP algorithm is closely related to the non-degeneracy of the underlying $p \times q$ Schur function S. Recall that an $S \in \mathcal{S}_{p\times q}(\mathbb{D})$ is said to be *non-degenerate of order* r for some $r \in \mathbb{N}_0$ if $\mathbf{I}_{(r+1)q} - (\mathbf{S}_r^{(S)})^*\mathbf{S}_r^{(S)}$ is a non-singular matrix, where

$$\mathbf{S}_r^{(S)} := \begin{pmatrix} \mathbf{A}_0 & 0 & \cdots & 0 \\ \mathbf{A}_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \mathbf{A}_r & \cdots & \mathbf{A}_1 & \mathbf{A}_0 \end{pmatrix}$$

and the sequence $(\mathbf{A}_j)_{j=0}^r$ of complex $p\times q$ matrices is given by the Taylor expansion of S via $S(w) = \sum_{j=0}^{\infty} \mathbf{A}_j w^j$, $w \in \mathbb{D}$.

Proposition 2.2. *If $V \in \mathcal{S}_{p\times q}(\mathbb{D})$ then the SNP algorithm can be carried out at least $n+1$ times for V (and for any choice of points $\zeta_0, \zeta_1, \ldots, \zeta_{n+1} \in \mathbb{D}$) if and only if the matrix $\mathbf{P}_{\Delta[V]}$ (resp., $\widetilde{\mathbf{P}}_{\Delta[V]}$) is non-singular.*

Proof. Because of [18, Corollary 2.7 and Theorem 5.5] we know that the SNP algorithm can be carried out at least $n+1$ times for V if and only if the $p \times q$ Schur function V is non-degenerate of order n. Since [21, Corollary 3.2] yields that V is non-degenerate of order n if and only if the matrix $\mathbf{P}_{\Delta[V]}$ (resp., $\widetilde{\mathbf{P}}_{\Delta[V]}$) is non-singular, the assertion follows. □

In the next section, we shall treat sequences of rational matrix-valued functions which are closely related to the SNP algorithm and to the dual pairs of orthogonal functions discussed in [22]. For a fixed sequence $(\alpha_k)_{k=0}^{\tau}$ of points belonging $\mathbb{D}$ and some $k \in \mathbb{N}_{0,\tau}$ with $\tau \in \mathbb{N}_0$ or $\tau = \infty$ the notation $\breve{\mathcal{R}}_{\alpha,k}$ stands for the space of rational functions x that admit for some complex polynomial p of degree not greater than k the representation

$$x = \frac{p}{\pi_{\alpha,k}},$$

where the polynomial $\pi_{\alpha,k}$ of degree not greater than $k+1$ is given by

$$\pi_{\alpha,k}(v) = \prod_{j=0}^{k}(1 - \overline{\alpha_j}v), \quad v \in \mathbb{C}.$$

Let $k \in \mathbb{N}_{0,\tau}$. Then $\breve{\mathcal{R}}_{\alpha,k}^{p\times q}$ denotes the set of all complex $p\times q$ matrix functions each entry of which belongs to $\breve{\mathcal{R}}_{\alpha,k}$. Moreover, we use (with $\frac{1}{0} := \infty$) the settings

$$\mathbb{P}_{\alpha,k} := \bigcup_{j=0}^{k}\left\{\frac{1}{\overline{\alpha_j}}\right\} \quad \text{and} \quad \mathbb{Z}_{\alpha,k} := \bigcup_{j=0}^{k}\{\alpha_j\}.$$

Note that $\mathbb{Z}_{\alpha,k} \subset \mathbb{D}$ such that $\mathbb{D} \cup \mathbb{T} \subset \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$, where $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

The following transform of a rational matrix-valued function into another plays a key role in the sequel (cf. [6], [7], [19], and [22]). For each $X \in \breve{\mathcal{R}}^{p\times q}_{\alpha,k}$, we define the *adjoint rational matrix-valued function* $X^{[\alpha,k]}$ of X (with respect to $\alpha_0, \alpha_1, \ldots, \alpha_k$) as the rational matrix-valued function which is uniquely determined via the formula

$$X^{[\alpha,k]}(v) = \frac{1}{v} B_{\alpha,k}(v) \left(X\left(\frac{1}{\overline{v}}\right)\right)^*, \quad v \in \mathbb{C} \setminus (\mathbb{P}_{\alpha,k} \cup \mathbb{Z}_{\alpha,k} \cup \{0\}), \tag{2.9}$$

where $B_{\alpha,k}$ stands for the *Blaschke product* (of degree $k+1$) concerning the points $\alpha_0, \alpha_1, \ldots, \alpha_k$, i.e.

$$B_{\alpha,k} := \prod_{j=0}^{k} b_{\alpha_j}. \tag{2.10}$$

Some information on the interplay between $X^{[\alpha,k]}$ and the underlying rational matrix-valued function X can be found in [19, Section 2]. Note that in [19] it was assumed that $\alpha_0 = 0$, but it is not hard to adapt this to the present situation. For instance, it holds that if $X \in \breve{\mathcal{R}}^{p\times q}_{\alpha,k}$ then $X^{[\alpha,k]} \in \breve{\mathcal{R}}^{q\times p}_{\alpha,k}$ and $(X^{[\alpha,k]})^{[\alpha,k]} = X$.

3. Some basics on SNP sequences of rational matrix functions

In this section, we consider particular systems of rational matrix-valued functions which are closely related to the SNP algorithm explained in the previous section. In fact, we study sequences of rational matrix-valued functions defined by a sequence of points belonging to $\mathbb{D}$ and a sequence of strictly contractive $p \times q$ matrices. The following results can be seen as a rational extension of the matrix polynomial case treated in [17, Section 3] and as a matrix extension of the scalar rational case discussed in [7, Section 3].

Let $\tau \in \mathbb{N}_0$ or $\tau = \infty$, let $(\alpha_k)_{k=0}^{\tau}$ be a sequence of points belonging to $\mathbb{D}$, and let $(\mathbf{F}_k)_{k=0}^{\tau}$ be a sequence of strictly contractive $p \times q$ matrices. Then we define sequences of rational matrix-valued functions $(O_k)_{k=0}^{\tau}$ and $(Q_k)_{k=0}^{\tau}$ by the relations

$$O_0(v) := \frac{\sqrt{1-|\alpha_0|^2}}{1-\overline{\alpha_0}v}(\mathbf{I}_q - \mathbf{F}_0^*\mathbf{F}_0)^{-\frac{1}{2}}\mathbf{F}_0^*, \quad Q_0(v) := \frac{\sqrt{1-|\alpha_0|^2}}{1-\overline{\alpha_0}v}(\mathbf{I}_p - \mathbf{F}_0\mathbf{F}_0^*)^{-\frac{1}{2}} \tag{3.1}$$

for each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,0}$ and recursively via

$$O_k(v) := \sqrt{\frac{1-|\alpha_k|^2}{1-|\alpha_{k-1}|^2}} \frac{1-\overline{\alpha_{k-1}}v}{1-\overline{\alpha_k}v}(\mathbf{I}_q - \mathbf{F}_k^*\mathbf{F}_k)^{-\frac{1}{2}}\Big(b_{\alpha_{k-1}}(v)O_{k-1}(v) + \mathbf{F}_k^* Q_{k-1}^{[\alpha,k-1]}(v)\Big),$$

$$Q_k(v) := \sqrt{\frac{1-|\alpha_k|^2}{1-|\alpha_{k-1}|^2}} \frac{1-\overline{\alpha_{k-1}}v}{1-\overline{\alpha_k}v}\Big(b_{\alpha_{k-1}}(v)Q_{k-1}(v) + O_{k-1}^{[\alpha,k-1]}(v)\mathbf{F}_k^*\Big)(\mathbf{I}_p - \mathbf{F}_k\mathbf{F}_k^*)^{-\frac{1}{2}}$$

for each $k \in \mathbb{N}_{1,\tau}$ and each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$, where $O_{k-1}^{[\alpha,k-1]}$ and $Q_{k-1}^{[\alpha,k-1]}$ stands for the adjoint rational matrix-valued function of O_{k-1} and Q_{k-1}, respectively (as in (2.9), but relating to the points $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$). We call $[(O_k)_{k=0}^{\tau}, (Q_k)_{k=0}^{\tau}]$ the *first SNP pair of rational matrix functions corresponding to* $(\alpha_k, \mathbf{F}_k)_{k=0}^{\tau}$. Similarly, we bring in sequences of rational matrix-valued functions $(P_k)_{k=0}^{\tau}$ and $(R_k)_{k=0}^{\tau}$ by

$$P_0(v) := \frac{\sqrt{1-|\alpha_0|^2}}{1-\overline{\alpha_0}v}(\mathbf{I}_q - \mathbf{F}_0^*\mathbf{F}_0)^{-\frac{1}{2}}, \quad R_0(v) := \frac{\sqrt{1-|\alpha_0|^2}}{1-\overline{\alpha_0}v}\mathbf{F}_0^*(\mathbf{I}_p - \mathbf{F}_0\mathbf{F}_0^*)^{-\frac{1}{2}} \tag{3.2}$$

for each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,0}$ and recursively via

$$P_k(v) := \sqrt{\frac{1-|\alpha_k|^2}{1-|\alpha_{k-1}|^2}}\frac{1-\overline{\alpha_{k-1}}v}{1-\overline{\alpha_k}v}(\mathbf{I}_q - \mathbf{F}_k^*\mathbf{F}_k)^{-\frac{1}{2}}\Big(b_{\alpha_{k-1}}(v)P_{k-1}(v) + \mathbf{F}_k^* R_{k-1}^{[\alpha,k-1]}(v)\Big),$$

$$R_k(v) := \sqrt{\frac{1-|\alpha_k|^2}{1-|\alpha_{k-1}|^2}}\frac{1-\overline{\alpha_{k-1}}v}{1-\overline{\alpha_k}v}\Big(b_{\alpha_{k-1}}(v)R_{k-1}(v) + P_{k-1}^{[\alpha,k-1]}(v)\mathbf{F}_k^*\Big)(\mathbf{I}_p - \mathbf{F}_k\mathbf{F}_k^*)^{-\frac{1}{2}}$$

for each $k \in \mathbb{N}_{1,\tau}$ and each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$. We call $[(P_k)_{k=0}^\tau, (R_k)_{k=0}^\tau]$ the *second SNP pair of rational matrix functions corresponding to* $(\alpha_k, \mathbf{F}_k)_{k=0}^\tau$.

We remark that $O_k \in \breve{\mathcal{R}}_{\alpha,k}^{q\times p}$, $Q_k \in \breve{\mathcal{R}}_{\alpha,k}^{p\times p}$, $P_k \in \breve{\mathcal{R}}_{\alpha,k}^{q\times q}$, and $R_k \in \breve{\mathcal{R}}_{\alpha,k}^{q\times p}$ for each $k \in \mathbb{N}_{0,\tau}$. Moreover, in the scalar situation $p = q = 1$ we have the identities $O_k = R_k$ and $P_k = Q_k$ for each $k \in \mathbb{N}_{0,\tau}$. These scalar rational function were studied in [7], whereas the particular case of polynomials (i.e. the special choice $\alpha_k = 0$ for all $k \in \mathbb{N}_{0,\tau}$) already occurred in [27].

In view of (2.5) and [12, Lemma 1.1.12], by setting

$$\mathbf{\Theta}_k(v) := \begin{pmatrix} b_{\alpha_k}(v)Q_k(v) & O_k^{[\alpha,k]}(v) \\ b_{\alpha_k}(v)R_k(v) & P_k^{[\alpha,k]}(v) \end{pmatrix}$$
(3.3)
$$\left(\text{resp., } \mathbf{\Xi}_k(v) := \begin{pmatrix} b_{\alpha_k}(v)P_k(v) & b_{\alpha_k}(v)O_k(v) \\ R_k^{[\alpha,k]}(v) & Q_k^{[\alpha,k]}(v) \end{pmatrix}\right)$$

and (for technical reasons)

$$\eta_k := \begin{cases} -1 & \text{if } \alpha_k = 0, \\ \dfrac{\overline{\alpha_k}}{|\alpha_k|} & \text{if } \alpha_k \neq 0, \end{cases} \tag{3.4}$$

for each $k \in \mathbb{N}_{0,\tau}$, the recurrence formulae above can be written in matrix form as

$$\mathbf{\Theta}_k(v) = \sqrt{\frac{1-|\alpha_k|^2}{1-|\alpha_{k-1}|^2}}\frac{1-\overline{\alpha_{k-1}}v}{1-\overline{\alpha_k}v}\mathbf{\Theta}_{k-1}(v)\widehat{\mathbf{\Theta}}_k(v)$$
(3.5)
$$\left(\text{resp., } \mathbf{\Xi}_k(v) = \sqrt{\frac{1-|\alpha_k|^2}{1-|\alpha_{k-1}|^2}}\frac{1-\overline{\alpha_{k-1}}v}{1-\overline{\alpha_k}v}\widehat{\mathbf{\Xi}}_k(v)\mathbf{\Xi}_{k-1}(v)\right)$$

for each $k \in \mathbb{N}_{1,\tau}$ and each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$, where

$$\widehat{\mathbf{\Theta}}_k(v) := \mathbf{H}_{\mathbf{F}_k}\begin{pmatrix} b_{\alpha_k}(v)\mathbf{I}_p & 0 \\ 0 & \eta_k\overline{\eta_{k-1}}\,\mathbf{I}_q \end{pmatrix}$$
(3.6)
$$\left(\text{resp., } \widehat{\mathbf{\Xi}}_k(v) := \begin{pmatrix} b_{\alpha_k}(v)\mathbf{I}_q & 0 \\ 0 & \eta_k\overline{\eta_{k-1}}\,\mathbf{I}_p \end{pmatrix}\mathbf{H}_{\mathbf{F}_k^*}\right).$$

For later reference we use (3.6) also for $k = 0$, setting by definition $\eta_{-1} := -1$.

Proposition 3.1. *Let $\tau \in \mathbb{N}$ or $\tau = \infty$, let $(\mathbf{F}_k)_{k=0}^\tau$ be a sequence of strictly contractive $p \times q$ matrices, let $(\alpha_k)_{k=0}^\tau$ be a sequence of points belonging to $\mathbb{D}$, and let*

$$c_k := \frac{1-\overline{\alpha_k}\alpha_{k-1}}{\sqrt{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)}}, \quad k \in \mathbb{N}_{1,\tau}.$$

Furthermore, let $(O_k)_{k=0}^\tau$ and $(Q_k)_{k=0}^\tau$ be sequences of functions such that O_0, Q_0 are defined as in (3.1) *and that $O_k \in \breve{\mathcal{R}}_{\alpha,k}^{q\times p}$ and $Q_k \in \breve{\mathcal{R}}_{\alpha,k}^{p\times p}$ for all $k \in \mathbb{N}_{1,\tau}$. Then*

$[(O_k)_{k=0}^\tau, (Q_k)_{k=0}^\tau]$ *is the first SNP pair of rational matrix functions corresponding to* $(\alpha_k, \mathbf{F}_k)_{k=0}^\tau$ *if and only if, for each* $k \in \mathbb{N}_{1,\tau}$ *and each* $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$*, the following backward recurrence relations hold:*

$$\eta_k \overline{\eta_{k-1}} O_k(v) - \mathbf{F}_k^* Q_k^{[\alpha,k]}(v) = c_k\big(b_{\alpha_k}(v) - b_{\alpha_k}(\alpha_{k-1})\big)(\mathbf{I}_q - \mathbf{F}_k^*\mathbf{F}_k)^{\frac{1}{2}} O_{k-1}(v),$$

$$\eta_k \overline{\eta_{k-1}} Q_k(v) - O_k^{[\alpha,k]}(v)\mathbf{F}_k^* = c_k\big(b_{\alpha_k}(v) - b_{\alpha_k}(\alpha_{k-1})\big) Q_{k-1}(v)(\mathbf{I}_p - \mathbf{F}_k\mathbf{F}_k^*)^{\frac{1}{2}}.$$

Moreover, if $(P_k)_{k=0}^\tau$ *and* $(R_k)_{k=0}^\tau$ *are sequences such that* P_0*,* R_0 *are defined as in* (3.2) *and that* $P_k \in \breve{\mathcal{R}}_{\alpha,k}^{q\times q}$ *and* $R_k \in \breve{\mathcal{R}}_{\alpha,k}^{q\times p}$ *for all* $k \in \mathbb{N}_{1,\tau}$*. Then* $[(P_k)_{k=0}^\tau, (R_k)_{k=0}^\tau]$ *is the second SNP pair of rational matrix functions corresponding to* $(\alpha_k, \mathbf{F}_k)_{k=0}^\tau$ *if and only if, for each* $k \in \mathbb{N}_{1,\tau}$ *and* $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$*, the following relations hold:*

$$\eta_k \overline{\eta_{k-1}} P_k(v) - \mathbf{F}_k^* R_k^{[\alpha,k]}(v) = c_k\big(b_{\alpha_k}(v) - b_{\alpha_k}(\alpha_{k-1})\big)(\mathbf{I}_q - \mathbf{F}_k^*\mathbf{F}_k)^{\frac{1}{2}} P_{k-1}(v),$$

$$\eta_k \overline{\eta_{k-1}} R_k(v) - P_k^{[\alpha,k]}(v)\mathbf{F}_k^* = c_k\big(b_{\alpha_k}(v) - b_{\alpha_k}(\alpha_{k-1})\big) R_{k-1}(v)(\mathbf{I}_p - \mathbf{F}_k\mathbf{F}_k^*)^{\frac{1}{2}}.$$

Proof. Let $k \in \mathbb{N}_{1,\tau}$ and let $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$. By virtue of (3.3), (3.6), and (2.3) (see also [12, Lemma 1.1.12]), we get that (3.5) is equivalent to the relation

$$\begin{pmatrix} Q_k(v) & \overline{\eta_k}\eta_{k-1} O_k^{[\alpha,k]}(v) \\ R_k(v) & \overline{\eta_k}\eta_{k-1} P_k^{[\alpha,k]}(v) \end{pmatrix} \mathbf{H}_{-\mathbf{F}_k} = d_k \frac{1-\overline{\alpha_{k-1}}v}{1-\overline{\alpha_k}v} \mathbf{\Theta}_{k-1}(v)$$

$$\left(\text{resp., } \mathbf{H}_{-\mathbf{F}_k^*} \begin{pmatrix} P_k(v) & O_k(v) \\ \overline{\eta_k}\eta_{k-1} R_k^{[\alpha,k]}(v) & \overline{\eta_k}\eta_{k-1} Q_k^{[\alpha,k]}(v) \end{pmatrix} = d_k \frac{1-\overline{\alpha_{k-1}}v}{1-\overline{\alpha_k}v} \mathbf{\Xi}_{k-1}(v) \right),$$

where $d_k := \sqrt{\frac{1-|\alpha_k|^2}{1-|\alpha_{k-1}|^2}}$. Hence, by considering the first column of $\mathbf{\Theta}_{k-1}(v)$ and the first row of $\mathbf{\Xi}_{k-1}(v)$, using

$$\overline{\eta_k}\eta_{k-1} \frac{1-\overline{\alpha_k}\alpha_{k-1}}{1-|\alpha_k|^2}\big(b_{\alpha_k}(v) - b_{\alpha_k}(\alpha_{k-1})\big) = \frac{1-\overline{\alpha_{k-1}}v}{1-\overline{\alpha_k}v} b_{\alpha_{k-1}}(v),$$

one can finally conclude the assertion. □

Observe that the backward recurrence relations in Proposition 3.1 for the first SNP pair of rational matrix functions corresponding to $(\alpha_k, \mathbf{F}_k)_{k=0}^\tau$ have the same form as these for the second SNP pair of rational matrix functions. It comes across an analogous concordance as in the case of the forward recursions defining such pairs. However, the size of the matrix functions involved as well as the initial conditions (3.1) and (3.2) are different.

A key tool in the proof of the descriptions (1.3) and (1.4) for the solution set $\mathcal{S}_\Delta$ of Problem (MNP) is an application of some well-known results on Potapov's **J**-theory (see, e.g., [25] and [26]). Recall that if r is a positive integer and if $\mathbf{J}$ is a complex $r \times r$ signature matrix (i.e. unitary and Hermitian) then a complex $r \times r$ matrix $\mathbf{A}$ is called **J**-*contractive* (resp., **J**-*unitary*), if

$$\mathbf{J} \geq \mathbf{A}^*\mathbf{J}\mathbf{A} \quad \big(\text{resp., } \mathbf{J} = \mathbf{A}^*\mathbf{J}\mathbf{A}\big).$$

In our case we shall use the special $(p+q) \times (p+q)$ signature matrices

$$\mathbf{j}_{pq} := \begin{pmatrix} \mathbf{I}_p & 0 \\ 0 & -\mathbf{I}_q \end{pmatrix} \quad \text{and} \quad \mathbf{j}_{qp} := \begin{pmatrix} \mathbf{I}_q & 0 \\ 0 & -\mathbf{I}_p \end{pmatrix}.$$

Henceforth in this section, $[(O_k)_{k=0}^\tau, (Q_k)_{k=0}^\tau]$ (resp., $[(P_k)_{k=0}^\tau, (R_k)_{k=0}^\tau]$) stands always for the first (resp., second) SNP pair of rational matrix functions corresponding to $(\alpha_k, \mathbf{F}_k)_{k=0}^\tau$, where $(\alpha_k)_{k=0}^\tau$ is some sequence of points belonging to $\mathbb{D}$

and $(\mathbf{F}_k)_{k=0}^{\tau}$ is some sequence of strictly contractive $p \times q$ matrices. Furthermore, we use the notations given by (3.3) and (3.6).

Theorem 3.2. *For each $k \in \mathbb{N}_{0,\tau}$ and each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$,*

$$\mathbf{\Theta}_k(v) = \frac{\sqrt{1-|\alpha_k|^2}}{1-\overline{\alpha_k}v} \widehat{\mathbf{\Theta}}_0(v)\widehat{\mathbf{\Theta}}_1(v) \cdots \widehat{\mathbf{\Theta}}_k(v),$$

$$\mathbf{\Xi}_k(v) = \frac{\sqrt{1-|\alpha_k|^2}}{1-\overline{\alpha_k}v} \widehat{\mathbf{\Xi}}_k(v)\widehat{\mathbf{\Xi}}_{k-1}(v) \cdots \widehat{\mathbf{\Xi}}_0(v).$$

Moreover, if $v \in \mathbb{D}$ (resp., $v \in \mathbb{T}$) then the matrix $\frac{1-\overline{\alpha_k}v}{\sqrt{1-|\alpha_k|^2}} \mathbf{\Theta}_k(v)$ is $\mathbf{j}_{pq}$-contractive (resp., $\mathbf{j}_{pq}$-unitary) and $\frac{1-\overline{\alpha_k}v}{\sqrt{1-|\alpha_k|^2}} \mathbf{\Xi}_k(v)$ is $\mathbf{j}_{qp}$-contractive (resp., $\mathbf{j}_{qp}$-unitary).

Proof. Let $k \in \mathbb{N}_{0,\tau}$ and let $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$. We prove only the above expressions with respect to $\mathbf{\Theta}_k(v)$. A proof for $\mathbf{\Xi}_k(v)$ is similar. As an easy consequence of (3.5) and the choice of O_0, P_0, Q_0, and R_0 given in (3.1) and (3.2) we obtain

$$\mathbf{\Theta}_k(v) = \frac{\sqrt{1-|\alpha_k|^2}}{1-\overline{\alpha_k}v} \widehat{\mathbf{\Theta}}_0(v)\widehat{\mathbf{\Theta}}_1(v) \cdots \widehat{\mathbf{\Theta}}_k(v). \tag{3.7}$$

It remains to prove that if $v \in \mathbb{D}$ (resp., $v \in \mathbb{T}$) then

$$\mathbf{j}_{pq} \geq \left(\frac{1-\overline{\alpha_k}v}{\sqrt{1-|\alpha_k|^2}} \mathbf{\Theta}_k(v) \right)^* \mathbf{j}_{pq} \left(\frac{1-\overline{\alpha_k}v}{\sqrt{1-|\alpha_k|^2}} \mathbf{\Theta}_k(v) \right)$$
$$\left(\text{resp., } \mathbf{j}_{pq} = \left(\frac{1-\overline{\alpha_k}v}{\sqrt{1-|\alpha_k|^2}} \mathbf{\Theta}_k(v) \right)^* \mathbf{j}_{pq} \left(\frac{1-\overline{\alpha_k}v}{\sqrt{1-|\alpha_k|^2}} \mathbf{\Theta}_k(v) \right) \right).$$

Taking [12, Lemma 1.3.13] and (3.6) into account, this follows immediately from (3.7) in combination with (2.4) and the fact that if $v \in \mathbb{D}$ (resp., $v \in \mathbb{T}$) then

$$\mathbf{j}_{pq} \geq \begin{pmatrix} b_{\alpha_j}(v)\mathbf{I}_p & 0 \\ 0 & u\mathbf{I}_q \end{pmatrix}^* \mathbf{j}_{pq} \begin{pmatrix} b_{\alpha_j}(v)\mathbf{I}_p & 0 \\ 0 & u\mathbf{I}_q \end{pmatrix}$$
$$\left(\text{resp., } \mathbf{j}_{pq} = \begin{pmatrix} b_{\alpha_j}(v)\mathbf{I}_p & 0 \\ 0 & u\mathbf{I}_q \end{pmatrix}^* \mathbf{j}_{pq} \begin{pmatrix} b_{\alpha_j}(v)\mathbf{I}_p & 0 \\ 0 & u\mathbf{I}_q \end{pmatrix} \right)$$

for each $j \in \mathbb{N}_{0,k}$ and some $u \in \mathbb{T}$. □

In view of (3.3) and some well-known results on $\mathbf{j}_{pq}$-contractive matrices (see, e.g., [12, Theorem 1.6.1]), Theorem 3.2 yields particularly the following result.

Corollary 3.3. *For each $k \in \mathbb{N}_{0,\tau}$ and each $v \in \mathbb{D} \cup \mathbb{T}$, the matrices $Q_k^{[\alpha,k]}(v)$ and $P_k^{[\alpha,k]}(v)$ are non-singular and the matrices*

$$\big(Q_k^{[\alpha,k]}(v)\big)^{-1} R_k^{[\alpha,k]}(v), \quad b_{\alpha_k}(v) O_k(v) \big(Q_k^{[\alpha,k]}(v)\big)^{-1},$$

$$O_k^{[\alpha,k]}(v) \big(P_k^{[\alpha,k]}(v)\big)^{-1}, \quad b_{\alpha_k}(v) \big(P_k^{[\alpha,k]}(v)\big)^{-1} R_k(v)$$

are strictly contractive.

Taking Corollary 3.3 into account, the next statement is an easy conclusion of Proposition 3.1 and (3.5) with the special choice $v = \alpha_{k-1}$.

Corollary 3.4. *For each $k \in \mathbb{N}_{1,\tau}$, the matrices $Q_k^{[\alpha,k]}(\alpha_{k-1})$ and $P_k^{[\alpha,k]}(\alpha_{k-1})$ are non-singular, the identities*

$$\mathbf{F}_k^* = \eta_k \overline{\eta_{k-1}} O_k(\alpha_{k-1}) \big(Q_k^{[\alpha,k]}(\alpha_{k-1})\big)^{-1}, \quad \mathbf{F}_k^* = \eta_k \overline{\eta_{k-1}} \big(P_k^{[\alpha,k]}(\alpha_{k-1})\big)^{-1} R_k(\alpha_{k-1}),$$

$$(\mathbf{I}_p - \mathbf{F}_k \mathbf{F}_k^*)^{\frac{1}{2}} = \eta_k \overline{\eta_{k-1}} \frac{\sqrt{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)}}{1-\overline{\alpha_k}\alpha_{k-1}} Q_{k-1}^{[\alpha,k-1]}(\alpha_{k-1}) \big(Q_k^{[\alpha,k]}(\alpha_{k-1})\big)^{-1},$$

$$(\mathbf{I}_q - \mathbf{F}_k^* \mathbf{F}_k)^{\frac{1}{2}} = \eta_k \overline{\eta_{k-1}} \frac{\sqrt{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)}}{1-\overline{\alpha_k}\alpha_{k-1}} \big(P_k^{[\alpha,k]}(\alpha_{k-1})\big)^{-1} P_{k-1}^{[\alpha,k-1]}(\alpha_{k-1})$$

are satisfied, and particularly $\mathbf{F}_k = 0 \iff O_k(\alpha_{k-1}) = 0 \iff R_k(\alpha_{k-1}) = 0$.

Note that, if we use the settings $\alpha_{-1} := 0$, $\eta_{-1} := -1$, $Q_{-1}^{[\alpha,-1]}(\alpha_{-1}) := \mathbf{I}_p$, and $P_{-1}^{[\alpha,-1]}(\alpha_{-1}) := \mathbf{I}_q$ then the relations in Corollary 3.4 hold also in the case $k = 0$.

Proposition 3.5. *For each $k \in \mathbb{N}_{0,\tau}$ and each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$,*

$$\mathbf{\Xi}_k(v) \begin{pmatrix} 0 & \mathbf{I}_q \\ -\mathbf{I}_p & 0 \end{pmatrix} \mathbf{\Theta}_k(v) = -\eta_k \frac{1-|\alpha_k|^2}{(1-\overline{\alpha_k}v)^2} B_{\alpha,k}(v) \begin{pmatrix} 0 & \mathbf{I}_q \\ -\mathbf{I}_p & 0 \end{pmatrix},$$

where $\mathbf{\Theta}_k(v)$ and $\mathbf{\Xi}_k(v)$ are defined as in (3.3)*, the number η_k is given as in* (3.4)*, and $B_{\alpha,k}$ is given as in* (2.10) *with respect to $\alpha_0, \alpha_1, \dots, \alpha_k$.*

Proof. Let $k \in \mathbb{N}_{0,\tau}$ and let $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$. Furthermore, let $\eta_{-1} := -1$. A straightforward calculation yields

$$\begin{pmatrix} b_{\alpha_j}(v)\mathbf{I}_p & 0 \\ 0 & \eta_j\overline{\eta_{j-1}}\,\mathbf{I}_q \end{pmatrix} \begin{pmatrix} 0 & \mathbf{I}_q \\ -\mathbf{I}_p & 0 \end{pmatrix} \begin{pmatrix} b_{\alpha_j}(v)\mathbf{I}_q & 0 \\ 0 & \eta_j\overline{\eta_{j-1}}\,\mathbf{I}_p \end{pmatrix} = \eta_j\overline{\eta_{j-1}} b_{\alpha_j}(v) \begin{pmatrix} 0 & \mathbf{I}_q \\ -\mathbf{I}_p & 0 \end{pmatrix}$$

for each $j \in \mathbb{N}_{0,k}$. Using this in combination with the fact that [12, Lemma 1.1.12] implies the identity

$$\mathbf{H}_{\mathbf{F}_j^*} \begin{pmatrix} 0 & \mathbf{I}_q \\ -\mathbf{I}_p & 0 \end{pmatrix} \mathbf{H}_{\mathbf{F}_j} = \begin{pmatrix} 0 & \mathbf{I}_q \\ -\mathbf{I}_p & 0 \end{pmatrix}$$

for each $j \in \mathbb{N}_{0,k}$, based on the decomposition of $\mathbf{\Theta}_k(v)$ and $\mathbf{\Xi}_k(v)$ according to Theorem 3.2, one can finally conclude the assertion. □

The next result is an easy consequence of Proposition 3.5, (3.3), and

$$\begin{pmatrix} 0 & \mathbf{I}_q \\ -\mathbf{I}_p & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & -\mathbf{I}_p \\ \mathbf{I}_q & 0 \end{pmatrix}.$$

Corollary 3.6. *For each $k \in \mathbb{N}_{0,\tau}$ and each $v \in \mathbb{C} \setminus (\mathbb{P}_{\alpha,k} \cup \mathbb{Z}_{\alpha,k})$, the matrices $\mathbf{\Theta}_k(v)$, $\mathbf{\Xi}_k(v)$ are non-singular and*

$$\big(\mathbf{\Theta}_k(v)\big)^{-1} = \frac{(1-\overline{\alpha_k}v)^2}{\eta_k(1-|\alpha_k|^2)B_{\alpha,k}(v)} \begin{pmatrix} -Q_k^{[\alpha,k]}(v) & R_k^{[\alpha,k]}(v) \\ b_{\alpha_k}(v)O_k(v) & -b_{\alpha_k}(v)P_k(v) \end{pmatrix},$$

$$\big(\mathbf{\Xi}_k(v)\big)^{-1} = \frac{(1-\overline{\alpha_k}v)^2}{\eta_k(1-|\alpha_k|^2)B_{\alpha,k}(v)} \begin{pmatrix} -P_k^{[\alpha,k]}(v) & b_{\alpha_k}(v)R_k(v) \\ O_k^{[\alpha,k]}(v) & -b_{\alpha_k}(v)Q_k(v) \end{pmatrix}.$$

Taking (3.3) into account, Corollary 3.6 yields by considering the corresponding block entries and using a continuity argument the identities below.

Corollary 3.7. *For each $k \in \mathbb{N}_{0,\tau}$ and each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$,*

$$O_k(v)Q_k(v) = P_k(v)R_k(v), \quad Q_k^{[\alpha,k]}(v)O_k^{[\alpha,k]}(v) = R_k^{[\alpha,k]}(v)P_k^{[\alpha,k]}(v),$$

$$P_k^{[\alpha,k]}(v)O_k(v) = R_k(v)Q_k^{[\alpha,k]}(v), \quad O_k^{[\alpha,k]}(v)P_k(v) = Q_k(v)R_k^{[\alpha,k]}(v),$$

$$b_{\alpha_k}(v)\Big(O_k^{[\alpha,k]}(v)O_k(v) - Q_k(v)Q_k^{[\alpha,k]}(v)\Big) = \eta_k \frac{1-|\alpha_k|^2}{(1-\overline{\alpha_k}v)^2} B_{\alpha,k}(v)\mathbf{I}_p,$$

$$b_{\alpha_k}(v)\Big(R_k(v)R_k^{[\alpha,k]}(v) - P_k^{[\alpha,k]}(v)P_k(v)\Big) = \eta_k \frac{1-|\alpha_k|^2}{(1-\overline{\alpha_k}v)^2} B_{\alpha,k}(v)\mathbf{I}_q,$$

$$b_{\alpha_k}(v)\Big(O_k(v)O_k^{[\alpha,k]}(v) - P_k(v)P_k^{[\alpha,k]}(v)\Big) = \eta_k \frac{1-|\alpha_k|^2}{(1-\overline{\alpha_k}v)^2} B_{\alpha,k}(v)\mathbf{I}_q,$$

$$b_{\alpha_k}(v)\Big(R_k^{[\alpha,k]}(v)R_k(v) - Q_k^{[\alpha,k]}(v)Q_k(v)\Big) = \eta_k \frac{1-|\alpha_k|^2}{(1-\overline{\alpha_k}v)^2} B_{\alpha,k}(v)\mathbf{I}_p.$$

Since $\alpha_0, \alpha_1 \ldots, \alpha_{k-1}$ are the zeroes of the function $B_{\alpha,k-1}$ if $k \in \mathbb{N}_{1,\tau}$ (cf. (1.5) and (2.10)), the last four equalities in Corollary 3.7 imply particularly the following.

Corollary 3.8. *For each $k \in \mathbb{N}_{1,\tau}$ and each $j \in \mathbb{N}_{0,k-1}$,*

$$O_k^{[\alpha,k]}(\alpha_j)O_k(\alpha_j) = Q_k(\alpha_j)Q_k^{[\alpha,k]}(\alpha_j), \quad R_k(\alpha_j)R_k^{[\alpha,k]}(\alpha_j) = P_k^{[\alpha,k]}(\alpha_j)P_k(\alpha_j),$$

$$O_k(\alpha_j)O_k^{[\alpha,k]}(\alpha_j) = P_k(\alpha_j)P_k^{[\alpha,k]}(\alpha_j), \quad R_k^{[\alpha,k]}(\alpha_j)R_k(\alpha_j) = Q_k^{[\alpha,k]}(\alpha_j)Q_k(\alpha_j).$$

Remark 3.9. If $\mathbf{F}_0 = 0$ then obviously

$$O_0^{[\alpha,0]}(\alpha_0) = 0, \quad R_0^{[\alpha,0]}(\alpha_0) = 0$$

and, for each $k \in \mathbb{N}_{1,\tau}$, by an application of Corollary 3.8 in combination with $\det Q_k^{[\alpha,k]}(\alpha_0) \neq 0$ and $\det P_k^{[\alpha,k]}(\alpha_0) \neq 0$ which follows by Corollary 3.3, one can inductively derive from the recurrence relations (use, e.g., (3.5)) that

$$O_k^{[\alpha,k]}(\alpha_0) = 0, \quad Q_k(\alpha_0) = 0, \quad R_k^{[\alpha,k]}(\alpha_0) = 0, \quad P_k(\alpha_0) = 0.$$

Proposition 3.10. *For each $k \in \mathbb{N}_{0,\tau}$ and each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$,*

$$\det Q_k(v) = \left(\frac{\sqrt{1-|\alpha_k|^2}}{1-\overline{\alpha_k}v}\right)^{p-q} \det P_k(v),$$

$$\det Q_k^{[\alpha,k]}(v) = \left(\frac{-\eta_k\sqrt{1-|\alpha_k|^2}}{1-\overline{\alpha_k}v}\right)^{p-q} \det P_k^{[\alpha,k]}(v).$$

Proof. To simplify notation, for each $k \in \mathbb{N}_{0,\tau}$ we set

$$r_k(v) := \frac{\sqrt{1-|\alpha_k|^2}}{1-\overline{\alpha_k}v}, \quad v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}.$$

Because of (2.9), for each $k \in \mathbb{N}_{0,\tau}$, one can easily see (cf. [19, Remark 2.6]) that

$$\det Q_k(v) = \big(r_k(v)\big)^{p-q} \det P_k(v), \quad v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}, \tag{3.8}$$

is tantamount to

$$\det Q_k^{[\alpha,k]}(v) = \big(-\eta_k r_k(v)\big)^{p-q} \det P_k^{[\alpha,k]}(v), \quad v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}. \tag{3.9}$$

Therefore, we only have to prove one of them. If $k = 0$ then it follow from the initial conditions (3.1) and (3.2) by using some standard calculation rules of determinants (see, e.g., [12, Lemma 1.1.8]) that

$$\det Q_0(v) = \frac{\big(r_0(v)\big)^p}{\sqrt{\det(\mathbf{I}_p - \mathbf{F}_0\mathbf{F}_0^*)}} = \frac{\big(r_0(v)\big)^p}{\sqrt{\det(\mathbf{I}_q - \mathbf{F}_0^*\mathbf{F}_0)}} = \big(r_0(v)\big)^{p-q} \det P_0(v)$$

for each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,0}$. Now we assume that, for some integer $k \in \mathbb{N}_{1,\tau}$ and each point $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k-1}$, we have

$$\det Q_{k-1}(v) = \big(r_{k-1}(v)\big)^{p-q} \det P_{k-1}(v)$$

and hence also

$$\det Q_{k-1}^{[\alpha,k-1]}(v) = \big(-\eta_{k-1} r_{k-1}(v)\big)^{p-q} \det P_{k-1}^{[\alpha,k-1]}(v). \tag{3.10}$$

Let $v \in \mathbb{D} \cup \mathbb{T}$. By Corollary 3.3 we know that $Q_{k-1}^{[\alpha,k-1]}(v)$ and $P_{k-1}^{[\alpha,k-1]}(v)$ are non-singular matrices, so that Corollary 3.7 yields the identity

$$O_{k-1}(v)\big(Q_{k-1}^{[\alpha,k-1]}(v)\big)^{-1} = \big(P_{k-1}^{[\alpha,k-1]}(v)\big)^{-1} R_{k-1}(v).$$

Thus, using the recursions of SNP pairs of rational matrix functions (see (3.5)), some elementary calculation rules of determinants, and (3.10) we obtain

$$\begin{aligned}
&\det Q_k^{[\alpha,k]}(v)\\
&= \left(\eta_k \overline{\eta_{k-1}} \frac{r_k(v)}{r_{k-1}(v)}\right)^p \frac{\det\big(Q_{k-1}^{[\alpha,k-1]}(v) + b_{\alpha_{k-1}}(v)\mathbf{F}_k O_{k-1}(v)\big)}{\sqrt{\det(\mathbf{I}_p - \mathbf{F}_k\mathbf{F}_k^*)}}\\
&= \left(\frac{\eta_k r_k(v)}{\eta_{k-1} r_{k-1}(v)}\right)^p \frac{\det\Big(\mathbf{I}_p + b_{\alpha_{k-1}}(v)\mathbf{F}_k O_{k-1}(v)\big(Q_{k-1}^{[\alpha,k-1]}(v)\big)^{-1}\Big)\det Q_{k-1}^{[\alpha,k-1]}(v)}{\sqrt{\det(\mathbf{I}_p - \mathbf{F}_k\mathbf{F}_k^*)}}\\
&= \left(\frac{\eta_k r_k(v)}{\eta_{k-1} r_{k-1}(v)}\right)^p \frac{\det\Big(\mathbf{I}_p + b_{\alpha_{k-1}}(v)\mathbf{F}_k \big(P_{k-1}^{[\alpha,k-1]}(v)\big)^{-1} R_{k-1}(v)\Big)\det Q_{k-1}^{[\alpha,k-1]}(v)}{\sqrt{\det(\mathbf{I}_p - \mathbf{F}_k\mathbf{F}_k^*)}}\\
&= \left(\frac{\eta_k r_k(v)}{\eta_{k-1} r_{k-1}(v)}\right)^p \frac{\det\Big(\mathbf{I}_q + b_{\alpha_{k-1}}(v)\big(P_{k-1}^{[\alpha,k-1]}(v)\big)^{-1} R_{k-1}(v)\mathbf{F}_k\Big)\det Q_{k-1}^{[\alpha,k-1]}(v)}{\sqrt{\det(\mathbf{I}_q - \mathbf{F}_k^*\mathbf{F}_k)}}\\
&= \big(-\eta_{k-1} r_{k-1}(v)\big)^{p-q}\left(\frac{\eta_k r_k(v)}{\eta_{k-1} r_{k-1}(v)}\right)^p \frac{\det\big(P_{k-1}^{[\alpha,k-1]}(v) + b_{\alpha_{k-1}}(v) R_{k-1}(v)\mathbf{F}_k\big)}{\sqrt{\det(\mathbf{I}_q - \mathbf{F}_k^*\mathbf{F}_k)}}\\
&= \big(-\eta_k r_k(v)\big)^{p-q} \det P_k^{[\alpha,k]}(v).
\end{aligned}$$

Since $\det Q_k^{[\alpha,k]}$, $\det P_k^{[\alpha,k]}$, and r_k are rational functions, we can conclude (3.9). Hence, since (3.9) implies (3.8), for each $k \in \mathbb{N}_{0,\tau}$ and each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$ the assertion is inductively shown. □

In the square matrix case $p = q$, Proposition 3.10 leads in combination with the first both identities of Corollary 3.7, the statement of regularity pointed out in Corollary 3.3, and a continuity argument to the following relations as well.

Corollary 3.11. *If $p = q$, then for each $k \in \mathbb{N}_{0,\tau}$ and each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$,*

$$\det O_k(v) = \det R_k(v) \quad \textit{and} \quad \det O_k^{[\alpha,k]}(v) = \det R_k^{[\alpha,k]}(v).$$

4. Christoffel-Darboux formulae

In the present section, we shall show that, like in the case of orthogonal rational matrix-valued functions (cf. [19, Section 5] and [22, Section 4]), also arbitrary SNP pairs of rational matrix functions fulfill some Christoffel-Darboux formulae. To prove these, we give first certain auxiliary identities. Here and in the sequel, $[(O_k)_{k=0}^{\tau},(Q_k)_{k=0}^{\tau}]$ (resp., $[(P_k)_{k=0}^{\tau},(R_k)_{k=0}^{\tau}]$) stands again for the first (resp., second) SNP pair of rational matrix functions corresponding to $(\alpha_k,\mathbf{F}_k)_{k=0}^{\tau}$, where $(\alpha_k)_{k=0}^{\tau}$ is some sequence of points belonging to $\mathbb{D}$ and $(\mathbf{F}_k)_{k=0}^{\tau}$ is some sequence of strictly contractive $p\times q$ matrices. For technical reasons, if $k\in\mathbb{N}_{0,\tau}$ and if $v,w\in\mathbb{C}\setminus\left\{\frac{1}{\overline{\alpha_k}}\right\}$, we also use in view of (1.5) the notation

$$\mathfrak{k}_{\alpha_k}(v,w):=1-b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}=\frac{(1-|\alpha_k|^2)(1-v\overline{w})}{(1-v\overline{\alpha_k})(1-\alpha_k\overline{w})}. \tag{4.1}$$

Lemma 4.1. *For all $k\in\mathbb{N}_{1,\tau}$ and all $v,w\in\mathbb{C}\setminus\mathbb{P}_{\alpha,k}$, the following relations hold:*

$$\mathfrak{k}_{\alpha_{k-1}}(w,v)\Big(\big(Q_k^{[\alpha,k]}(v)\big)^*Q_k^{[\alpha,k]}(w)-\big(O_k(v)\big)^*O_k(w)\Big)$$
$$=\mathfrak{k}_{\alpha_k}(w,v)\Big(\big(Q_{k-1}^{[\alpha,k-1]}(v)\big)^*Q_{k-1}^{[\alpha,k-1]}(w)-\overline{b_{\alpha_{k-1}}(v)}b_{\alpha_{k-1}}(w)\big(O_{k-1}(v)\big)^*O_{k-1}(w)\Big),$$

$$\mathfrak{k}_{\alpha_{k-1}}(v,w)\Big(O_k^{[\alpha,k]}(v)\big(O_k^{[\alpha,k]}(w)\big)^*-Q_k(v)\big(Q_k(w)\big)^*\Big)$$
$$=\mathfrak{k}_{\alpha_k}(v,w)\Big(O_{k-1}^{[\alpha,k-1]}(v)\big(O_{k-1}^{[\alpha,k-1]}(w)\big)^*-b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)}Q_{k-1}(v)\big(Q_{k-1}(w)\big)^*\Big),$$

$$\mathfrak{k}_{\alpha_{k-1}}(w,v)\Big(\big(R_k^{[\alpha,k]}(v)\big)^*R_k^{[\alpha,k]}(w)-\big(P_k(v)\big)^*P_k(w)\Big)$$
$$=\mathfrak{k}_{\alpha_k}(w,v)\Big(\big(R_{k-1}^{[\alpha,k-1]}(v)\big)^*R_{k-1}^{[\alpha,k-1]}(w)-\overline{b_{\alpha_{k-1}}(v)}b_{\alpha_{k-1}}(w)\big(P_{k-1}(v)\big)^*P_{k-1}(w)\Big),$$

$$\mathfrak{k}_{\alpha_{k-1}}(v,w)\Big(P_k^{[\alpha,k]}(v)\big(P_k^{[\alpha,k]}(w)\big)^*-R_k(v)\big(R_k(w)\big)^*\Big)$$
$$=\mathfrak{k}_{\alpha_k}(v,w)\Big(P_{k-1}^{[\alpha,k-1]}(v)\big(P_{k-1}^{[\alpha,k-1]}(w)\big)^*-b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)}R_{k-1}(v)\big(R_{k-1}(w)\big)^*\Big),$$

$$\mathfrak{k}_{\alpha_{k-1}}(w,v)\Big(\big(Q_k^{[\alpha,k]}(v)\big)^*R_k^{[\alpha,k]}(w)-\big(O_k(v)\big)^*P_k(w)\Big)$$
$$=\mathfrak{k}_{\alpha_k}(w,v)\Big(\big(Q_{k-1}^{[\alpha,k-1]}(v)\big)^*R_{k-1}^{[\alpha,k-1]}(w)-\overline{b_{\alpha_{k-1}}(v)}b_{\alpha_{k-1}}(w)\big(O_{k-1}(v)\big)^*P_{k-1}(w)\Big),$$

$$\mathfrak{k}_{\alpha_{k-1}}(v,w)\Big(O_k^{[\alpha,k]}(v)\big(P_k^{[\alpha,k]}(w)\big)^*-Q_k(v)\big(R_k(w)\big)^*\Big)$$
$$=\mathfrak{k}_{\alpha_k}(v,w)\Big(O_{k-1}^{[\alpha,k-1]}(v)\big(P_{k-1}^{[\alpha,k-1]}(w)\big)^*-b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)}Q_{k-1}(v)\big(R_{k-1}(w)\big)^*\Big).$$

Proof. Taking (3.3), (3.6), and (4.1) into account, the assertion is an easy consequence of (3.5) and (2.4). We give an example of the straightforward calculation involved by proving the first identity. Let $k\in\mathbb{N}_{1,\tau}$ and let $v,w\in\mathbb{C}\setminus\mathbb{P}_{\alpha,k}$. Because of (4.1) it follows

$$\mathfrak{k}_{\alpha_k}(w,v)=\mathfrak{k}_{\alpha_{k-1}}(w,v)\frac{(1-|\alpha_k|^2)(1-\alpha_{k-1}\overline{v})(1-\overline{\alpha_{k-1}}w)}{(1-|\alpha_{k-1}|^2)(1-\alpha_k\overline{v})(1-\overline{\alpha_k}w)}.$$

Consequently, by considering the lower $p \times (q+p)$ row of $\big(\mathbf{\Xi}_k(v)\big)^*$ and the right $(q+p) \times p$ column of $\mathbf{\Xi}_k(w)$ we get

$$
\begin{aligned}
&\mathfrak{k}_{\alpha_{k-1}}(w,v)\Big(\big(Q_k^{[\alpha,k]}(v)\big)^* Q_k^{[\alpha,k]}(w) - \big(O_k(v)\big)^* O_k(w)\Big) \\
&\quad = -\,\mathfrak{k}_{\alpha_{k-1}}(w,v) \begin{pmatrix} O_k(v) \\ \overline{\eta_k}\eta_{k-1} Q_k^{[\alpha,k]}(v) \end{pmatrix}^* \mathbf{j}_{qp} \begin{pmatrix} O_k(w) \\ \overline{\eta_k}\eta_{k-1} Q_k^{[\alpha,k]}(w) \end{pmatrix} \\
&\quad = -\,\mathfrak{k}_{\alpha_{k-1}}(w,v) \left(\sqrt{\frac{1-|\alpha_k|^2}{1-|\alpha_{k-1}|^2}}\, \frac{1-\overline{\alpha_{k-1}}v}{1-\overline{\alpha_k}v}\, \mathbf{H}_{\mathbf{F}_k^*} \begin{pmatrix} b_{\alpha_{k-1}}(v) O_{k-1}(v) \\ Q_{k-1}^{[\alpha,k-1]}(v) \end{pmatrix} \right)^* \mathbf{j}_{qp} \times \\
&\qquad\qquad \times \left(\sqrt{\frac{1-|\alpha_k|^2}{1-|\alpha_{k-1}|^2}}\, \frac{1-\overline{\alpha_{k-1}}w}{1-\overline{\alpha_k}w}\, \mathbf{H}_{\mathbf{F}_k^*} \begin{pmatrix} b_{\alpha_{k-1}}(w) O_k(w) \\ Q_{k-1}^{[\alpha,k-1]}(w) \end{pmatrix} \right) \\
&\quad = -\,\mathfrak{k}_{\alpha_k}(w,v) \begin{pmatrix} b_{\alpha_{k-1}}(v) O_{k-1}(v) \\ Q_{k-1}^{[\alpha,k-1]}(v) \end{pmatrix}^* \mathbf{j}_{qp} \begin{pmatrix} b_{\alpha_{k-1}}(w) O_k(w) \\ Q_{k-1}^{[\alpha,k-1]}(w) \end{pmatrix} \\
&\quad = \mathfrak{k}_{\alpha_k}(w,v)\Big(\big(Q_{k-1}^{[\alpha,k-1]}(v)\big)^* Q_{k-1}^{[\alpha,k-1]}(w) - \overline{b_{\alpha_{k-1}}(v)}\, b_{\alpha_{k-1}}(w) \big(O_{k-1}(v)\big)^* O_{k-1}(w)\Big),
\end{aligned}
$$

which is the first identity. □

Theorem 4.2. *For all $k \in \mathbb{N}_{0,\tau}$ and all $v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$, the following Christoffel-Darboux formulae hold:*

$$
\begin{aligned}
&\big(1 - \overline{b_{\alpha_k}(v)}\, b_{\alpha_k}(w)\big) \sum_{j=0}^{k} \big(O_j(v)\big)^* O_j(w) \\
&\quad = \big(Q_k^{[\alpha,k]}(v)\big)^* Q_k^{[\alpha,k]}(w) - \overline{b_{\alpha_k}(v)}\, b_{\alpha_k}(w) \big(O_k(v)\big)^* O_k(w) - \frac{1-|\alpha_k|^2}{(1-\alpha_k\overline{v})(1-\overline{\alpha_k}w)} \mathbf{I}_p,
\end{aligned}
$$

$$
\begin{aligned}
&\big(1 - b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}\,\big) \sum_{j=0}^{k} Q_j(v) \big(Q_j(w)\big)^* \\
&\quad = O_k^{[\alpha,k]}(v) \big(O_k^{[\alpha,k]}(w)\big)^* - b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}\, Q_k(v) \big(Q_k(w)\big)^* + \frac{1-|\alpha_k|^2}{(1-\overline{\alpha_k}v)(1-\alpha_k\overline{w})} \mathbf{I}_p,
\end{aligned}
$$

$$
\begin{aligned}
&\big(1 - \overline{b_{\alpha_k}(v)}\, b_{\alpha_k}(w)\big) \sum_{j=0}^{k} \big(P_j(v)\big)^* P_j(w) \\
&\quad = \big(R_k^{[\alpha,k]}(v)\big)^* R_k^{[\alpha,k]}(w) - \overline{b_{\alpha_k}(v)}\, b_{\alpha_k}(w) \big(P_k(v)\big)^* P_k(w) + \frac{1-|\alpha_k|^2}{(1-\alpha_k\overline{v})(1-\overline{\alpha_k}w)} \mathbf{I}_q,
\end{aligned}
$$

$$
\begin{aligned}
&\big(1 - b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}\,\big) \sum_{j=0}^{k} R_j(v) \big(R_j(w)\big)^* \\
&\quad = P_k^{[\alpha,k]}(v) \big(P_k^{[\alpha,k]}(w)\big)^* - b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}\, R_k(v) \big(R_k(w)\big)^* - \frac{1-|\alpha_k|^2}{(1-\overline{\alpha_k}v)(1-\alpha_k\overline{w})} \mathbf{I}_q,
\end{aligned}
$$

$$
\begin{aligned}
&\big(1 - \overline{b_{\alpha_k}(v)}\, b_{\alpha_k}(w)\big) \sum_{j=0}^{k} \big(O_j(v)\big)^* P_j(w) \\
&\quad = \big(Q_k^{[\alpha,k]}(v)\big)^* R_k^{[\alpha,k]}(w) - \overline{b_{\alpha_k}(v)}\, b_{\alpha_k}(w) \big(O_k(v)\big)^* P_k(w),
\end{aligned}
$$

$$\big(1 - b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}\,\big)\sum_{j=0}^{k} Q_j(v)\big(R_j(w)\big)^* \\ = O_k^{[\alpha,k]}(v)\big(P_k^{[\alpha,k]}(w)\big)^* - b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}Q_k(v)\big(R_k(w)\big)^*.$$

Proof. In the particular case $k = 0$, the formulae follow immediately from the initial conditions (3.1) and (3.2) by using some elementary properties of strictly contractive matrices (see, e.g., [12, Lemma 1.1.12]). Using Lemma 4.1, the proof follows by induction. As an example we prove the first Christoffel-Darboux formula. Let $v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,0}$. According to (3.1) and (2.9), we have

$$\begin{aligned}\big(O_0(v)\big)^* O_0(w) &= \Big(\frac{\sqrt{1-|\alpha_0|^2}}{1-\overline{\alpha_0}v}(\mathbf{I}_q - \mathbf{F}_0^*\mathbf{F}_0)^{-\frac{1}{2}}\mathbf{F}_0^*\Big)^*\Big(\frac{\sqrt{1-|\alpha_0|^2}}{1-\overline{\alpha_0}w}(\mathbf{I}_q - \mathbf{F}_0^*\mathbf{F}_0)^{-\frac{1}{2}}\mathbf{F}_0^*\Big) \\ &= \frac{1-|\alpha_0|^2}{(1-\alpha_0\overline{v})(1-\overline{\alpha_0}w)}\mathbf{F}_0(\mathbf{I}_q - \mathbf{F}_0^*\mathbf{F}_0)^{-1}\mathbf{F}_0^*\end{aligned}$$

and

$$\begin{aligned}&\big(Q_0^{[\alpha,0]}(v)\big)^* Q_0^{[\alpha,0]}(w) \\ &= \Big(\frac{-\eta_0\sqrt{1-|\alpha_0|^2}}{1-\overline{\alpha_0}v}(\mathbf{I}_p - \mathbf{F}_0\mathbf{F}_0^*)^{-\frac{1}{2}}\Big)^*\Big(\frac{-\eta_0\sqrt{1-|\alpha_0|^2}}{1-\overline{\alpha_0}w}(\mathbf{I}_p - \mathbf{F}_0\mathbf{F}_0^*)^{-\frac{1}{2}}\Big) \\ &= \frac{1-|\alpha_0|^2}{(1-\alpha_0\overline{v})(1-\overline{\alpha_0}w)}(\mathbf{I}_p - \mathbf{F}_0\mathbf{F}_0^*)^{-1}.\end{aligned}$$

Hence, because

$$\begin{aligned}\mathbf{F}_0(\mathbf{I}_q - \mathbf{F}_0^*\mathbf{F}_0)^{-1}\mathbf{F}_0^* &= (\mathbf{I}_p - \mathbf{F}_0\mathbf{F}_0^*)^{-1}\mathbf{F}_0\mathbf{F}_0^* \\ &= (\mathbf{I}_p - \mathbf{F}_0\mathbf{F}_0^*)^{-1}(\mathbf{F}_0\mathbf{F}_0^* - \mathbf{I}_p + \mathbf{I}_p) \\ &= -\mathbf{I}_p + (\mathbf{I}_p - \mathbf{F}_0\mathbf{F}_0^*)^{-1},\end{aligned}$$

we get

$$\begin{aligned}&\big(1 - \overline{b_{\alpha_0}(v)}b_{\alpha_0}(w)\big)\sum_{j=0}^{0}\big(O_j(v)\big)^* O_j(w) \\ &= \big(O_0(v)\big)^* O_0(w) - \overline{b_{\alpha_0}(v)}b_{\alpha_0}(w)\big(O_0(v)\big)^* O_0(w) \\ &= \frac{1-|\alpha_0|^2}{(1-\alpha_0\overline{v})(1-\overline{\alpha_0}w)}\mathbf{F}_0(\mathbf{I}_q - \mathbf{F}_0^*\mathbf{F}_0)^{-1}\mathbf{F}_0^* - \overline{b_{\alpha_0}(v)}b_{\alpha_0}(w)\big(O_0(v)\big)^* O_0(w) \\ &= \big(Q_0^{[\alpha,0]}(v)\big)^* Q_0^{[\alpha,0]}(w) - \overline{b_{\alpha_0}(v)}b_{\alpha_0}(w)\big(O_0(v)\big)^* O_0(w) - \frac{1-|\alpha_0|^2}{(1-\alpha_0\overline{v})(1-\overline{\alpha_0}w)}\mathbf{I}_p.\end{aligned}$$

Thus, for the case $k = 0$ the first identity is verified. Now we assume that, for all $k \in \mathbb{N}_{1,\tau}$ and all $v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k-1}$, the formula

$$\begin{aligned}&\big(1 - \overline{b_{\alpha_{k-1}}(v)}b_{\alpha_{k-1}}(w)\big)\sum_{j=0}^{k-1}\big(O_j(v)\big)^* O_j(w) \\ &= \big(Q_{k-1}^{[\alpha,k-1]}(v)\big)^* Q_{k-1}^{[\alpha,k-1]}(w) - \overline{b_{\alpha_{k-1}}(v)}b_{\alpha_{k-1}}(w)\big(O_{k-1}(v)\big)^* O_{k-1}(w) \\ &\quad - \frac{1-|\alpha_{k-1}|^2}{(1-\alpha_{k-1}\overline{v})(1-\overline{\alpha_{k-1}}w)}\mathbf{I}_p\end{aligned}$$

is already proved. Therefore, taking (4.1) into account, an application of the first equality in Lemma 4.1 implies

$$\begin{aligned}
&\big(1-\overline{b_{\alpha_k}(v)}b_{\alpha_k}(w)\big)\sum_{j=0}^{k}\big(O_j(v)\big)^*O_j(w)\\
&=\frac{\mathfrak{k}_{\alpha_k}(w,v)}{\mathfrak{k}_{\alpha_{k-1}}(w,v)}\big(1-\overline{b_{\alpha_{k-1}}(v)}b_{\alpha_{k-1}}(w)\big)\sum_{j=0}^{k-1}\big(O_j(v)\big)^*O_j(w)+\mathfrak{k}_{\alpha_k}(w,v)\big(O_k(v)\big)^*O_k(w)\\
&=\frac{\mathfrak{k}_{\alpha_k}(w,v)}{\mathfrak{k}_{\alpha_{k-1}}(w,v)}\Big(\big(Q_{k-1}^{[\alpha,k-1]}(v)\big)^*Q_{k-1}^{[\alpha,k-1]}(w)-\overline{b_{\alpha_{k-1}}(v)}b_{\alpha_{k-1}}(w)\big(O_{k-1}(v)\big)^*O_{k-1}(w)\Big)\\
&\quad-\frac{\mathfrak{k}_{\alpha_k}(w,v)}{\mathfrak{k}_{\alpha_{k-1}}(w,v)}\frac{1-|\alpha_{k-1}|^2}{(1-\alpha_{k-1}\overline{v})(1-\overline{\alpha_{k-1}}w)}\mathbf{I}_p+\mathfrak{k}_{\alpha_k}(w,v)\big(O_k(v)\big)^*O_k(w)\\
&=\big(Q_k^{[\alpha,k]}(v)\big)^*Q_k^{[\alpha,k]}(w)-\big(O_k(v)\big)^*O_k(w)-\frac{1-|\alpha_k|^2}{(1-\alpha_k\overline{v})(1-\overline{\alpha_k}w)}\mathbf{I}_p\\
&\quad+\big(O_k(v)\big)^*O_k(w)-\overline{b_{\alpha_k}(v)}b_{\alpha_k}(w)\big(O_k(v)\big)^*O_k(w)\\
&=\big(Q_k^{[\alpha,k]}(v)\big)^*Q_k^{[\alpha,k]}(w)-\overline{b_{\alpha_k}(v)}b_{\alpha_k}(w)\big(O_k(v)\big)^*O_k(w)-\frac{1-|\alpha_k|^2}{(1-\alpha_k\overline{v})(1-\overline{\alpha_k}w)}\mathbf{I}_p
\end{aligned}$$

firstly for $v,w\in\mathbb{C}\setminus\mathbb{P}_{\alpha,k}$ satisfying $\overline{v}w\neq 1$. Applying a continuity argument one can get that this identity is actually fulfilled for all $v,w\in\mathbb{C}\setminus\mathbb{P}_{\alpha,k}$. □

Obviously, the formulae in Theorem 4.2 can be restated as follows:

Corollary 4.3. *For all $k\in\mathbb{N}_{1,\tau}$ and all $v,w\in\mathbb{C}\setminus\mathbb{P}_{\alpha,k}$, the following Christoffel-Darboux formulae hold:*

$$\begin{aligned}
&\big(1-\overline{b_{\alpha_k}(v)}b_{\alpha_k}(w)\big)\sum_{j=0}^{k-1}\big(O_j(v)\big)^*O_j(w)\\
&\quad=\big(Q_k^{[\alpha,k]}(v)\big)^*Q_k^{[\alpha,k]}(w)-\big(O_k(v)\big)^*O_k(w)-\frac{1-|\alpha_k|^2}{(1-\alpha_k\overline{v})(1-\overline{\alpha_k}w)}\mathbf{I}_p,\\
&\big(1-b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}\,\big)\sum_{j=0}^{k-1}Q_j(v)\big(Q_j(w)\big)^*\\
&\quad=O_k^{[\alpha,k]}(v)\big(O_k^{[\alpha,k]}(w)\big)^*-Q_k(v)\big(Q_k(w)\big)^*+\frac{1-|\alpha_k|^2}{(1-\overline{\alpha_k}v)(1-\alpha_k\overline{w})}\mathbf{I}_p,\\
&\big(1-\overline{b_{\alpha_k}(v)}b_{\alpha_k}(w)\big)\sum_{j=0}^{k-1}\big(P_j(v)\big)^*P_j(w)\\
&\quad=\big(R_k^{[\alpha,k]}(v)\big)^*R_k^{[\alpha,k]}(w)-\big(P_k(v)\big)^*P_k(w)+\frac{1-|\alpha_k|^2}{(1-\alpha_k\overline{v})(1-\overline{\alpha_k}w)}\mathbf{I}_q,\\
&\big(1-b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}\,\big)\sum_{j=0}^{k-1}R_j(v)\big(R_j(w)\big)^*\\
&\quad=P_k^{[\alpha,k]}(v)\big(P_k^{[\alpha,k]}(w)\big)^*-R_k(v)\big(R_k(w)\big)^*-\frac{1-|\alpha_k|^2}{(1-\overline{\alpha_k}v)(1-\alpha_k\overline{w})}\mathbf{I}_q,\\
&\big(1-\overline{b_{\alpha_k}(v)}b_{\alpha_k}(w)\big)\sum_{j=0}^{k-1}\big(O_j(v)\big)^*P_j(w)=\big(Q_k^{[\alpha,k]}(v)\big)^*R_k^{[\alpha,k]}(w)-\big(O_k(v)\big)^*P_k(w),
\end{aligned}$$

$$\big(1 - b_{\alpha_k}(v)\overline{b_{\alpha_k}(w)}\,\big)\sum_{j=0}^{k-1} Q_j(v)\big(R_j(w)\big)^* = O_k^{[\alpha,k]}(v)\big(P_k^{[\alpha,k]}(w)\big)^* - Q_k(v)\big(R_k(w)\big)^*.$$

Using the same argumentation as for orthogonal rational matrix-valued functions (cf. [19, Section 7]), we may conclude from the (first and the fourth) Christoffel-Darboux formulae stated in Corollary 4.3 with $v = w$ that the following statement holds (cf. Corollary 3.3).

Corollary 4.4. *Let $k \in \mathbb{N}_{1,\tau}$. For each $v \in \mathbb{D} \cup \mathbb{T}$, the matrices $Q_k^{[\alpha,k]}(v)$ and $P_k^{[\alpha,k]}(v)$ are non-singular and the complex $q \times p$ matrices*

$$O_k(v)\big(Q_k^{[\alpha,k]}(v)\big)^{-1} \quad \text{and} \quad \big(P_k^{[\alpha,k]}(v)\big)^{-1}R_k(v)$$

are strictly contractive.

Remark 4.5. Let $k \in \mathbb{N}_{1,\tau}$. Since $X^{[\alpha,k]}(\alpha_k) = 0 \iff X \in \breve{\mathcal{R}}_{\alpha,k-1}^{p\times q}$ if $X \in \breve{\mathcal{R}}_{\alpha,k}^{p\times q}$, it follows from Corollary 4.4 that $Q_k \in \breve{\mathcal{R}}_{\alpha,k}^{p\times p} \setminus \breve{\mathcal{R}}_{\alpha,k-1}^{p\times p}$ and $P_k \in \breve{\mathcal{R}}_{\alpha,k}^{q\times q} \setminus \breve{\mathcal{R}}_{\alpha,k-1}^{q\times q}$. Furthermore, $O_k \in \breve{\mathcal{R}}_{\alpha,k-1}^{q\times p}$ is possible (cf. Remark 3.9) and Theorem 4.2 implies

$$\begin{aligned}
O_k \in \breve{\mathcal{R}}_{\alpha,k-1}^{q\times p} \quad &\iff \quad \sum_{j=0}^{k} Q_j(v)\big(Q_j(\alpha_k)\big)^* = \frac{1}{1-\overline{\alpha_k}v}\mathbf{I}_p, \quad v \in \mathbb{C}\setminus\mathbb{P}_{\alpha,k},\\
&\iff \quad \sum_{j=0}^{k} Q_j(\alpha_k)\big(Q_j(\alpha_k)\big)^* = \frac{1}{1-|\alpha_k|^2}\mathbf{I}_p\\
&\iff \quad \sum_{j=0}^{k} R_j(v)\big(Q_j(\alpha_k)\big)^* = 0, \quad v \in \mathbb{C}\setminus\mathbb{P}_{\alpha,k},\\
&\iff \quad \sum_{j=0}^{k} R_j(\alpha_k)\big(Q_j(\alpha_k)\big)^* = 0.
\end{aligned}$$

Similarly, the case $R_k \in \breve{\mathcal{R}}_{\alpha,k-1}^{q\times p}$ is possible and Theorem 4.2 shows

$$\begin{aligned}
R_k \in \breve{\mathcal{R}}_{\alpha,k-1}^{q\times p} \quad &\iff \quad \sum_{j=0}^{k} \big(P_j(\alpha_k)\big)^* P_j(v) = \frac{1}{1-\overline{\alpha_k}v}\mathbf{I}_q, \quad v \in \mathbb{C}\setminus\mathbb{P}_{\alpha,k},\\
&\iff \quad \sum_{j=0}^{k} \big(P_j(\alpha_k)\big)^* P_j(\alpha_k) = \frac{1}{1-|\alpha_k|^2}\mathbf{I}_q\\
&\iff \quad \sum_{j=0}^{k} \big(P_j(\alpha_k)\big)^* O_j(v) = 0, \quad v \in \mathbb{C}\setminus\mathbb{P}_{\alpha,k},\\
&\iff \quad \sum_{j=0}^{k} \big(P_j(\alpha_k)\big)^* O_j(\alpha_k) = 0.
\end{aligned}$$

Remark 4.6. Let $k \in \mathbb{N}_{1,\tau}$. Based on the fifth or the sixth Christoffel-Darboux formulae stated in Corollary 4.3 (cf. the reasoning demonstrated in [19, Lemma 6.5]), for each $v \in \mathbb{C}\setminus\mathbb{P}_{\alpha,k}$, one can derive that

$$P_k^{[\alpha,k]}(v)O_k(v) = R_k(v)Q_k^{[\alpha,k]}(v) \quad \text{and} \quad O_k^{[\alpha,k]}(v)P_k(v) = Q_k(v)R_k^{[\alpha,k]}(v).$$

5. A characterization of SNP sequences

In the previous section (see, e.g., Theorem 4.2), we have explained that SNP sequences of rational matrix functions fulfill some Christoffel-Darboux formulae. In the present section we study now an inverse question. Roughly speaking, we shall see that (cf. [22, Section 5] for the case of certain orthogonal rational matrix-valued functions) the realization of Christoffel-Darboux formulae is in a way also a sufficient condition for systems of rational matrix functions to be SNP sequences. First, we give some information on the connection between the different types of Christoffel-Darboux formulae.

Remark 5.1. Let $\tau \in \mathbb{N}$ or $\tau = \infty$ and let $(\alpha_k)_{k=0}^{\tau}$ be a sequence of points belonging to $\mathbb{D}$. Furthermore, let $k \in \mathbb{N}_{1,\tau}$ and let $O_j \in \breve{\mathcal{R}}_{\alpha,j}^{q\times p}$, $Q_j \in \breve{\mathcal{R}}_{\alpha,j}^{p\times p}$, $P_j \in \breve{\mathcal{R}}_{\alpha,j}^{q\times q}$, and $R_j \in \breve{\mathcal{R}}_{\alpha,j}^{q\times p}$ for each $j \in \mathbb{N}_{0,k}$. Clearly, the following statements are equivalent:

(i) The first (resp., second, third, fourth, fifth, or sixth) Christoffel-Darboux formula in Theorem 4.2 is satisfied for all $v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$.
(ii) The first (resp., second, third, fourth, fifth, or sixth) Christoffel-Darboux formula in Corollary 4.3 is satisfied for all $v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$.

Lemma 5.2. *Let $\tau \in \mathbb{N}$ or $\tau = \infty$ and let $(\alpha_k)_{k=0}^{\tau}$ be a sequence of points belonging to $\mathbb{D}$. Furthermore, let $k \in \mathbb{N}_{1,\tau}$ and let $O_j \in \breve{\mathcal{R}}_{\alpha,j}^{q\times p}$, $Q_j \in \breve{\mathcal{R}}_{\alpha,j}^{p\times p}$, $P_j \in \breve{\mathcal{R}}_{\alpha,j}^{q\times q}$, and $R_j \in \breve{\mathcal{R}}_{\alpha,j}^{q\times p}$ for $j \in \{k-1, k\}$. The following statements are equivalent:*

(i) *The first (resp., third, or fifth) identity of* Lemma 4.1 *is satisfied for all $v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$.*
(ii) *The second (resp., fourth, or sixth) identity of* Lemma 4.1 *is satisfied for all $v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$.*

Proof. Taking (2.9) into account, the assertion follows by a straightforward calculation. As an example, we demonstrate this fact on the basis of the first and the second identities in Lemma 4.1. If we fix the complex number $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$ then, in view of (4.1), (2.9), and forming the adjoint with respect to the other variable w and the underlying $k+2$ points $\alpha_0, \alpha_1, \ldots, \alpha_k, \alpha_{k-1}$, one can see that the first identity of Lemma 4.1 is equivalent to the equality

$$\begin{aligned}&\big(b_{\alpha_{k-1}}(w)-b_{\alpha_{k-1}}(v)\big)\Big(Q_k(w)Q_k^{[\alpha,k]}(v)-O_k^{[\alpha,k]}(w)O_k(v)\Big)\\&\quad=\big(b_{\alpha_k}(w)-b_{\alpha_k}(v)\big)\Big(b_{\alpha_{k-1}}(w)Q_{k-1}(w)Q_{k-1}^{[\alpha,k-1]}(v)-b_{\alpha_{k-1}}(v)O_{k-1}^{[\alpha,k-1]}(w)O_{k-1}(v)\Big)\end{aligned}$$

for all $v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$. Since, by fixing now the complex number $w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$ and forming the adjoint, this relation is equivalent to

$$\begin{aligned}&-\mathfrak{k}_{\alpha_{k-1}}(v,w)\Big(Q_k(v)\big(Q_k(w)\big)^*-O_k^{[\alpha,k]}(v)\big(O_k^{[\alpha,k]}(w)\big)^*\Big)\\&\quad=-\mathfrak{k}_{\alpha_k}(v,w)\Big(b_{\alpha_{k-1}}(v)\overline{b_{\alpha_{k-1}}(w)}Q_{k-1}(v)\big(Q_{k-1}(w)\big)^*-O_{k-1}^{[\alpha,k-1]}(v)\big(O_{k-1}^{[\alpha,k-1]}(w)\big)^*\Big)\end{aligned}$$

for all $v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$. Therefore, we obtain eventually the equivalence of the first and the second identity of Lemma 4.1. □

Lemma 5.3. *Let $\tau \in \mathbb{N}$ or $\tau = \infty$ and let $(\alpha_k)_{k=0}^{\tau}$ be a sequence of points belonging to $\mathbb{D}$. Furthermore, let $(O_k)_{k=0}^{\tau}$, $(Q_k)_{k=0}^{\tau}$, $(P_k)_{k=0}^{\tau}$, and $(R_k)_{k=0}^{\tau}$ be sequences of rational matrix-valued functions such that O_0, Q_0, P_0, and R_0 are defined as in*

(3.1) *and* (3.2) *and that* $O_k \in \breve{\mathcal{R}}_{\alpha,k}^{q\times p}$, $Q_k \in \breve{\mathcal{R}}_{\alpha,k}^{p\times p}$, $P_k \in \breve{\mathcal{R}}_{\alpha,k}^{q\times q}$, *and* $R_k \in \breve{\mathcal{R}}_{\alpha,k}^{q\times p}$ *for each* $k \in \mathbb{N}_{1,\tau}$. *The following statements are equivalent:*

(i) *For all* $k \in \mathbb{N}_{1,\tau}$ *and all* $v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$, *the first (resp., second, third, fourth, fifth, or sixth) identity of* Lemma 4.1 *is fulfilled.*

(ii) *For all* $k \in \mathbb{N}_{1,\tau}$ *and all* $v, w \in \mathbb{C}\setminus\mathbb{P}_{\alpha,k}$, *the first (resp., second, third, fourth, fifth, or sixth) Christoffel-Darboux formula in* Theorem 4.2 *is fulfilled.*

Proof. By using the same arguments as in the proof of Theorem 4.2, one can inductively show that (i) implies (ii). It remains to verify that (ii) implicates also (i). Exemplarily, we show this with respect to the first identity of Lemma 4.1 and the first Christoffel-Darboux formula in Theorem 4.2. As already explained in the proof of Theorem 4.2 in detail, the initial condition (3.1) leads to the validity of the first Christoffel-Darboux formula in Theorem 4.2 for $k = 0$ and all $v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,0}$. Consequently, for all $k \in \mathbb{N}_{1,\tau}$ and all $v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$, from (4.1), Remark 5.1, the first formula in Corollary 4.3, and the first formula in Theorem 4.2 it follows

$$
\begin{aligned}
&\mathfrak{k}_{\alpha_{k-1}}(w,v)\Big(\big(Q_k^{[\alpha,k]}(v)\big)^* Q_k^{[\alpha,k]}(w) - \big(O_k(v)\big)^* O_k(w)\Big)\\
&= \mathfrak{k}_{\alpha_{k-1}}(w,v)\Big(\big(Q_k^{[\alpha,k]}(v)\big)^* Q_k^{[\alpha,k]}(w) - \big(O_k(v)\big)^* O_k(w) - \frac{1-|\alpha_k|^2}{(1-\alpha_k\overline{v})(1-\overline{\alpha_k}w)}\mathbf{I}_p\Big)\\
&\quad + \frac{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)(1-\overline{v}w)}{(1-\alpha_k\overline{v})(1-\overline{\alpha_k}w)(1-\alpha_{k-1}\overline{v})(1-\overline{\alpha_{k-1}}w)}\mathbf{I}_p\\
&= \mathfrak{k}_{\alpha_{k-1}}(w,v)\mathfrak{k}_{\alpha_k}(w,v)\sum_{j=0}^{k-1}\big(O_j(v)\big)^* O_j(w) + \mathfrak{k}_{\alpha_k}(w,v)\frac{1-|\alpha_{k-1}|^2}{(1-\alpha_{k-1}\overline{v})(1-\overline{\alpha_{k-1}}w)}\mathbf{I}_p\\
&= \mathfrak{k}_{\alpha_k}(w,v)\Big(\big(Q_{k-1}^{[\alpha,k-1]}(v)\big)^* Q_{k-1}^{[\alpha,k-1]}(w) - \overline{b_{\alpha_{k-1}}(v)}b_{\alpha_{k-1}}(w)\big(O_{k-1}(v)\big)^* O_{k-1}(w)\Big).
\end{aligned}
$$

Thus, regarding the first kind of identities it is shown that also (ii) yields (i). □

Theorem 5.4. *Let* $\tau \in \mathbb{N}_0$ *or* $\tau = \infty$, *let* $(\alpha_k)_{k=0}^{\tau}$ *be a sequence of points belonging to* $\mathbb{D}$, *and let* $(O_k)_{k=0}^{\tau}$, $(Q_k)_{k=0}^{\tau}$, $(P_k)_{k=0}^{\tau}$, *and* $(R_k)_{k=0}^{\tau}$ *be sequences of rational matrix-valued functions such that the following five conditions are satisfied:*

(I) $O_k \in \breve{\mathcal{R}}_{\alpha,k}^{q\times p}$, $Q_k \in \breve{\mathcal{R}}_{\alpha,k}^{p\times p}$, $P_k \in \breve{\mathcal{R}}_{\alpha,k}^{q\times q}$, *and* $R_k \in \breve{\mathcal{R}}_{\alpha,k}^{q\times p}$ *for each* $k \in \mathbb{N}_{0,\tau}$.

(II) *If* $k \in \mathbb{N}_{0,\tau}$ *then the first or the second Christoffel-Darboux formula in* Theorem 4.2 *is fulfilled for all* $v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$.

(III) *If* $k \in \mathbb{N}_{0,\tau}$ *then the third or the fourth Christoffel-Darboux formula in* Theorem 4.2 *is fulfilled for all* $v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$.

(IV) *If* $k \in \mathbb{N}_{0,\tau}$ *then the fifth or the sixth Christoffel-Darboux formula in* Theorem 4.2 *is fulfilled for all* $v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$.

(V) $Q_0(\alpha_0) \geq 0$ *and* $P_0(\alpha_0) \geq 0$ *as well as the relations*

$$\frac{\eta_k\overline{\eta_{k-1}}}{1-\overline{\alpha_k}\alpha_{k-1}}\big(Q_k^{[\alpha,k]}(\alpha_{k-1})\big)^* Q_{k-1}^{[\alpha,k-1]}(\alpha_{k-1}) \geq 0,$$

$$\frac{\eta_k\overline{\eta_{k-1}}}{1-\overline{\alpha_k}\alpha_{k-1}} P_{k-1}^{[\alpha,k-1]}(\alpha_{k-1})\big(P_k^{[\alpha,k]}(\alpha_{k-1})\big)^* \geq 0$$

are fulfilled for each $k \in \mathbb{N}_{1,\tau}$.

Then, for each $k \in \mathbb{N}_{0,\tau}$ and each $v \in \mathbb{D} \cup \mathbb{T}$, the matrices $Q_k^{[\alpha,k]}(v)$ and $P_k^{[\alpha,k]}(v)$ are non-singular and if we put

$$\mathbf{F}_k := \overline{\eta_k}\eta_{k-1}\Big(O_k(\alpha_{k-1})\big(Q_k^{[\alpha,k]}(\alpha_{k-1})\big)^{-1}\Big)^*, \quad k \in \mathbb{N}_{0,\tau},$$

where $\alpha_{-1} := 0$ and $\eta_{-1} := -1$, then $[(O_k)_{k=0}^{\tau}, (Q_k)_{k=0}^{\tau}]$ is the first SNP pair of rational matrix functions corresponding to $(\alpha_k, \mathbf{F}_k)_{k=0}^{\tau}$ and $[(P_k)_{k=0}^{\tau}, (R_k)_{k=0}^{\tau}]$ is the second SNP pair of rational matrix functions corresponding to $(\alpha_k, \mathbf{F}_k)_{k=0}^{\tau}$.

Proof. First, we remark that in view of the conditions (I), (II), (III), and (IV) an application of Lemma 5.3 and Lemma 5.2 yields that all Christoffel-Darboux formula stated in Theorem 4.2 are satisfied for all $k \in \mathbb{N}_{0,\tau}$ and all $v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$. We consider now the case $k = 0$. From the first and the fifth Christoffel-Darboux formulae stated in Theorem 4.2 it follows

$$\big(O_0(v)\big)^* O_0(w) = \big(Q_0^{[\alpha,0]}(v)\big)^* Q_0^{[\alpha,0]}(w) - \frac{1-|\alpha_0|^2}{(1-\alpha_0\overline{v})(1-\overline{\alpha_0}w)}\mathbf{I}_p \tag{5.1}$$

and

$$\big(P_0(w)\big)^* O_0(v) = \big(R_0^{[\alpha,0]}(w)\big)^* Q_0^{[\alpha,0]}(v) \tag{5.2}$$

for all $v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,0}$. In particular, we can infer from (5.1) that the matrix $Q_0^{[\alpha,0]}(v)$ is non-singular for each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,0}$ and that by setting

$$\mathbf{F}_0 := -\overline{\eta_0}\Big(O_0(0)\big(Q_0^{[\alpha,0]}(0)\big)^{-1}\Big)^* \tag{5.3}$$

a strictly contractive $p \times q$ matrix is defined. Similarly, using the fourth Christoffel-Darboux formula stated in Theorem 4.2 one can see that the matrix $P_0^{[\alpha,0]}(v)$ is non-singular for each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,0}$. Because of (I) there is a $\mathbf{Q} \in \mathbb{C}^{p \times p}$ such that

$$Q_0(v) = \frac{\sqrt{1-|\alpha_0|^2}}{1-\overline{\alpha_0}v}\mathbf{Q} \tag{5.4}$$

for each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,0}$. Since (5.1) and (5.3) leads particularly to

$$\mathbf{I}_p - \mathbf{F}_0\mathbf{F}_0^* = (1-|\alpha_0|^2)\Big(Q_0^{[\alpha,0]}(0)\big(Q_0^{[\alpha,0]}(0)\big)^*\Big)^{-1},$$

in view of (5.4), (2.9), and $Q_0(\alpha_0) \geq 0$ we get $\mathbf{Q} = (\mathbf{I}_p - \mathbf{F}_0\mathbf{F}_0^*)^{-\frac{1}{2}}$, i.e.

$$Q_0(v) = \frac{\sqrt{1-|\alpha_0|^2}}{1-\overline{\alpha_0}v}(\mathbf{I}_p - \mathbf{F}_0\mathbf{F}_0^*)^{-\frac{1}{2}} \tag{5.5}$$

for each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,0}$. Moreover, (5.2) and (2.9) result in

$$P_0^{[\alpha,0]}(w)O_0(v) = R_0(w)Q_0^{[\alpha,0]}(v) \tag{5.6}$$

for all $v, w \in \mathbb{C} \setminus \mathbb{P}_{\alpha,0}$. Consequently, by virtue of (5.3) we have

$$\mathbf{F}_0 = -\overline{\eta_0}\Big(\big(P_0^{[\alpha,0]}(0)\big)^{-1}R_0(0)\Big)^*$$

which implies in combination with (5.6), (5.5), (2.9), and some elementary properties of strictly contractive matrices (see, e.g., [12, Lemma 1.1.12]) then

$$
\begin{aligned}
O_0(v) &= \big(P_0^{[\alpha,0]}(0)\big)^{-1} R_0(0) Q_0^{[\alpha,0]}(v) \\
&= \big(-\overline{\eta_0}\,\mathbf{F}_0^*\big)\Big(\frac{-\eta_0\sqrt{1-|\alpha_0|^2}}{1-\overline{\alpha_0}v}(\mathbf{I}_p-\mathbf{F}_0\mathbf{F}_0^*)^{-\frac{1}{2}}\Big) \qquad (5.7)\\
&= \frac{\sqrt{1-|\alpha_0|^2}}{1-\overline{\alpha_0}v}(\mathbf{I}_q-\mathbf{F}_0^*\mathbf{F}_0)^{-\frac{1}{2}}\mathbf{F}_0^*
\end{aligned}
$$

for each $v \in \mathbb{C}\setminus\mathbb{P}_{\alpha,0}$. Thus, the initial condition (3.1) is proved. Particularly for the case $\tau = 0$ it is shown that $[(O_k)_{k=0}^{\tau}, (Q_k)_{k=0}^{\tau}]$ is the first SNP pair of rational matrix functions corresponding to $(\alpha_k, \mathbf{F}_k)_{k=0}^{\tau}$. Now let $\tau \in \mathbb{N}$ or $\tau = \infty$ and let $k \in \mathbb{N}_{1,\tau}$. Since the assumptions include that the first and the fourth Christoffel-Darboux formula stated in Theorem 4.2 are satisfied, we get that the matrices $Q_k^{[\alpha,k]}(v)$ and $P_k^{[\alpha,k]}(v)$ are non-singular for each $v \in \mathbb{D}\cup\mathbb{T}$ and that by setting

$$
\mathbf{F}_k := \overline{\eta_k}\eta_{k-1}\Big(O_k(\alpha_{k-1})\big(Q_k^{[\alpha,k]}(\alpha_{k-1})\big)^{-1}\Big)^* \qquad (5.8)
$$

a strictly contractive $p \times q$ matrix is defined (note Remark 5.1 and Corollary 4.4). Obviously, from (V) it follows

$$
\frac{\eta_k\overline{\eta_{k-1}}}{1-\overline{\alpha_k}\alpha_{k-1}} Q_{k-1}^{[\alpha,k-1]}(\alpha_{k-1})\big(Q_k^{[\alpha,k]}(\alpha_{k-1})\big)^{-1} \geq 0. \qquad (5.9)
$$

Furthermore, in view of $b_{\alpha_{k-1}}(\alpha_{k-1}) = 0$, Lemma 5.3, and the first identity of Lemma 4.1 we obtain

$$
\begin{aligned}
&\big(Q_k^{[\alpha,k]}(\alpha_{k-1})\big)^* Q_k^{[\alpha,k]}(v) - \big(O_k(\alpha_{k-1})\big)^* O_k(v) \\
&= \big(1-\overline{b_{\alpha_{k-1}}(\alpha_{k-1})}b_{\alpha_{k-1}}(v)\big)\Big(\big(Q_k^{[\alpha,k]}(\alpha_{k-1})\big)^* Q_k^{[\alpha,k]}(v) - \big(O_k(\alpha_{k-1})\big)^* O_k(v)\Big) \qquad (5.10)\\
&= \big(1-\overline{b_{\alpha_k}(\alpha_{k-1})}b_{\alpha_k}(v)\big)\big(Q_{k-1}^{[\alpha,k-1]}(\alpha_{k-1})\big)^* Q_{k-1}^{[\alpha,k-1]}(v)
\end{aligned}
$$

for each $v \in \mathbb{C}\setminus\mathbb{P}_{\alpha,k}$. In particular, (5.10), (5.8), (4.1), and (5.9) lead to

$$
\begin{aligned}
&(\mathbf{I}_p-\mathbf{F}_k\mathbf{F}_k^*)^{\frac{1}{2}} \\
&= \bigg(\mathbf{I}_p - \Big(\big(Q_k^{[\alpha,k]}(\alpha_{k-1})\big)^*\Big)^{-1}\big(O_k(\alpha_{k-1})\big)^* O_k(\alpha_{k-1})\big(Q_k^{[\alpha,k]}(\alpha_{k-1})\big)^{-1}\bigg)^{\frac{1}{2}} \\
&= \bigg(\frac{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)}{|1-\overline{\alpha_k}\alpha_{k-1}|^2}\mathbf{Q}_k^*\mathbf{Q}_k\bigg)^{\frac{1}{2}} \qquad (5.11)\\
&= \eta_k\overline{\eta_{k-1}}\frac{\sqrt{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)}}{1-\overline{\alpha_k}\alpha_{k-1}} Q_{k-1}^{[\alpha,k-1]}(\alpha_{k-1})\big(Q_k^{[\alpha,k]}(\alpha_{k-1})\big)^{-1},
\end{aligned}
$$

where $\mathbf{Q}_k := Q_{k-1}^{[\alpha,k-1]}(\alpha_{k-1})\big(Q_k^{[\alpha,k]}(\alpha_{k-1})\big)^{-1}$ for technical reasons. With a view to (2.9), the relation (5.10) implies

$$
Q_k(v)Q_k^{[\alpha,k]}(\alpha_{k-1}) - O_k^{[\alpha,k]}(v)O_k(\alpha_{k-1}) = \big(b_{\alpha_k}(v)-b_{\alpha_k}(\alpha_{k-1})\big)Q_{k-1}(v)Q_{k-1}^{[\alpha,k-1]}(\alpha_{k-1})
$$

for each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$. Consequently, an application of (5.8) and (5.11) yields

$$\begin{aligned}
&\eta_k\overline{\eta_{k-1}}Q_k(v) - O_k^{[\alpha,k]}(v)\mathbf{F}_k^* \\
&= \eta_k\overline{\eta_{k-1}}Q_k(v) - \eta_k\overline{\eta_{k-1}}O_k^{[\alpha,k]}(v)O_k(\alpha_{k-1})\big(Q_k^{[\alpha,k]}(\alpha_{k-1})\big)^{-1} \\
&= \eta_k\overline{\eta_{k-1}}\big(b_{\alpha_k}(v)-b_{\alpha_k}(\alpha_{k-1})\big)Q_{k-1}(v)Q_{k-1}^{[\alpha,k-1]}(\alpha_{k-1})\big(Q_k^{[\alpha,k]}(\alpha_{k-1})\big)^{-1} \\
&= \frac{1-\overline{\alpha_k}\alpha_{k-1}}{\sqrt{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)}}\big(b_{\alpha_k}(v) - b_{\alpha_k}(\alpha_{k-1})\big)Q_{k-1}(v)(\mathbf{I}_p - \mathbf{F}_k\mathbf{F}_k^*)^{\frac{1}{2}}
\end{aligned} \tag{5.12}$$

for each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$. In addition, by the sixth Christoffel-Darboux formula stated in Theorem 4.2, Lemma 5.3, and $b_{\alpha_{k-1}}(\alpha_{k-1}) = 0$ it follows that

$$\begin{aligned}
&O_k^{[\alpha,k]}(v)\big(P_k^{[\alpha,k]}(\alpha_{k-1})\big)^* - Q_k(v)\big(R_k(\alpha_{k-1})\big)^* \\
&\quad = \big(1 - b_{\alpha_k}(v)\overline{b_{\alpha_k}(\alpha_{k-1})}\,\big)O_{k-1}^{[\alpha,k-1]}(v)\big(P_{k-1}^{[\alpha,k-1]}(\alpha_{k-1})\big)^*
\end{aligned}$$

for each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$. By forming the adjoint rational matrix functions, we get

$$P_k^{[\alpha,k]}(\alpha_{k-1})O_k(v) - R_k(\alpha_{k-1})Q_k^{[\alpha,k]}(v) = \big(b_{\alpha_k}(v) - b_{\alpha_k}(\alpha_{k-1})\big)P_{k-1}^{[\alpha,k-1]}(\alpha_{k-1})O_{k-1}(v)$$

for each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$. In particular, we see

$$P_k^{[\alpha,k]}(\alpha_{k-1})O_k(\alpha_{k-1}) = R_k(\alpha_{k-1})Q_k^{[\alpha,k]}(\alpha_{k-1})$$

so that (5.8) results in

$$\mathbf{F}_k = \overline{\eta_k}\eta_{k-1}\Big(\big(P_k^{[\alpha,k]}(\alpha_{k-1})\big)^{-1}R_k(\alpha_{k-1})\Big)^*$$

and similar to (5.11) one can conclude that

$$(\mathbf{I}_q - \mathbf{F}_k^*\mathbf{F}_k)^{\frac{1}{2}} = \eta_k\overline{\eta_{k-1}}\frac{\sqrt{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)}}{1-\overline{\alpha_k}\alpha_{k-1}}\big(P_k^{[\alpha,k]}(\alpha_{k-1})\big)^{-1}P_{k-1}^{[\alpha,k-1]}(\alpha_{k-1}).$$

To summarize the considerations after (5.12), in analogy to (5.12) we obtain

$$\begin{aligned}
&\eta_k\overline{\eta_{k-1}}O_k(v) - \mathbf{F}_k^*Q_k^{[\alpha,k]}(v) \\
&= \frac{1-\overline{\alpha_k}\alpha_{k-1}}{\sqrt{(1-|\alpha_k|^2)(1-|\alpha_{k-1}|^2)}}\big(b_{\alpha_k}(v) - b_{\alpha_k}(\alpha_{k-1})\big)(\mathbf{I}_q - \mathbf{F}_k^*\mathbf{F}_k)^{\frac{1}{2}}O_{k-1}(v)
\end{aligned} \tag{5.13}$$

for each $v \in \mathbb{C} \setminus \mathbb{P}_{\alpha,k}$. In the end, by virtue of (I), (5.5), (5.7), (5.12), (5.13), and Proposition 3.1 one can conclude that $[(O_k)_{k=0}^{\tau}, (Q_k)_{k=0}^{\tau}]$ is the first SNP pair of rational matrix functions corresponding to $(\alpha_k, \mathbf{F}_k)_{k=0}^{\tau}$. A similar argumentation can be used for the second SNP pair. □

6. Solution of Problem (MNP) in the non-degenerate case

In this section, we shall show that the solution set $\mathcal{S}_\Delta$ of Problem (MNP) can be parameterized by the linear fractional matrix transformations (1.3) and (1.4), where O_n, Q_n, P_n, and R_n are elements of some SNP pairs of rational matrix functions with n defined as in (1.2), if Δ (as in (2.1)) is given such that $\mathbf{P}_\Delta > 0$.

With the interpolation points $\beta_0, \beta_1, \ldots, \beta_m$ in Δ we form here a sequence $(\alpha_k)_{k=0}^n$ in which β_j appears according to its multiplicity $l_j + 1$ times for each $j \in \mathbb{N}_{0,m}$. For instance, we can choose $\alpha_k := \gamma_k$ with

$$\gamma_k := \beta_j \quad \text{if} \quad j + \sum_{r=0}^{j-1} l_r \le k \le \sum_{r=0}^{j} l_r, \quad j \in \mathbb{N}_{0,m}.$$

However, in the following, it is not essential that equal points are successors, i.e. for an arbitrary bijective mapping $\wp$ of $\{0, 1, \ldots, n\}$ onto itself we can put

$$\alpha_k := \gamma_{\wp(k)}, \quad k \in \mathbb{N}_{0,n}. \tag{6.1}$$

In the sequel, $[(O_k)_{k=0}^n, (Q_k)_{k=0}^n]$ (resp., $[(P_k)_{k=0}^n, (R_k)_{k=0}^n]$) stands for the first (resp., second) SNP pair of rational matrix functions corresponding to $(\alpha_k, \mathbf{F}_k)_{k=0}^n$, where $(\mathbf{F}_k)_{k=0}^n$ is a certain sequence of strictly contractive $p \times q$ matrices.

Remark 6.1. In view of Corollary 3.3 and some elementary properties of strictly contractive matrices (see, e.g., [12, Remark 1.1.2 and part (a) of Lemma 1.1.13]) one can conclude that, for every function $S \in \mathcal{S}_{p\times q}(\mathbb{D})$ and every point $w \in \mathbb{D}$, the matrices $P_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)R_n(w)S(w)$ and $Q_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)S(w)O_n(w)$ are both non-singular. Moreover, Corollary 3.3 implies in combination with the second identity in Corollary 3.7 that by setting

$$V_\circ(w) := O_n^{[\alpha,n]}(w)\big(P_n^{[\alpha,n]}(w)\big)^{-1}, \quad w \in \mathbb{D}, \tag{6.2}$$

a function belonging to $\mathcal{S}_{p\times q}(\mathbb{D})$ is well-defined, where

$$V_\circ(w) = \big(Q_n^{[\alpha,n]}(w)\big)^{-1} R_n^{[\alpha,n]}(w), \quad w \in \mathbb{D}.$$

Lemma 6.2. *Let $S \in \mathcal{S}_{p\times q}(\mathbb{D})$. Then the matrix-valued function*

$$V(w) := \Big(O_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)Q_n(w)S(w)\Big)\Big(P_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)R_n(w)S(w)\Big)^{-1}, \quad w \in \mathbb{D},$$

belongs to $\mathcal{S}_{p\times q}(\mathbb{D})$, it admits the representation

$$V(w) = \Big(Q_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)S(w)O_n(w)\Big)^{-1}\Big(R_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)S(w)P_n(w)\Big), \quad w \in \mathbb{D},$$

and $\Delta^{[V]} = \Delta^{[V_\circ]}$, where $V_\circ$ is the matrix function given by (6.2). Moreover, the matrices $-P_n(w) + O_n(w)V(w)$ and $-Q_n(w) + V(w)R_n(w)$ are non-singular for each $w \in \mathbb{D} \setminus \mathbb{Z}_{\alpha,n}$, whereby the function S can be recovered via

$$S(w) = \frac{1}{b_{\alpha_n}(w)}\Big(R_n^{[\alpha,n]}(w) - Q_n^{[\alpha,n]}(w)V(w)\Big)\Big(-P_n(w) + O_n(w)V(w)\Big)^{-1}, \tag{6.3}$$

$$S(w) = \frac{1}{b_{\alpha_n}(w)}\Big(-Q_n(w) + V(w)R_n(w)\Big)^{-1}\Big(O_n^{[\alpha,n]}(w) - V(w)P_n^{[\alpha,n]}(w)\Big). \tag{6.4}$$

Proof. Let $w \in \mathbb{D}$. In addition to the definition of V, we use in the following

$$W(w) := \Big(Q_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)S(w)O_n(w)\Big)^{-1}\Big(R_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)S(w)P_n(w)\Big).$$

In view of Remark 6.1 and the choice of V (resp., W) this matrix-valued function is not only well-defined but also holomorphic in $\mathbb{D}$, where we can also write

$$V(w) = \mathfrak{S}_{\mathbf{\Theta}(w)}\big(S(w)\big) \quad \Big(\text{resp., } W(w) = \mathfrak{T}_{\mathbf{\Xi}(w)}\big(S(w)\big)\Big),$$

by using (2.2) with

$$\begin{gathered}\mathbf{\Theta}(w) := \frac{1-\overline{\alpha_n}w}{\sqrt{1-|\alpha_n|^2}}\begin{pmatrix} b_{\alpha_n}(w)Q_n(w) & O_n^{[\alpha,n]}(w) \\ b_{\alpha_n}(w)R_n(w) & P_n^{[\alpha,n]}(w)\end{pmatrix} \\ \left(\text{resp., } \mathbf{\Xi}(w) := \frac{1-\overline{\alpha_n}w}{\sqrt{1-|\alpha_n|^2}}\begin{pmatrix} b_{\alpha_n}(w)P_n(w) & b_{\alpha_n}(w)O_n(w) \\ R_n^{[\alpha,n]}(w) & Q_n^{[\alpha,n]}(w)\end{pmatrix}\right).\end{gathered} \tag{6.5}$$

Because of (6.5) and Theorem 3.2 the complex $(p+q)\times(p+q)$ matrix $\mathbf{\Theta}(w)$ is $\mathbf{j}_{pq}$-contractive (resp., $\mathbf{\Xi}(w)$ is $\mathbf{j}_{qp}$-contractive). Therefore, from a well-known result on linear fractional matrix transformations (cf. [12, Theorem 1.6.1]) we see that $V(w)$ (resp., $W(w)$) is a contractive $p\times q$ matrix, since $S(w)$ is a contractive $p\times q$ matrix. Hence, V (resp., W) belongs to $\mathcal{S}_{p\times q}(\mathbb{D})$. Moreover, taking into account that V and W are particularly holomorphic in $\mathbb{D}$ and that Proposition 3.5 yields in combination with [12, Proposition 1.6.1] the identity

$$\mathfrak{S}_{\mathbf{\Theta}(v)}\big(S(v)\big)=\mathfrak{T}_{\mathbf{\Xi}(v)}\big(S(v)\big),\quad v\in\mathbb{D}\setminus\mathbb{Z}_{\alpha,n},$$

by a continuity argument one can find that

$$V(w)=\Big(Q_n^{[\alpha,n]}(w)+b_{\alpha_n}(w)S(w)O_n(w)\Big)^{-1}\Big(R_n^{[\alpha,n]}(w)+b_{\alpha_n}(w)S(w)P_n(w)\Big).$$

Furthermore, by setting

$$N(v):=P_n^{[\alpha,n]}(w)+b_{\alpha_n}(w)R_n(w)S(w),$$

Remark 6.1 leads in combination with Corollary 3.7 to the relation

$$\begin{aligned}
&V(w)-V_\circ(w)\\
&=\Big(O_n^{[\alpha,n]}(w)+b_{\alpha_n}(w)Q_n(w)S(w)\Big)\big(N(v)\big)^{-1}-\big(Q_n^{[\alpha,n]}(w)\big)^{-1}R_n^{[\alpha,n]}(w)\\
&=\big(Q_n^{[\alpha,n]}(w)\big)^{-1}\Big(Q_n^{[\alpha,n]}(w)O_n^{[\alpha,n]}(w)+b_{\alpha_n}(w)Q_n^{[\alpha,n]}(w)Q_n(w)S(w)\\
&\qquad-R_n^{[\alpha,n]}(w)P_n^{[\alpha,n]}(w)-b_{\alpha_n}(w)R_n^{[\alpha,n]}(w)R_n(w)S(w)\Big)\big(N(v)\big)^{-1}\\
&=-\eta_n\frac{1-|\alpha_n|^2}{(1-\overline{\alpha_n}w)^2}B_{\alpha,n}(w)\big(Q_n^{[\alpha,n]}(w)\big)^{-1}S(w)\big(N(v)\big)^{-1}.
\end{aligned}\tag{6.6}$$

Since the Blaschke product $B_{\alpha,n}$ has a zero of order l_j+1 at the point β_j for each $j\in\mathbb{N}_{0,m}$ (note (6.1), (2.10), and (1.5)), one can finally conclude

$$V^{(t)}(\beta_k)=V_\circ^{(t)}(\beta_k),\quad t\in\mathbb{N}_{0,l_k},\ k\in\mathbb{N}_{0,m},$$

i.e. $\Delta^{[V]}=\Delta^{[V_\circ]}$. Now let $w\in\mathbb{D}\setminus\mathbb{Z}_{\alpha,n}$. Since (6.5) and Corollary 3.6 imply that the matrix $\mathbf{\Theta}(w)$ $\big($resp., $\mathbf{\Xi}(w)\big)$ is non-singular, where

$$\big(\mathbf{\Theta}(w)\big)^{-1}=\frac{1-\overline{\alpha_n}w}{\eta_nB_{\alpha,n}(w)\sqrt{1-|\alpha_n|^2}}\begin{pmatrix}-Q_n^{[\alpha,n]}(w) & R_n^{[\alpha,n]}(w)\\ b_{\alpha_n}(w)O_n(w) & -b_{\alpha_n}(w)P_n(w)\end{pmatrix}\tag{6.7}$$

$$\left(\text{resp., }\big(\mathbf{\Xi}(w)\big)^{-1}=\frac{1-\overline{\alpha_n}w}{\eta_nB_{\alpha,n}(w)\sqrt{1-|\alpha_n|^2}}\begin{pmatrix}-P_n^{[\alpha,n]}(w) & b_{\alpha_n}(w)R_n(w)\\ O_n^{[\alpha,n]}(w) & -b_{\alpha_n}(w)Q_n(w)\end{pmatrix}\right),$$

and since a basic result on inverting linear fractional matrix transformations $\big($see, e.g., [12, Proposition 1.6.2]$\big)$ yields

$$S(w)=\mathfrak{S}_{(\mathbf{\Theta}(w))^{-1}}\big(V(w)\big)\quad\Big(\text{resp., }S(w)=\mathfrak{T}_{(\mathbf{\Xi}(w))^{-1}}\big(V(w)\big)\Big),$$

one can reason that the matrices $-P_n(w)+O_n(w)V(w)$ and $-Q_n(w)+V(w)R_n(w)$ are non-singular and that the value $S(w)$ can be recovered via (6.3) and (6.4). □

Lemma 6.3. *If* $[(\widetilde{O}_k)_{k=0}^n, (\widetilde{Q}_k)_{k=0}^n]$ *(resp.,* $[(\widetilde{P}_k)_{k=0}^n, (\widetilde{R}_k)_{k=0}^n]$*) is the first (resp., second) SNP pair of rational matrix functions corresponding to* $(\alpha_k, \widetilde{\mathbf{F}}_k)_{k=0}^n$*, with some sequence* $(\widetilde{\mathbf{F}}_k)_{k=0}^n$ *of strictly contractive* $p \times q$ *matrices, so that* $\Delta^{[\widetilde{V}_\circ]} = \Delta^{[V_\circ]}$*, where* $V_\circ$ *is defined as in* (6.2) *and the matrix-valued function* $\widetilde{V}_\circ$ *similarly by*

$$\widetilde{V}_\circ(w) := \widetilde{O}_n^{[\alpha,n]}(w)\big(\widetilde{P}_n^{[\alpha,n]}(w)\big)^{-1}, \quad w \in \mathbb{D}, \tag{6.8}$$

then the equality $\widetilde{\mathbf{F}}_k = \mathbf{F}_k$ *holds for each* $k \in \mathbb{N}_{0,n}$*.*

Proof. By virtue of (3.1), (3.2), and (2.9) we have for each $w \in \mathbb{D}$ the relations

$$\widetilde{\mathbf{F}}_0 = \widetilde{O}_0^{[\alpha,0]}(w)\big(\widetilde{P}_0^{[\alpha,0]}(w)\big)^{-1}, \quad \mathbf{F}_0 = O_0^{[\alpha,0]}(w)\big(P_0^{[\alpha,0]}(w)\big)^{-1}, \tag{6.9}$$

and so $\widetilde{\mathbf{F}}_0 = \mathbf{F}_0$ follows evidently if $n = 0$. Now let $n > 0$. In view of the recursions defining SNP pairs of rational matrix functions, (2.9), and $\Delta^{[\widetilde{V}_\circ]} = \Delta^{[V_\circ]}$ we obtain that the values of the matrix-valued functions

$$\Big(\widetilde{O}_{n-1}^{[\alpha,n-1]}(w)+b_{\alpha_{n-1}}(w)\widetilde{Q}_{n-1}(w)\widetilde{\mathbf{F}}_n\Big)\Big(\widetilde{P}_{n-1}^{[\alpha,n-1]}(w)+b_{\alpha_{n-1}}(w)\widetilde{R}_{n-1}(w)\widetilde{\mathbf{F}}_n\Big)^{-1}, \quad w \in \mathbb{D},$$

$$\Big(O_{n-1}^{[\alpha,n-1]}(w)+b_{\alpha_{n-1}}(w)Q_{n-1}(w)\mathbf{F}_n\Big)\Big(P_{n-1}^{[\alpha,n-1]}(w)+b_{\alpha_{n-1}}(w)R_{n-1}(w)\mathbf{F}_n\Big)^{-1}, \quad w \in \mathbb{D},$$

and their derivatives up to the order l_j at the point β_j for each $j \in \mathbb{N}_{0,m}$ coincide. Because of Lemma 6.2, a successive continuation of this procedure yields that, for each $k \in \mathbb{N}_{1,n}$, the values of the matrix-valued functions

$$\Big(\widetilde{O}_{k-1}^{[\alpha,k-1]}(w)+b_{\alpha_{k-1}}(w)\widetilde{Q}_{k-1}(w)\widetilde{\mathbf{F}}_k\Big)\Big(\widetilde{P}_{k-1}^{[\alpha,k-1]}(w)+b_{\alpha_{k-1}}(w)\widetilde{R}_{k-1}(w)\widetilde{\mathbf{F}}_k\Big)^{-1}, \quad w \in \mathbb{D},$$

$$\Big(O_{k-1}^{[\alpha,k-1]}(w)+b_{\alpha_{k-1}}(w)Q_{k-1}(w)\mathbf{F}_k\Big)\Big(P_{k-1}^{[\alpha,k-1]}(w)+b_{\alpha_{k-1}}(w)R_{k-1}(w)\mathbf{F}_k\Big)^{-1}, \quad w \in \mathbb{D},$$

and their derivatives up to the order $r_{j,k} - 1$ at any point β_j contained in the sequence $(\alpha_\ell)_{\ell=0}^k$ (where $r_{j,k}$ stands for the number how many times) coincide and in particular that

$$\widetilde{O}_0^{[\alpha,0]}(\alpha_0)\big(\widetilde{P}_0^{[\alpha,0]}(\alpha_0)\big)^{-1} = O_0^{[\alpha,0]}(\alpha_0)\big(P_0^{[\alpha,0]}(\alpha_0)\big)^{-1}. \tag{6.10}$$

In the following, by induction on k, we verify that $\widetilde{\mathbf{F}}_k = \mathbf{F}_k$ holds for each $k \in \mathbb{N}_{0,n}$. In the case of $k = 0$, the equalities (6.9) and (6.10) supply immediately

$$\widetilde{\mathbf{F}}_0 = \mathbf{F}_0.$$

Now let $k \in \mathbb{N}_{1,n}$ and we assume that the identity $\widetilde{\mathbf{F}}_j = \mathbf{F}_j$ is already proved for each $j \in \mathbb{N}_{0,k-1}$. In view of this induction assumption, the definition of SNP pairs of rational matrix functions, and the considerations below of the proof we obtain that the values of the matrix-valued functions

$$\Big(O_{k-1}^{[\alpha,k-1]}(w)+b_{\alpha_{k-1}}(w)Q_{k-1}(w)\widetilde{\mathbf{F}}_k\Big)\Big(P_{k-1}^{[\alpha,k-1]}(w)+b_{\alpha_{k-1}}(w)R_{k-1}(w)\widetilde{\mathbf{F}}_k\Big)^{-1}, \quad w \in \mathbb{D},$$

$$\Big(O_{k-1}^{[\alpha,k-1]}(w)+b_{\alpha_{k-1}}(w)Q_{k-1}(w)\mathbf{F}_k\Big)\Big(P_{k-1}^{[\alpha,k-1]}(w)+b_{\alpha_{k-1}}(w)R_{k-1}(w)\mathbf{F}_k\Big)^{-1}, \quad w \in \mathbb{D},$$

and their derivatives up to the order $r_{j,k} - 1$ at any point β_j contained in the sequence $(\alpha_\ell)_{\ell=0}^k$ (where $r_{j,k}$ stands again for the number how many times) coincide on the one hand and on the other hand, by setting

$$N_k(w) := P_{k-1}^{[\alpha,k-1]}(w) + b_{\alpha_{k-1}}(w)R_{k-1}(w)\mathbf{F}_k,$$
$$\widetilde{N}_k(w) := Q_{k-1}^{[\alpha,k-1]}(w) + b_{\alpha_{k-1}}(w)\widetilde{\mathbf{F}}_k O_{k-1}(w),$$

Remark 6.1, Lemma 6.2, and Corollary 3.7 provide

$$\begin{aligned}
&\Big(O_{k-1}^{[\alpha,k-1]}(w)+b_{\alpha_{k-1}}(w)Q_{k-1}(w)\mathbf{F}_k\Big)\Big(P_{k-1}^{[\alpha,k-1]}(w)+b_{\alpha_{k-1}}(w)R_{k-1}(w)\mathbf{F}_k\Big)^{-1}\\
&\quad-\Big(O_{k-1}^{[\alpha,k-1]}(w)+b_{\alpha_{k-1}}(w)Q_{k-1}(w)\widetilde{\mathbf{F}}_k\Big)\Big(P_{k-1}^{[\alpha,k-1]}(w)+b_{\alpha_{k-1}}(w)R_{k-1}(w)\widetilde{\mathbf{F}}_k\Big)^{-1}\\
&=\Big(O_{k-1}^{[\alpha,k-1]}(w)+b_{\alpha_{k-1}}(w)Q_{k-1}(w)\mathbf{F}_k\Big)\big(N_k(w)\big)^{-1}\\
&\quad-\big(\widetilde{N}_k(w)\big)^{-1}\Big(R_{k-1}^{[\alpha,k-1]}(w)+b_{\alpha_{k-1}}(w)\widetilde{\mathbf{F}}_kP_{k-1}(w)\Big)\\
&=\big(\widetilde{N}_k(w)\big)^{-1}\Big(Q_{k-1}^{[\alpha,k-1]}(w)O_{k-1}^{[\alpha,k-1]}(w)+b_{\alpha_{k-1}}(w)\widetilde{\mathbf{F}}_kO_{k-1}(w)O_{k-1}^{[\alpha,k-1]}(w)\\
&\quad+b_{\alpha_{k-1}}(w)Q_{k-1}^{[\alpha,k-1]}(w)Q_{k-1}(w)\mathbf{F}_k+\big(b_{\alpha_{k-1}}(w)\big)^2\widetilde{\mathbf{F}}_kO_{k-1}(w)Q_{k-1}(w)\mathbf{F}_k\\
&\quad-b_{\alpha_{k-1}}(w)R_{k-1}^{[\alpha,k-1]}(w)R_{k-1}(w)\mathbf{F}_k-\big(b_{\alpha_{k-1}}(w)\big)^2\widetilde{\mathbf{F}}_kP_{k-1}(w)R_{k-1}(w)\mathbf{F}_k\\
&\quad-R_{k-1}^{[\alpha,k-1]}(w)P_{k-1}^{[\alpha,k-1]}(w)-b_{\alpha_{k-1}}(w)\widetilde{\mathbf{F}}_kP_{k-1}(w)P_{k-1}^{[\alpha,k-1]}(w)\Big)\big(N_k(w)\big)^{-1}\\
&=\eta_{k-1}\frac{1-|\alpha_{k-1}|^2}{(1-\overline{\alpha_{k-1}}w)^2}B_{\alpha,k-1}(w)\big(\widetilde{N}_k(w)\big)^{-1}(\widetilde{\mathbf{F}}_k-\mathbf{F}_k)\big(N_k(w)\big)^{-1}
\end{aligned}$$

for each $w\in\mathbb{D}$. Since $\eta_{k-1}(1-|\alpha_{k-1}|^2)\neq 0$ and since the Blaschke product $B_{\alpha,k-1}$ has in view of (6.1), (2.10), and (1.5) only k zeroes (at the points $\alpha_0,\alpha_1,\dots,\alpha_{k-1}$), one can finally conclude $\widetilde{\mathbf{F}}_k=\mathbf{F}_k$. □

Now we are able to prove the main result of this paper, i.e. that in the non-degenerate case the set $\mathcal{S}_\Delta$ of all solutions of Problem (MNP) is given by a linear fractional matrix transformation of the form stated in (1.3) and (1.4).

Theorem 6.4. *Let Δ be a data set as in* (2.1) *whereby $\mathbf{P}_\Delta>0$ (resp., $\widetilde{\mathbf{P}}_\Delta>0$). Furthermore, let $V_\bullet\in\mathcal{S}_\Delta$, let $(\alpha_k)_{k=0}^n$ be given as in* (6.1), *and let $(\mathbf{S}_k)_{k=0}^n$ be the sequence of SNP parameters associated with $[V_\bullet,(\alpha_k)_{k=0}^n]$. If $V\in\mathcal{S}_{p\times q}(\mathbb{D})$ then the following statements are equivalent:*

(i) *$V\in\mathcal{S}_\Delta$.*

(ii) *$(\mathbf{S}_k)_{k=0}^n$ is the sequence of SNP parameters associated with $[V,(\alpha_k)_{k=0}^n]$.*

In addition, if $[(O_k)_{k=0}^n,(Q_k)_{k=0}^n]$ (resp., $[(P_k)_{k=0}^n,(R_k)_{k=0}^n]$) stands for the first (resp., second) SNP pair of rational matrix functions corresponding to $(\alpha_k,\mathbf{F}_k)_{k=0}^n$, where $\mathbf{F}_0:=\mathbf{S}_0$ and $\mathbf{F}_k:=-\eta_{k-1}\mathbf{S}_k$ for each $k\in\mathbb{N}_{1,n}$ then (i) *is equivalent to:*

(iii) *There is an $S\in\mathcal{S}_{p\times q}(\mathbb{D})$ so that V admits, for each $w\in\mathbb{D}$, the description*

$$V(w)=\Big(O_n^{[\alpha,n]}(w)+b_{\alpha_n}(w)Q_n(w)S(w)\Big)\Big(P_n^{[\alpha,n]}(w)+b_{\alpha_n}(w)R_n(w)S(w)\Big)^{-1}.$$

(iv) *There is an $\widetilde{S}\in\mathcal{S}_{p\times q}(\mathbb{D})$ so that V admits, for each $w\in\mathbb{D}$, the description*

$$V(w)=\Big(Q_n^{[\alpha,n]}(w)+b_{\alpha_n}(w)\widetilde{S}(w)O_n(w)\Big)^{-1}\Big(R_n^{[\alpha,n]}(w)+b_{\alpha_n}(w)\widetilde{S}(w)P_n(w)\Big).$$

Moreover, if (i) *is satisfied then $S=\widetilde{S}$ and the $p\times q$ Schur function S can be recovered, for each $w\in\mathbb{D}\setminus\mathbb{Z}_{\alpha,n}$, via the formulae* (6.3) *and* (6.4).

Proof. Let $w\in\mathbb{D}$. Note that, by $\mathbf{P}_\Delta>0$ (resp., $\widetilde{\mathbf{P}}_\Delta>0$) and Proposition 2.2, for any solution of Problem (MNP) the SNP algorithm can be carried out (at least) $n+1$ times. Consequently, we can always suppose in the following a given $p\times q$ Schur function V for which the SNP algorithm can be carried out (at least) $n+1$

times. In particular (cf. (2.6) and (2.8)), we find SNP parameters $\widetilde{\mathbf{S}}_0, \widetilde{\mathbf{S}}_1, \dots, \widetilde{\mathbf{S}}_n$ associated with the pair $[V, (\alpha_k)_{k=0}^n]$ which are strictly contractive $p \times q$ matrices and a $p \times q$ Schur function V_{n+1} such that the relation

$$V(w) = \mathfrak{S}_{\mathbf{\Phi}(w)}\big(V_{n+1}(w)\big) \tag{6.11}$$

is satisfied, where

$$\mathbf{\Phi}(w) := \overset{\curvearrowright}{\prod_{k=0}^{n}} \widetilde{\mathbf{\Phi}}_k(w) \quad \Big(:= \widetilde{\mathbf{\Phi}}_0(w)\widetilde{\mathbf{\Phi}}_1(w)\cdots\widetilde{\mathbf{\Phi}}_n(w)\Big)$$

and (by using (2.5))

$$\widetilde{\mathbf{\Phi}}_k(w) := \mathbf{H}_{\widetilde{\mathbf{S}}_k}\begin{pmatrix} b_{\alpha_k}(w)\mathbf{I}_p & 0 \\ 0 & \mathbf{I}_q \end{pmatrix}, \quad k \in \mathbb{N}_{0,n}.$$

By virtue of (2.5) and (3.4), with $\eta_{-1} := -1$ one can also write

$$\mathbf{\Phi}(w) = \left(\overset{\curvearrowright}{\prod_{k=0}^{n}} \mathbf{H}_{-\eta_{k-1}\widetilde{\mathbf{S}}_k} \begin{pmatrix} b_{\alpha_k}(w)\mathbf{I}_p & 0 \\ 0 & \eta_k\overline{\eta_{k-1}}\,\mathbf{I}_q \end{pmatrix} \right)\begin{pmatrix} \mathbf{I}_p & 0 \\ 0 & -\overline{\eta_n}\,\mathbf{I}_q \end{pmatrix}. \tag{6.12}$$

According to Section 3, we define now $[(\widetilde{O}_k)_{k=0}^n, (\widetilde{Q}_k)_{k=0}^n]$ (resp., $[(\widetilde{P}_k)_{k=0}^n, (\widetilde{R}_k)_{k=0}^n]$) as the first (resp., second) SNP pair of rational matrix functions corresponding to $(\alpha_k, \widetilde{\mathbf{F}}_k)_{k=0}^n$ with $\widetilde{\mathbf{F}}_k := -\eta_{k-1}\widetilde{\mathbf{S}}_k$ for each $k \in \mathbb{N}_{0,n}$. Thus, setting

$$\widetilde{\mathbf{\Theta}}_k(w) := \mathbf{H}_{\widetilde{\mathbf{F}}_k}\begin{pmatrix} b_{\alpha_k}(w)\mathbf{I}_p & 0 \\ 0 & \eta_k\overline{\eta_{k-1}}\,\mathbf{I}_q \end{pmatrix}, \quad k \in \mathbb{N}_{0,n},$$

Theorem 3.2 yields the identity

$$\begin{pmatrix} b_{\alpha_n}(w)\widetilde{Q}_n(w) & \widetilde{O}_n^{[\alpha,n]}(w) \\ b_{\alpha_n}(w)\widetilde{R}_n(w) & \widetilde{P}_n^{[\alpha,n]}(w) \end{pmatrix} = \frac{\sqrt{1-|\alpha_n|^2}}{1-\overline{\alpha_n}w}\widetilde{\mathbf{\Theta}}_0(w)\widetilde{\mathbf{\Theta}}_1(w)\cdots\widetilde{\mathbf{\Theta}}_n(w),$$

while (6.12) implies

$$\mathbf{\Phi}(w) = \left(\overset{\curvearrowright}{\prod_{k=0}^{n}} \widetilde{\mathbf{\Theta}}_k(w) \right)\begin{pmatrix} \mathbf{I}_p & 0 \\ 0 & -\overline{\eta_n}\,\mathbf{I}_q \end{pmatrix}.$$

Hence, by (6.11), Remark 6.1, (2.2), and [12, Proposition 1.6.3], we see that

$$V(w) = \Big(\widetilde{O}_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)\widetilde{Q}_n(w)S(w)\Big)\Big(\widetilde{P}_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)\widetilde{R}_n(w)S(w)\Big)^{-1} \tag{6.13}$$

with $S(w) := -\eta_n V_{n+1}(w)$. Obviously, $S \in \mathcal{S}_{p\times q}(\mathbb{D})$ because $V_{n+1} \in \mathcal{S}_{p\times q}(\mathbb{D})$. Moreover, via the construction of the rational matrix functions $\widetilde{O}_n$ and $\widetilde{P}_n$, Lemma 6.2 provides that the $p \times q$ Schur function $\widetilde{V}_\circ$ given as in (6.8) fulfills $\Delta^{[V]} = \Delta^{[\widetilde{V}_\circ]}$. In particular, since $V_\bullet \in \mathcal{S}_\Delta$, the considerations above supply that

$$V_\bullet(w) = \Big(O_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)Q_n(w)S_\bullet(w)\Big)\Big(P_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)R_n(w)S_\bullet(w)\Big)^{-1}$$

for some $S_\bullet \in \mathcal{S}_{p\times q}(\mathbb{D})$ and that the $p \times q$ Schur function $V_\circ$ given as in (6.2) fulfills the identity $\Delta^{[V_\bullet]} = \Delta^{[V_\circ]}$. Consequently, if $V \in \mathcal{S}_\Delta$ then

$$\Delta^{[\widetilde{V}_\circ]} = \Delta^{[V]} = \Delta^{[V_\bullet]} = \Delta^{[V_\circ]}$$

and hence Lemma 6.3 yields $\widetilde{\mathbf{F}}_k = \mathbf{F}_k$ (i.e. $\widetilde{\mathbf{S}}_k = \mathbf{S}_k$) for each $k \in \mathbb{N}_{0,n}$. Therefore, (i) implicates (ii) and in addition (6.13) leads to (1.3). Conversely, if V admits the representation (1.3) for some $S \in \mathcal{S}_{p\times q}(\mathbb{D})$ then from Lemma 6.2 one can get

$$\Delta^{[V]} = \Delta^{[V_\circ]} = \Delta^{[V_\bullet]},$$

i.e. $V \in \mathcal{S}_\Delta$. Furthermore, if (ii) is fulfilled then (6.13) implies that V admits the representation (1.3) with $S(w) := -\eta_n V_{n+1}(w)$. So it is proved that the statements (i), (ii), and (iii) are equivalent. Based on it, the remaining part of the assertion is an easy consequence of Lemma 6.2. □

Observe that the equivalence of (i) and (ii) in Theorem 6.4 is closely related to [12, Corollary 3.8.1] which is a matrix version of Schur's classical result that, for each $l \in \mathbb{N}$, there is a one-to-one correspondence between the first l Taylor coefficients of a Schur function at the point zero and the first l corresponding Schur parameters. Clearly, applying appropriate conformal mappings of the open unit disk $\mathbb{D}$ onto itself, one can obtain a similar result with respect to arbitrarily points $\beta_0, \beta_1, \ldots, \beta_m \in \mathbb{D}$. Nevertheless, it seems to be really hard and unwieldy to derive the equivalence of (i) and (ii) directly from [12, Corollary 3.8.1], since the underlying sequence $(\alpha_k)_{k=0}^n$ has only to fulfill (6.1) and hence the points $\alpha_0, \alpha_1, \ldots, \alpha_n$ are not strictly in the order as $(\gamma_k)_{k=0}^n$. Furthermore, in the particular case of the finite Taylor coefficient problem at the point zero for matrix-valued Schur functions, i.e. if $m := 0$ and if $\beta_0 := 0$ in Problem (MNP) is chosen, the equivalence of (i), (iii), and (iv) in Theorem 6.4 leads to [17, Theorem 14].

7. On Weyl matrix balls associated with Problem (MNP)

Let Δ be a data set as in (2.1) such that $\mathbf{P}_\Delta > 0$ (resp., $\widetilde{\mathbf{P}}_\Delta > 0$). For a fixed point $w \in \mathbb{D}$, we study below the geometric structure of the set

$$\mathfrak{K}_{\Delta,w} := \{V(w) \colon\ V \in \mathcal{S}_\Delta\}. \tag{7.1}$$

Clearly, we have

$$\mathfrak{K}_{\Delta,\beta_k} = \{\mathbf{V}_{k0}\}, \quad k \in \mathbb{N}_{0,m}, \tag{7.2}$$

where $\mathbf{V}_{k0}$ is the prescribed value in Problem (MNP). In general, similar as in the cases of the Taylor coefficient problem at the point zero and the Nevanlinna-Pick problem for matrix-valued Schur functions (cf. [9, Section 5], [20, Section 2], and [17, Theorem 16]), the set $\mathfrak{K}_{\Delta,w}$ is a so-called *Weyl matrix ball* $\mathfrak{K}(\mathbf{M};\mathbf{L},\mathbf{R})$ with certain complex $p\times q$ matrix $\mathbf{M}$, complex $p\times p$ matrix $\mathbf{L}$, and complex $q\times q$ matrix $\mathbf{R}$, i.e. the set of all complex $p\times q$ matrices $\mathbf{X}$ fulfilling the equality $\mathbf{X} = \mathbf{M}+\mathbf{LKR}$ for some contractive $p\times q$ matrix $\mathbf{K}$. We compute now appropriate parameters $\mathbf{M}$, $\mathbf{L}$, and $\mathbf{R}$ based on Theorem 6.4.

Proposition 7.1. *Under the assumptions of* Theorem 6.4*, if $w \in \mathbb{D}$ is fixed then the set $\mathfrak{K}_{\Delta,w}$ in* (7.1) *can be described by*

$$\mathfrak{K}_{\Delta,w} = \mathfrak{K}\Big(\mathbf{M}_{n,w}; |B_{\alpha,n}(w)|\mathbf{L}_{n,w}^{\frac{1}{2}}, \mathbf{R}_{n,w}^{\frac{1}{2}}\Big), \tag{7.3}$$

where

$$\mathbf{M}_{n,w} := \frac{|1-\overline{\alpha_n}w|^2}{1-|\alpha_n|^2}\mathbf{L}_{n,w}\Big(\big(Q_n^{[\alpha,n]}(w)\big)^* R_n^{[\alpha,n]}(w) - |b_{\alpha_n}(w)|^2\big(O_n(w)\big)^* P_n(w)\Big),$$

$$\mathbf{L}_{n,w} := \frac{1-|\alpha_n|^2}{|1-\overline{\alpha_n}w|^2}\Big(\big(Q_n^{[\alpha,n]}(w)\big)^*Q_n^{[\alpha,n]}(w) - |b_{\alpha_n}(w)|^2\big(O_n(w)\big)^*O_n(w)\Big)^{-1},$$

$$\mathbf{R}_{n,w} := \frac{1-|\alpha_n|^2}{|1-\overline{\alpha_n}w|^2}\Big(P_n^{[\alpha,n]}(w)\big(P_n^{[\alpha,n]}(w)\big)^* - |b_{\alpha_n}(w)|^2R_n(w)\big(R_n(w)\big)^*\Big)^{-1}.$$

Moreover, the complex $p \times q$ *matrix* $\mathbf{M}_{n,w}$ *admits also the representation*

$$\mathbf{M}_{n,w} = \frac{|1-\overline{\alpha_n}w|^2}{1-|\alpha_n|^2}\Big(O_n^{[\alpha,n]}(w)\big(P_n^{[\alpha,n]}(w)\big)^* - |b_{\alpha_n}(w)|^2Q_n(w)\big(R_n(w)\big)^*\Big)\mathbf{R}_{n,w}.$$

Proof. Let $w \in \mathbb{D}$. In view of Corollary 3.3 and $\alpha_n \in \mathbb{D}$ we see that the matrices $\mathbf{L}_{n,w}$ and $\mathbf{R}_{n,w}$ are not only well defined but also positive Hermitian. Moreover, if we set

$$\mathbf{K} := -\overline{b_{\alpha_n}(w)}\Big(O_n(w)\big(Q_n^{[\alpha,n]}(w)\big)^{-1}\Big)^*$$

then Corollary 3.3 implies in combination with Corollary 3.7 the representation

$$\mathbf{K} = -\overline{b_{\alpha_n}(w)}\Big(\big(P_n^{[\alpha,n]}(w)\big)^{-1}R_n(w)\Big)^*.$$

Therefore, an application of Theorem 6.4 yields the equality

$$\begin{aligned}\mathbf{M}_{n,w} &= \frac{|1-\overline{\alpha_n}w|^2}{1-|\alpha_n|^2}\mathbf{L}_{n,w}\Big(\big(Q_n^{[\alpha,n]}(w)\big)^*R_n^{[\alpha,n]}(w) - |b_{\alpha_n}(w)|^2\big(O_n(w)\big)^*P_n(w)\Big)\\ &= \Big(Q_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)\mathbf{K}O_n(w)\Big)^{-1}\Big(R_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)\mathbf{K}P_n(w)\Big)\\ &= \Big(O_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)Q_n(w)\mathbf{K}\Big)\Big(P_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)R_n(w)\mathbf{K}\Big)^{-1}\\ &= \frac{|1-\overline{\alpha_n}w|^2}{1-|\alpha_n|^2}\Big(O_n^{[\alpha,n]}(w)\big(P_n^{[\alpha,n]}(w)\big)^* - |b_{\alpha_n}(w)|^2Q_n(w)\big(R_n(w)\big)^*\Big)\mathbf{R}_{n,w}\end{aligned}$$

and

$$\mathbf{M}_{n,\beta_k} = \Big(Q_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)\mathbf{K}O_n(w)\Big)^{-1}\Big(R_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)\mathbf{K}P_n(w)\Big) = \mathbf{V}_{k0}$$

for each $k \in \mathbb{N}_{0,m}$. Hence, in view of (7.2) it follows

$$\mathfrak{K}_{\Delta,\beta_k} = \{\mathbf{V}_{k0}\} = \{\mathbf{M}_{n,\beta_k}\} = \mathfrak{K}\Big(\mathbf{M}_{n,\beta_k}; |B_{\alpha,n}(\beta_k)|\mathbf{L}_{n,\beta_k}^{\frac{1}{2}}, \mathbf{R}_{n,\beta_k}^{\frac{1}{2}}\Big)$$

for each $k \in \mathbb{N}_{0,m}$, i.e. (7.3) for the particular case $w = \beta_k$, since the Blaschke product $B_{\alpha,n}$ has a zero at β_k by definition. Now, let $w \in \mathbb{D} \setminus \{\beta_0, \beta_1, \dots, \beta_m\}$. If we define $\mathbf{\Theta}(w)$ as in (6.5) then Corollary 3.6 shows that the matrix $\mathbf{\Theta}(w)$ is non-singular and that $\big(\mathbf{\Theta}(w)\big)^{-1}$ is given by (6.7). Consequently, also the matrix

$$\mathbf{W} := \Big(\big(\mathbf{\Theta}(w)\big)^{-1}\Big)^*\begin{pmatrix}\mathbf{I}_p & 0\\ 0 & -\mathbf{I}_q\end{pmatrix}\big(\mathbf{\Theta}(w)\big)^{-1}$$

is non-singular and by partitioning $\mathbf{W}$ and $\mathbf{W}^{-1}$ into blocks as

$$\mathbf{W} = \begin{pmatrix}\mathbf{W}_{11} & \mathbf{W}_{12}\\ \mathbf{W}_{21} & \mathbf{W}_{22}\end{pmatrix}, \quad \mathbf{W}^{-1} = \begin{pmatrix}\mathbf{V}_{11} & \mathbf{V}_{12}\\ \mathbf{V}_{21} & \mathbf{V}_{22}\end{pmatrix}$$

with $\mathbf{W}_{11}, \mathbf{V}_{11}$ belonging to $\mathbb{C}^{p\times p}$, $\mathbf{W}_{12}, \mathbf{V}_{12}$ belonging to $\mathbb{C}^{p\times q}$, $\mathbf{W}_{21}, \mathbf{V}_{21}$ belonging to $\mathbb{C}^{q\times p}$, and $\mathbf{W}_{22}, \mathbf{V}_{22}$ belonging to $\mathbb{C}^{q\times q}$ we get

$$\mathbf{W}_{11} = \frac{|1-\overline{\alpha_n}w|^2}{|B_{\alpha,n}(w)|^2(1-|\alpha_n|^2)}\Big(\big(Q_n^{[\alpha,n]}(w)\big)^*Q_n^{[\alpha,n]}(w) - |b_{\alpha_n}(w)|^2\big(O_n(w)\big)^*O_n(w)\Big),$$

$$\mathbf{W}_{12} = -\frac{|1-\overline{\alpha_n}w|^2}{|B_{\alpha,n}(w)|^2(1-|\alpha_n|^2)}\Big(\big(Q_n^{[\alpha,n]}(w)\big)^*R_n^{[\alpha,n]}(w) - |b_{\alpha_n}(w)|^2\big(O_n(w)\big)^*P_n(w)\Big),$$

$$\mathbf{W}_{22} = \frac{|1-\overline{\alpha_n}w|^2}{|B_{\alpha,n}(w)|^2(1-|\alpha_n|^2)}\Big(\big(R_n^{[\alpha,n]}(w)\big)^*R_n^{[\alpha,n]}(w) - |b_{\alpha_n}(w)|^2\big(P_n(w)\big)^*P_n(w)\Big),$$

$$\mathbf{V}_{11} = -\frac{|1-\overline{\alpha_n}w|^2}{1-|\alpha_n|^2}\Big(O_n^{[\alpha,n]}(w)\big(O_n^{[\alpha,n]}(w)\big)^* - |b_{\alpha_n}(w)|^2Q_n(w)\big(Q_n(w)\big)^*\Big),$$

$$\mathbf{V}_{12} = -\frac{|1-\overline{\alpha_n}w|^2}{1-|\alpha_n|^2}\Big(O_n^{[\alpha,n]}(w)\big(P_n^{[\alpha,n]}(w)\big)^* - |b_{\alpha_n}(w)|^2Q_n(w)\big(R_n(w)\big)^*\Big),$$

and

$$\mathbf{V}_{22} = -\frac{|1-\overline{\alpha_n}w|^2}{1-|\alpha_n|^2}\Big(P_n^{[\alpha,n]}(w)\big(P_n^{[\alpha,n]}(w)\big)^* - |b_{\alpha_n}(w)|^2R_n(w)\big(R_n(w)\big)^*\Big).$$

Thus, taking into account the technical similarity with [17, Theorem 16], using a general result on linear fractional matrix transformations like in the proof of [17, Theorem 16] (see also [9, Appendix], [20, Theorem 1 in §2], and [13, Chapter 6]) we obtain (7.3) for that case. □

Remark 7.2. Let the assumptions of Theorem 6.4 be fulfilled and let $w \in \mathbb{D}$ be fixed. If, like in the proof of Proposition 7.1, we set

$$S_n(w) := -b_{\alpha_n}(w)O_n(w)\big(Q_n^{[\alpha,n]}(w)\big)^{-1}$$

then the matrix $\mathbf{M}_{n,w}$ admits the representations

$$\mathbf{M}_{n,w} = \Big(Q_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)\big(S_n(w)\big)^*O_n(w)\Big)^{-1}\Big(R_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)\big(S_n(w)\big)^*P_n(w)\Big),$$

$$\mathbf{M}_{n,w} = \Big(O_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)Q_n(w)\big(S_n(w)\big)^*\Big)\Big(P_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)R_n(w)\big(S_n(w)\big)^*\Big)^{-1}.$$

Moreover, by virtue of Corollary 3.3 and Corollary 3.7 it follows

$$\mathbf{L}_{n,w} = \frac{1-|\alpha_n|^2}{|1-\overline{\alpha_n}w|^2}\big(Q_n^{[\alpha,n]}(w)\big)^{-1}\Big(\mathbf{I}_p - \big(S_n(w)\big)^*S_n(w)\Big)^{-1}\Big(\big(Q_n^{[\alpha,n]}(w)\big)^{-1}\Big)^*,$$

$$\mathbf{R}_{n,w} = \frac{1-|\alpha_n|^2}{|1-\overline{\alpha_n}w|^2}\Big(\big(P_n^{[\alpha,n]}(w)\big)^{-1}\Big)^*\Big(\mathbf{I}_q - S_n(w)\big(S_n(w)\big)^*\Big)^{-1}\big(P_n^{[\alpha,n]}(w)\big)^{-1}.$$

In particular (keep in mind Proposition 3.10 and [12, Lemma 1.1.8]), we get

$$\det \mathbf{L}_{n,w} = \det \mathbf{R}_{n,w}. \tag{7.4}$$

Note that the center $\mathbf{M}$ of a matrix ball $\mathfrak{K}(\mathbf{M};\mathbf{L},\mathbf{R})$ is uniquely determined, but not the semi-radii $\mathbf{L}$ and $\mathbf{R}$ (see, e.g., [12, Corollary 1.5.1 and Theorem 1.5.2]). Consequently, the concrete order of the points in the underlying sequence $(\alpha_j)_{j=0}^n$ subject to (6.1) does obviously not have an influence on the shape of the matrix $\mathbf{M}_{n,w}$ given by Proposition 7.1. In view of (7.4) and [12, Theorem 1.5.2] one can see that also the choice of the matrices $\mathbf{L}_{n,w}$ and $\mathbf{R}_{n,w}$ according to Proposition 7.1 includes this property.

Remark 7.3. Let the assumptions of Theorem 6.4 be fulfilled. If $w \in \mathbb{D}$ is fixed and if S_w stands for the constant function on $\mathbb{D}$ with value $\big(S_n(w)\big)^*$, where $S_n(w)$ is defined as in Remark 7.2, then Theorem 6.4 and Remark 7.2 show that, by setting

$$V_w(v) := \Big(Q_n^{[\alpha,n]}(v) + b_{\alpha_n}(v)S_w(v)O_n(v)\Big)^{-1}\Big(R_n^{[\alpha,n]}(v) + b_{\alpha_n}(v)S_w(v)P_n(v)\Big)$$

for each $v \in \mathbb{D}$, a $p \times q$ Schur function is defined which belongs to $\mathcal{S}_\Delta$ and which satisfies additionally the identity $V_w(w) = \mathbf{M}_{n,w}$. Moreover, if we define the matrix functions $V_\bullet$ and $S_\bullet$, for each $w \in \mathbb{D}$, by

$$V_\bullet(w) := \mathbf{M}_{n,w} \quad \text{and} \quad S_\bullet(w) := \big(S_n(w)\big)^*$$

then $V_\bullet$ is continuous, satisfies the identity $V_\bullet(\beta_k) = \mathbf{V}_{k0}$ for each $k \in \mathbb{N}_{0,m}$, and admits the representation

$$V_\bullet(w) = \Big(Q_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)S_\bullet(w)O_n(w)\Big)^{-1}\Big(R_n^{[\alpha,n]}(w) + b_{\alpha_n}(w)S_\bullet(w)P_n(w)\Big)$$

for each $w \in \mathbb{D}$. However, since the matrix-valued function $S_\bullet$ is not holomorphic in $\mathbb{D}$ (in particular $S_\bullet \notin \mathcal{S}_{p\times q}(\mathbb{D})$), in view of Theorem 6.4 it follows $V_\bullet \notin \mathcal{S}_\Delta$.

Remark 7.4. Let the assumptions of Theorem 6.4 be fulfilled and let $w \in \mathbb{D}$ be fixed. Because of Theorem 4.2 and (4.1) we see that the complex matrices $\mathbf{M}_{n,w}$, $\mathbf{L}_{n,w}$, and $\mathbf{R}_{n,w}$ introduced in Proposition 7.1 can be recovered by

$$\mathbf{M}_{n,w} = \big(1-|w|^2\big)\mathbf{L}_{n,w}\left(\sum_{j=0}^{n}\big(O_j(w)\big)^* P_j(w)\right) = \big(1-|w|^2\big)\left(\sum_{j=0}^{n} Q_j(w)\big(R_j(w)\big)^*\right)\mathbf{R}_{n,w},$$

$$\mathbf{L}_{n,w} = \left(\mathbf{I}_p + \big(1-|w|^2\big)\sum_{j=0}^{n}\big(O_j(w)\big)^* O_j(w)\right)^{-1},$$

$$\mathbf{R}_{n,w} = \left(\mathbf{I}_q + \big(1-|w|^2\big)\sum_{j=0}^{n} R_j(w)\big(R_j(w)\big)^*\right)^{-1}.$$

Remark 7.5. Let the assumptions of Theorem 6.4 be fulfilled, let $k \in \mathbb{N}_{0,m}$, and let the underlying sequence $(\alpha_j)_{j=0}^n$ be satisfying the additional condition $\alpha_n = \beta_k$. Clearly (note (7.2) and Proposition 7.1), we have

$$\mathbf{V}_{k0} = 0 \iff \mathbf{M}_{n,\beta_k} = 0,$$

where in the case of $n = 0$ from (2.6), (3.1), and (3.2) it follows that $\mathbf{V}_{00} = 0$ if and only if O_0 (resp., R_0) is the complex $q \times p$ matrix function with value 0. Moreover, if $n \geq 1$ then, by virtue of Remark 7.4 and Remark 4.5, one can see that

$$\mathbf{V}_{k0} = 0 \iff O_n \in \breve{\mathcal{R}}^{q\times p}_{\alpha,n-1} \iff R_n \in \breve{\mathcal{R}}^{q\times p}_{\alpha,n-1}.$$

Let $k \in \mathbb{N}_{0,m}$. Since the set $\mathfrak{K}_{\Delta,\beta_k}$ defined by (7.1) contains because of (7.2) at least the required value $\mathbf{V}_{k0}$ from Problem (MNP), this set is extraneous. But following the idea of [29, Section 6] concerning complex-valued functions, if we consider in the non-degenerate case instead the set

$$\mathfrak{K}'_{\Delta,\beta_k} := \left\{\frac{1}{(l_k+1)!}V^{(l_k+1)}(\beta_k) :\ V \in \mathcal{S}_\Delta\right\}$$

then $\mathfrak{K}'_{\Delta,\beta_k}$ fills a matrix ball as well, where based on the matrices $\mathbf{L}_{n,\beta_k}$ and $\mathbf{R}_{n,\beta_k}$ defined in Proposition 7.1 one can calculate semi-radii of this matrix ball. This will be emphasized by the concluding statement of this paper.

Proposition 7.6. *Let the assumptions of* Theorem 6.4 *be fulfilled. Let* $k \in \mathbb{N}_{0,m}$*, let the underlying sequence* $(\alpha_j)_{j=0}^n$ *be satisfying the additional condition* $\alpha_n = \beta_k$*, and let the complex matrices* $\mathbf{L}_{n,\beta_k}$ *and* $\mathbf{R}_{n,\beta_k}$ *be defined as in* Proposition 7.1. *Furthermore, let*

$$\mathbf{M}'_{n,\beta_k} := \frac{1}{(l_k+1)!} V_\circ^{(l_k+1)}(\beta_k),$$

where the rational matrix-valued function $V_\circ$ *is given by* (6.2)*. Then*

$$\mathfrak{K}'_{\Delta,\beta_k} = \mathfrak{K}\Big(\mathbf{M}'_{n,\beta_k}\, ; \frac{1}{(1-|\beta_k|^2)^{l_k+1}} \big(\prod_{\substack{j=0\\ j\neq k}}^{m} |b_{\beta_j}(\beta_k)|^{l_j+1}\big)\mathbf{L}^{\frac{1}{2}}_{n,\beta_k}, \mathbf{R}^{\frac{1}{2}}_{n,\beta_k}\Big).$$

Proof. The definition of $\mathbf{L}_{n,\beta_k}$ $\big($resp., $\mathbf{R}_{n,\beta_k}\big)$ implies by virtue of $b_{\alpha_n}(\alpha_n) = 0$, $\alpha_n = \beta_k$, and the polar decomposition of a non-singular (quadratic) matrix the existence of a unitary $p \times p$ matrix $\mathbf{V}$ (resp., unitary $q \times q$ matrix $\mathbf{U}$) such that

$$\frac{1}{\sqrt{1-|\beta_k|^2}} \big(Q_n^{[\alpha,n]}(\beta_k)\big)^{-1} = \mathbf{L}^{\frac{1}{2}}_{n,\beta_k}\mathbf{V} \tag{7.5}$$
$$\Big(\text{resp., } \frac{1}{\sqrt{1-|\beta_k|^2}} \big(P_n^{[\alpha,n]}(\beta_k)\big)^{-1} = \mathbf{U}\mathbf{R}^{\frac{1}{2}}_{n,\beta_k}\Big)$$

is fulfilled. Moreover, taking into account Theorem 6.4, (6.1), (2.10), $b_{\alpha_n}(\alpha_n) = 0$, $\alpha_n = \beta_k$, and the fact that by setting $g := b_{\alpha_n}^{l_k+1}$ it follows

$$\frac{1}{(l_k+1)!} g^{(l_k+1)}(\alpha_n) = \frac{1}{\big(\overline{\eta_n}(|\alpha_n|^2-1)\big)^{l_k+1}},$$

based on (6.6), a straightforward calculation yields

$$\frac{1}{(l_k+1)!} V^{(l_k+1)}(\beta_k) - \frac{1}{(l_k+1)!} V_\circ^{(l_k+1)}(\beta_k)$$
$$= \frac{1}{\big(\overline{\eta_n}(|\beta_k|^2-1)\big)^{l_k+2}} \Bigg(\prod_{\substack{j=0\\ j\neq k}}^{m} \big(b_{\beta_j}(\beta_k)\big)^{l_j+1}\Bigg) \big(Q_n^{[\alpha,n]}(\beta_k)\big)^{-1} S(\beta_k) \big(P_n^{[\alpha,n]}(\beta_k)\big)^{-1}$$

for each $V \in \mathcal{S}_\Delta$ with some $S \in \mathcal{S}_{p\times q}(\mathbb{D})$ (and any $S \in \mathcal{S}_{p\times q}(\mathbb{D})$ can appear). Therefore, in view of (7.5), one can conclude the assertion. □

Finally, we point out that the rational matrix-valued functions O_n, Q_n, P_n, and R_n which occur in the linear fractional matrix transformations according to Theorem 6.4 can be constructed from the interpolation data Δ in Problem (MNP), but indirectly. One needs to determine the corresponding SNP parameters first which is not easy to do in general. A work around is the following. Since the definition of SNP sequences of rational matrix functions introduced in Section 3 is done with a view to orthogonal rational matrix-valued functions on $\mathbb{T}$ and the recurrence relations studied in [22], one can also use the theory of orthogonal rational matrix-valued functions to compute the corresponding SNP parameters or the functions O_n, Q_n, P_n, and R_n. This will be explained in detail in a forthcoming work.

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