

**Connection and comparison between
frequency shift time integration and a
spectral transformation preconditioner**

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Abstract

The numerical study of exterior acoustics problems is usually carried out in the frequency domain. Finite element analyses often require the solution of large scale algebraic linear systems. For very large problems, sometimes the time domain is used. Implicit time integration requires linear system solves, but these are often far easier than those from the frequency domain. This paper shows a connection between a spectral transformation preconditioner and frequency shift time integration. This preconditioner is close to the shifted Laplace preconditioner. The preconditioned iterative method appears to be faster than time integration.

Keywords : Preconditioning, spectral transformation, implicit Euler method, exterior acoustics, infinite elements.

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Connection and comparison between frequency shift time integration and a spectral transformation preconditioner

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SUMMARY

The numerical study of exterior acoustics problems is usually carried out in the frequency domain. Finite element analyses often require the solution of large scale algebraic linear systems. For very large problems, sometimes the time domain is used. Implicit time integration requires linear system solves, but these are often far easier than those from the frequency domain. This paper shows a connection between a spectral transformation preconditioner and frequency shift time integration. This preconditioner is close to the shifted Laplace preconditioner. The preconditioned iterative method appears to be faster than time integration. Copyright © 2007 John Wiley & Sons, Ltd.

KEY WORDS: Preconditioning, spectral transformation, implicit Euler method, exterior acoustics, infinite elements

1. Introduction

Linear systems of algebraic equations arising in the computation of the frequency response function of vibration models often take the form

$$Zx = f \quad \text{with} \quad Z = K + i\omega C - \omega^2 M \quad (1)$$

where K , C and M are real symmetric and positive (semi) definite and ω is a real positive value (the frequency). For finite element analyses with a variational formulation, M is positive definite and well-conditioned ; C is the damping matrix or results from admittance boundary conditions and is positive semi-definite ; K usually is positive semi-definite. For exterior acoustics, discretized by finite-infinite elements [1] [21] [24] [37] using spherical infinite elements, M is positive semi-definite due to a large zero block, and C and K are unsymmetric.

The dimensions of the matrices arising from the discretization of 3D models usually are a few 100,000 to over 1,000,000 and usually direct linear system solvers are used. In some situations,

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the memory requirements and computational cost of direct methods is prohibitive, since a computer with large RAM and fast disk are required, or a distributed machine should be used. Several efforts have been done in developing iterative solvers for acoustics [13] [31] [34] [17] [30] [12] [25]. This list is far from complete. Typically, Krylov methods such as GMRES [36] or BiCGStab [39] with simple preconditioning techniques [23] including Gauss-Seidel, Jacobi, and even ILU [33] do not work (well) for these problems. The goal is to solve (1) for realistic industrial problems using iterative methods. This point is important, since in our tests only simple geometries could be solved with (standard) incomplete factorizations.

It is known that modifying the main diagonal of Z improves the performance of ILU [23] [27]. This was analysed for finite difference problems; see [23] and the references therein. Magolu monga Made [27] [28] modifies the main diagonal of Z for incomplete factorization preconditioning. The incomplete factorizations (especially those with additional fill) may suffer from the fact that the matrix no longer is positive definite. Shifting the spectrum along the imaginary axis usually increases the performance of incomplete factorization methods [29]. A similar approach is studied in [7] [26] for the Helmholtz equation, i.e. with K real symmetric positive semi-definite and M symmetric positive-definite and $C = 0$. The preconditioner is multigrid for the matrix $K + \omega^2 M$. (In fact, multigrid is not defined for a matrix but for a set of nested discretizations. If the original problem is an operator of the form $k - \omega^2$, the preconditioner is multigrid for $k + \omega^2$.) A nice idea was proposed by [15], where the preconditioned equation is shifted over the imaginary axis, i.e. the matrix is preconditioned by the shifted Laplace equation. In this paper, we use a strongly related technique for the exterior Helmholtz equation discretized by infinite elements.

For large vibration problems, the time domain is often used, since less computer resources are required than for the frequency domain. A well designed finite element model usually leads to a stable system of second order ODE's

$$Kx(t) + C\dot{x}(t) + M\ddot{x}(t) = f(t) \quad (2)$$

where M is well conditioned and positive definite. Therefore, the linear systems that arise in implicit time integration methods are relatively easy to solve. When infinite elements are used, the matrices in (2) are non-symmetric, M can be singular and there are cases where the equations are unstable [1, 5, 2, 3, 9]. We have observed that infinite elements do not reduce the computational efficiency of the iterative linear systems solvers in the time domain.

The engineering literature has introduced the so-called frequency shift time integration or pseudo time integration method [38] [10] in the context of solving (1) for a frequency band. This idea is particularly appealing when (1) is solved for various values of ω . The inverse Fourier transform is used to map the solutions from the time domain to the frequency domain. These methods were proposed in a time that algebraic model reduction techniques were not yet used for the solution of vibration problems. For a review of such techniques, see [8]. This method shifts the spectrum of the second order differential (algebraic) equation along the imaginary axis.

The solution of (1) is the steady state solution of a complex valued system of differential equations. It is easy to show (see further) that in this context, time stepping is a stationary linear system solver where the time step is a parameter. This suggests to go one step further and use a Krylov subspace method instead.

In this paper, we do one more little step. The frequency shift time integration method solves a second order differential equation, and therefore writes the problem as a first order

problem of double size. We give arguments that using a Krylov method on Z with well chosen preconditioner is better than any of the time integration methods discussed above. The obtained preconditioner is close to the shifted Laplace preconditioner [15].

The plan of the paper is as follows. In §2, we review the mathematical model for exterior acoustics and briefly comment on the discretization by finite elements and infinite elements. Section 3 introduces the quadratic eigenvalue problem associated with (1). Section 4 discusses the frequency shift time integration method. In §5, we choose a preconditioner to (1) inspired by time integration, and in §6, we show the connection with the Helmholtz equation. In §7, we illustrate the theory in the paper by numerical examples. The goal is to illustrate that the detour via the time domain is helpful in the development of preconditioners.

2. Finite element discretization of acoustic problems

Exterior acoustics are usually modeled by the wave equation (3a) and often using the Sommerfeld radiation condition (3b) :

$$-\nabla^2 p + \frac{1}{c^2} \frac{\partial^2}{\partial t^2} p = 0 \quad \text{on } \Omega \quad (3a)$$

$$\frac{\partial}{\partial r} p + i \frac{\omega}{c} p \longrightarrow 0 \quad \text{as } r \rightarrow \infty \quad (3b)$$

where c is the speed of sound and Ω is the interior domain. The Sommerfeld condition ensures that energy propagates to an outward direction (and does not reflect on the boundary).

The governing equation for acoustics in the frequency domain is the Helmholtz equation (4a). The Sommerfeld condition is approximated by a *plane wave damper* (4c) :

$$-\nabla^2 p - \left(\frac{\omega}{c}\right)^2 p = 0 \quad \text{on } \Omega \quad (4a)$$

$$\nabla p \cdot n + \rho a_n = 0 \quad \text{on } \Sigma \quad (4b)$$

$$\nabla p \cdot n + i \frac{\omega}{c} p = 0 \quad \text{on } \Gamma \quad (4c)$$

where Ω is the interior domain and Γ the exterior boundary. Σ is the boundary with an acceleration boundary condition and n is the normal on Σ , and a_n is a prescribed acceleration boundary condition. The difficulty of solving this equation is in the stiffness in the first term of the Helmholtz equation, and, in addition, the term $-\left(\frac{\omega}{c}\right)^2 p$ that makes the system indefinite.

The discretization that is often used consists of finite elements in the finite domain and infinite elements for modelling the radiation towards infinity. The infinite elements modalize the acoustics in the far field, taking into account the Sommerfeld condition on Γ where Γ is e.g. a sphere whose radius is going to infinity. Special trial and weight functions are chosen for this purpose.

In this paper, we do not consider a very specific formulation, although numerical experiments use the conjugate formulation [4]. The initial formulations led to ill-conditioned matrices. More recent formulations use orthogonal polynomials to improve conditioning [11]. With the infinite elements, the equations in the time domain may become unstable [3], which makes time domain computations rather tricky. For spheroidal domains, the finite element — infinite element formulation leads to an index one stable system of differential algebraic equations [9].

3. The quadratic eigenvalue problem

The spectrum of the quadratic eigenvalue problem

$$\lambda^2 My + \lambda Cy + Ky = 0 \quad (5)$$

plays a central role in the analysis of the paper. In general, we assume that the eigenvalues λ lie in the left half complex plane.

Equation (5) can be written as the first order equation

$$\lambda Bp + Ap = 0 \quad (6)$$

with

$$B = \begin{bmatrix} C & M \\ I & 0 \end{bmatrix}, \quad A = \begin{bmatrix} K & 0 \\ 0 & -I \end{bmatrix} \quad \text{and} \quad p = \begin{pmatrix} y \\ \lambda y \end{pmatrix}.$$

We also introduce the eigenvalue problem

$$\mu^2 My + \mu \tilde{C}y + Zy = 0 \quad \text{with} \quad \tilde{C} = (C + i2\omega M). \quad (7)$$

The eigenvectors are the same as for (5), but the eigenvalues are $\mu_j = \lambda_j - i\omega$. In words, (7) corresponds to a shifted quadratic eigenvalue problem with shift $i\omega$. The corresponding first order eigenvalue problem reads

$$\mu \tilde{B}p + \tilde{A}p = 0$$

with

$$\tilde{B} = \begin{bmatrix} \tilde{C} & M \\ I & 0 \end{bmatrix}, \quad \tilde{A} = \begin{bmatrix} Z & 0 \\ 0 & -I \end{bmatrix}.$$

For the sequel, it is important to note that if λ_j lies in the left half complex plane, so does μ_j .

4. Frequency shift time integration

Equation (1) is related to the differential equation

$$M\ddot{y} + C\dot{y} + Ky = fe^{i\omega t}. \quad (8)$$

If the eigenvalues λ lie in the left half complex plane, (8) is stable. If the equation is stable, the solution converges to $xe^{i\omega t}$ for $t \rightarrow \infty$. When we let y take the form $ze^{i\omega t}$, then z satisfies

$$M\ddot{z} + \tilde{C}\dot{z} + Zz = f. \quad (9)$$

This is known as frequency shift time integration [38]. The solution of (1), x , is the steady state of (9).

Equation (9) can be rewritten as the first order equation

$$\tilde{B}\dot{p} + \tilde{A}p = b \quad (10)$$

with

$$p = \begin{pmatrix} z \\ \dot{z} \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} f \\ 0 \end{pmatrix}.$$

The solution p converges to $(x, 0)$. We denote by p_j the approximation of $p(t_j)$, computed by the time integration method, at time steps $t_j = j\delta$ for $j = 1, 2, \dots$, given $p(0)$. The value δ is the time step.

In the presence of infinite eigenvalues, there usually are eigenvalues with very large modulus, sometimes unstable, or infinite eigenvalues. This makes explicit methods less attractive, so these are not considered. Among the implicit methods, we limit ourselves to backward Euler and Crank-Nicholson, because they involve only one linear system solve per time step.

In the following, we treat integration methods as stationary iterative solvers. We will show the connection with time integration methods and iterative linear systems solvers for $\tilde{A}p = b$. We call a method stationary when there is a preconditioner P so that

$$p_{j+1} = p_j + Pr_j$$

where $r_j = b - \tilde{A}p_j$ is the residual and $e_j = \tilde{A}^{-1}r_j$ is the error. With the iteration matrix $G = I - P\tilde{A}$ [22, Chapter 10], we also have

$$\begin{aligned} e_{j+1} &= Ge_j \\ r_{j+1} &= \tilde{A}G\tilde{A}^{-1}r_j . \end{aligned} \tag{11}$$

The speed of convergence is determined by the spectral radius of G . A spectral radius larger than 1 implies divergence of the method. Note that the spectra of G and $\tilde{A}G\tilde{A}^{-1}$ are the same.

4.1. Backward Euler

Let p_j be given. Then p_{j+1} is computed from

$$\tilde{B}(p_{j+1} - p_j) = \delta(b - \tilde{A}p_{j+1}) \tag{12}$$

or

$$(\tilde{B} + \delta\tilde{A})(p_{j+1} - p_j) = \delta(b - \tilde{A}p_j)$$

from which we derive the iteration matrix

$$G_{BE} := (\tilde{B} + \delta\tilde{A})^{-1}\tilde{B} .$$

The spectrum of G_{BE} is $(-\mu_j\delta + 1)^{-1}$ where μ_j for $j = 1, \dots, 2n$ are the eigenvalues of (7). The condition of convergence corresponds to the stability condition of the backward Euler method : when (8) is stable, Backward Euler converges for any time step. The matrix G_{BE} is also known as the spectral transformation [14] or shift-and-invert transformation [35].

4.2. Crank-Nicholson

For the trapezoid rule or Crank-Nicholson,

$$\tilde{B}(p_{j+1} - p_j) = \delta(b - \tilde{A}(p_j + p_{j+1})/2)$$

or

$$(\tilde{B} + \delta\tilde{A})(p_{j+1} - p_j) = \delta(b - \tilde{A}p_j)$$

from which we derive the iteration matrix

$$G_{CN} := (\tilde{B} + \delta/2\tilde{A})^{-1}(\tilde{B} - \delta/2\tilde{A}) ,$$

which is known as the Cayley transformation [20] [32]. The spectrum of G_{CN} is $(2 + \mu_j \delta)/(2 - \mu_j \delta)$ where μ_j for $j = 1, \dots, 2n$ are the eigenvalues of (7). The condition of convergence corresponds to the stability condition of the backward Euler method : when (8) is strictly stable, Crank-Nicholson converges for any time step. If the problem has undamped harmonics, i.e. μ lies on the imaginary axis, $\|G_{CN}u\|_2 = \|u\|_2$. Crank-Nicholson keeps the harmonics that are already present in the initial solution. The solution never converges to a steady state. For this reason, we prefer the backward Euler method.

We have a brief look at the practical implications. For the implementation, the most expensive operations are the multiplications with \tilde{A} , and $(\tilde{B} + \delta\tilde{A})^{-1}$.

The multiplication with $(\tilde{B} + \delta\tilde{A})^{-1}$ requires the solution of the algebraic linear system

$$\begin{bmatrix} \tilde{C} + \delta Z & M \\ I & -\delta I \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Multiplying the first line with δ and the second line by M and adding to the first leads to

$$\begin{bmatrix} M + \delta\tilde{C} + \delta^2 Z & 0 \\ I & -\delta I \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \delta y_1 + M y_2 \\ y_2 \end{pmatrix}$$

Here is an algorithm :

Algorithm 1 (Solve with $\tilde{B} + \delta\tilde{A}$)

1. Solve $(M + \delta\tilde{C} + \delta^2 Z)x_1 = \delta y_1 + M y_2$
2. $x_2 = (x_1 - y_2)/\delta$

The major difficulty here is the solution with a linear system with the matrix

$$\tilde{Z} = M + \delta\tilde{C} + \delta^2 Z .$$

The choice of δ definitely influences the ease of solution.

When using an implicit time integration method, more time steps (or outer iterations) are needed for smaller δ , but the linear systems with \tilde{Z} become cheaper, because \tilde{Z} is closer to the discretized identity operator. When δ is larger, fewer outer iterations are needed, but \tilde{Z} gets closer to Z and thus harder to use in an iterative solver. So, there is an optimal value of δ .

If λ is an eigenvalue of (5) then

$$\theta = \frac{\delta^{-1}}{-\mu + \delta^{-1}} = \frac{-\delta^{-1}}{\lambda - (i\omega + \delta^{-1})}$$

is an eigenvalue of G_{BE} .

Theorem 1. *Let λ_j , $j = 1, \dots, 2n$ be the eigenvalues of the quadratic eigenvalue problem (5) and let $\text{real}(\lambda_j) > 0$. Let θ_j , $j = 1, \dots, 2n$ be the eigenvalues of G_{BE} . The mapping properties are summarized as follows (see Figure 1) :*

- The vertical line $\text{Re}(\lambda) = 0$ is mapped to the circle \mathcal{C}_1 centred in 0.5 and going through zero and one.
- The circle centred at $i\omega + \delta^{-1}$ going through $i\omega$ is mapped onto the unit circle in the θ plane. This implies that all (stable) eigenvalues λ are mapped inside the unit disk in the θ plane.
- Let γ be such that the interior of the disk \mathcal{C}_λ^* centred at $i\omega + \delta^{-1}$ with radius γ does not contain an eigenvalue λ , then G_{BE} does not have an eigenvalue outside the circle \mathcal{C}_θ^* centered at the origin with radius $\frac{\delta^{-1}}{\gamma}$. There always is a $\gamma > \delta^{-1}$ since $i\omega$ is no eigenvalue.

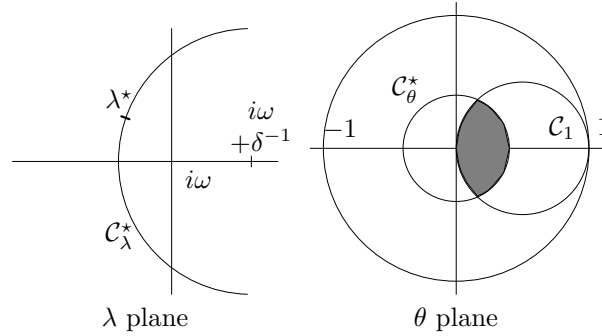


Figure 1. Mapping properties of spectral transformation

Proof. The proof is left to the reader. It consists of elementary geometric operations. \square

For λ_j far away from $i\omega$, $|\theta_j|$ is close to zero. So, most eigenvalues are mapped near zero for many practical problems, since the λ_j 's lie spread out along the imaginary axis. Only the λ_j 's near $i\omega$ are mapped close to one. There are only a few of those.

The speed of convergence of the iterative method, is the asymptotic reduction in residual norm for one iteration. This is the largest $|\theta_j|$. Following Theorem 1, the convergence speed is bounded from above by

$$\rho = (\delta\gamma)^{-1}. \quad (13)$$

If δ is large, ρ is small and convergence is attained in one step. Also γ plays a role. When γ is large, $i\omega$ is far from any eigenvalue and the system is well-conditioned. When there are eigenvalues near $i\omega$, $\gamma \simeq \delta^{-1}$, so ρ is close to one and convergence is slow.

In general, most θ lie near zero and only a few near one. This is a situation where Krylov methods perform significantly better than this stationary method. That is: we apply a Krylov method to \tilde{A} and use $(\tilde{B} + \delta\tilde{A})^{-1}$ as a preconditioner. The eigenvalues of G_{BE} lie on one side of the spectrum (the dark area in Figure 1) which also leads to a higher convergence rate in a Krylov method than the backward Euler method.

5. Further improvements

The backward Euler method solves the preconditioned system

$$\begin{bmatrix} \tilde{C} + \delta Z & M \\ I & -\delta I \end{bmatrix}^{-1} \begin{bmatrix} Z & 0 \\ 0 & -I \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} \tilde{C} + \delta Z & M \\ I & -\delta I \end{bmatrix}^{-1} \begin{pmatrix} f \\ 0 \end{pmatrix},$$

where $x_1 = x$ and $x_2 = 0$.

We compare this to the iterative solution of

$$\tilde{Z}^{-1}Zx = \tilde{Z}^{-1}f \quad (14)$$

with the stationary iterative scheme :

$$\tilde{Z}(x_{j+1} - x_j) = \delta^2(f - Zx_j). \quad (15)$$

It is easy to see that the field of values of

$$I - \delta^2 \tilde{Z}^{-1} Z = \tilde{Z}^{-1} (M + \delta \tilde{C})$$

is contained by the field of values of

$$(\tilde{B} + \delta A)^{-1} \tilde{B} = \begin{bmatrix} \tilde{Z}^{-1} (M + \delta \tilde{C}) & \delta \tilde{Z}^{-1} \tilde{C} \\ -\delta I & \delta I \end{bmatrix}.$$

The spectral radius of $(\tilde{B} + \delta A)^{-1} \tilde{B}$ is larger than the one of $I - \delta^2 \tilde{Z}^{-1} Z$. These are arguments to assume that (15) converges faster than (12) [23].

Another way of looking at the two methods is the following one : we have to compare the iteration scheme that maps the error at iteration j :

$$\begin{aligned} \tilde{Z} e_{j+1}^{(1)} &= \delta \tilde{C} e_j^{(1)} + \delta M e_j^{(2)} + M e_j^{(1)} \\ e_{j+1}^{(2)} &= \delta^{-1} (e_{j+1}^{(1)} - e_j^{(1)}) \end{aligned}$$

for the backward Euler method and

$$\tilde{Z} e_{j+1} = (M + \delta \tilde{C}) e_j$$

for (15). The difference lies in the $\delta M e_j^{(2)}$ term. If it is small, the two methods behave similarly. If it is large, the Euler method is slower. If it is of the same order as the other terms, the Euler method might be faster if terms cancel out.

Time integration methods for the frequency shift equation can be considered as stationary algebraic linear system solvers. In general, Krylov methods are more efficient than stationary solvers and therefore we advise the use of GMRES [36], BiCGStab [39], QMR [19] or another Krylov method [6].

6. Choice of δ and interpretation for the Helmholtz equation

Following (13), it is best to choose δ as large as possible : the solution is computed exactly in an infinite time-step. However, the iterative solves with \tilde{Z} are harder.

The difficulty with preconditioning Z mainly lies in the $-\omega^2 M$ term. Choosing $\delta = \omega^{-1}$ produces

$$\tilde{Z} = \omega^{-2} K + (1 + i) \omega^{-1} C + i 2M.$$

The term in $-\omega^2 M$ has disappeared.

Whereas Z is the discrete operator for (4), \tilde{Z} is obtained by replacing ω by $\omega - i/\delta$. This leads to the problem

$$\begin{aligned} -\nabla^2 p - \left(\frac{\omega - i/\delta}{c} \right)^2 p &= 0 \quad \text{on } \Omega \\ \nabla p \cdot n + \rho a_n &= 0 \quad \text{on } \Sigma \\ \nabla p \cdot n + \left(\frac{i\omega}{c} + \frac{1}{\delta c} \right) p &= 0 \quad \text{on } \Gamma \end{aligned}$$

or

$$\begin{aligned} \left\{ -\nabla^2 - \left(\frac{\omega}{c}\right)^2 \right\} p + \left(\frac{1}{\delta^2 c^2} + i \frac{2\omega}{\delta c^2} \right) p &= 0 \quad \text{on } \Omega \\ \nabla p \cdot n + \rho a_n &= 0 \quad \text{on } \Sigma \\ \left\{ \nabla p \cdot n + \frac{i\omega}{c} p \right\} + \frac{1}{\delta c} p &= 0 \quad \text{on } \Gamma . \end{aligned}$$

We add positive (mass) terms to the equations, which makes them, in general, easier to solve. When $\delta < \omega^{-1}$, the indefinite term $-\left(\frac{\omega}{c}\right)^2 p$ disappears from the Helmholtz equation, which makes it easier to solve. In brackets $\{\}$, we have the original equations. The real part becomes positive definite.

It was illustrated by [27] [28] that the improvement of the real part is very important for using incomplete factorizations.

The physical interpretation is that replacing the frequency ω by $\omega - i/\delta$ adds damping to the problem, which has a stabilizing effect to the equations : the eigenvalues λ are shifted by $-\delta^{-1}$, i.e. get a larger absolute damping term.

An important conclusion, however, is that $\delta = \omega^{-1}$ is small for large ω , which illustrates that, although preconditioner is easier, more time steps are required to attain the solution. It is thus to be expected that convergence is slow for high frequencies. This is illustrated by the numerical examples.

In [27], ILU for $Z + \gamma I$ is used as preconditioner where γ is a complex constant. This is a purely algebraic shift of the matrix. Denote by \mathcal{Z} the operator for the system (4c), and \mathcal{L} the Laplacian including the Sommerfeld boundary condition. In [15], ILU is applied to the discretization of $\mathcal{L} + \gamma$. In our approach, the spectrum of the Helmholtz operator with boundary conditions is shifted over a complex value γ . In terms of linear algebra, this can be seen as follows : our approach applies a shift to the quadratic eigenvalue problem (5).

7. Numerical examples

We illustrate the theory in the paper for two numerical examples. We used BiCGStab [39] as iterative scheme with ILU(1) preconditioning on \tilde{Z} .

The first is for a simple geometry for which the classical ILU method applied to Z performs slightly better than ILU for \tilde{Z} .

We have run the method for industrial applications, which we cannot report on. However, the second example, provided to us by Airbus, behaves as many acoustic applications, we have solved with the method. The examples were run with the acoustic simulation code ACTRAN [18] on a Dell Dimension 4550 running 32 bit Linux redhat. The stopping criterion was a residual norm

$$\|Zx - b\| \leq 10^{-8} .$$

In this section, we illustrate several aspects of the method described in this paper. First, we illustrate the choice of δ . Finally, we show the frequency dependence of the convergence behaviour.

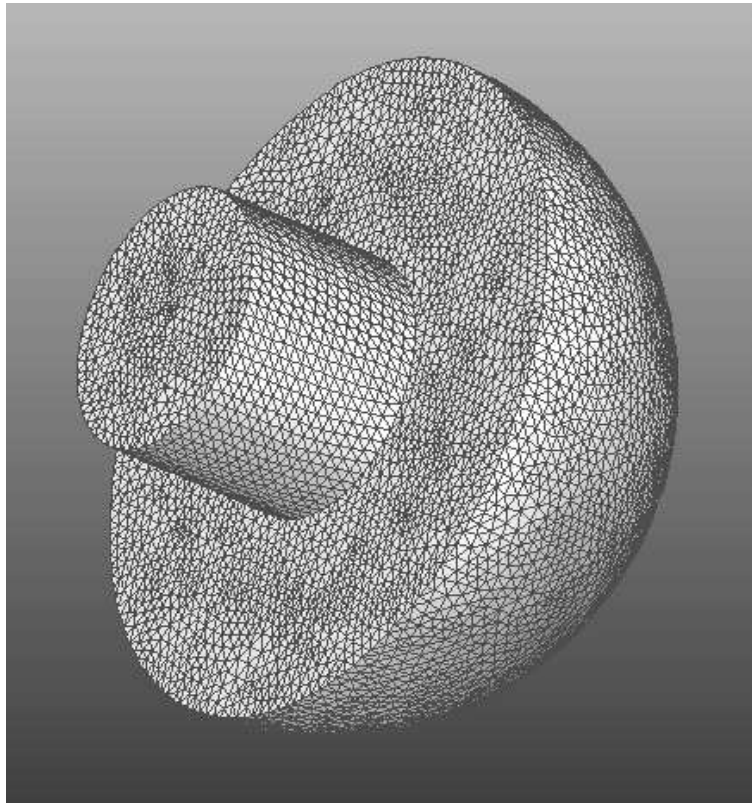


Figure 2. Mushroom shaped mesh

7.1. A simple problem: cube

The following are results for an acoustic box with coordinates $[-0.5, 0.5]^3$ covered with infinite elements for radiation towards infinity. A unit acoustic point source was put at coordinate $(0, 0, -0.3)$. The finite element mesh consisted of 64,000 linear hexahedral elements and 9600 infinite elements of order 5. The frequency was $\omega = 100$.

Using ILU(1) on Z , the solution was found with the desired accuracy in 103 iterations. Using ILU(1) on \tilde{Z} , the code required 101 iterations. There is practically no gain. Similar problems have shown no significant differences between both preconditioners.

7.2. Mushroom : choice of δ

The following problem originates from Airbus in a study of iterative methods for acoustics. The mesh is shown in Figure 2. It is a mushroom shaped volume, where the spherical part (half sphere) is covered by infinite elements. On the bottom of the mushroom is a unit acceleration boundary condition. The discrete mesh consists of 309,642 linear tetrahedral elements and 8764 infinite elements of order 5. The frequency was $\omega = 500$.

Table I. ILU preconditioning for different value of δ .

$\delta\omega$	iterations BiCGStab – ILU(1)	time (s)
0.2	no cvg	
0.5	2369	204
1	935	82
2	589	52
5	1533	133
7	3517	300
∞	no cvg	

Table II. Results preconditioning for different value of ω .

ω	iterations	time (s)
100	285	27
200	561	51
300	851	75
400	705	62
500	935	82
600	1365	118
700	1411	123

Table I shows the number of iterations for BiCGStab using the preconditioner for various values of δ . Note that $\delta = \infty$ implies $\tilde{Z} = Z$. The conclusion is that the optimal value is around $\delta = 2\omega^{-1}$. Timings include the ILU factorization and construction of \tilde{Z} from Z and C and M . When no convergence is reached after 10,000 iterations, we report *no cvg*.

7.3. Mushroom : influence of ω

Table II shows the number of iterations for different values of ω and $\delta = \omega^{-1}$. Timings include the ILU factorization and construction of \tilde{Z} from Z and C and M . The performance clearly degrades for larger ω 's. The fact that δ becomes smaller indicates smaller time steps and thus slower convergence.

7.4. Comparison with time integration

The poor performance of time integration can be explained by various factors : first, it is a stationary iterative method, so, in general, it is slower than a Krylov method ; second, the field of values of $(\tilde{B} + \tilde{A})^{-1}\tilde{A}$ is larger than the field of values of $\delta\tilde{Z}^{-1}Z$. Since the field of values is much larger for the first order problem, we also expect slower convergence.

The previous example is not very well suited for this since the governing equations lead to a slightly instable system of second order differential equations, which leads to complications for the numerical solution.

	time steps	matrix vector products
Time integration BiCGStab	1352	29751
BiCGStab		5801

Table III. Statistics for the mushroom mesh with admittance boundary condition

Table IV. Comparison for the mushroom infinite element model between different shifted preconditioners

Preconditioner	\tilde{Z}	\tilde{Z}_1	\tilde{Z}_2
Iterations	935	1387	2441

We illustrate this with the same example, where the infinite elements are now replaced by an admittance boundary condition. This leads to complex symmetric matrices with a stable system of second order differential equations. We used the backward Euler method where linear systems with \tilde{Z} are solved by BiCGStab with ILU preconditioning. We compare with BiCGStab applied to Z with ILU(0) preconditioning for \tilde{Z} . We found that time integration with time step $\delta = \omega^{-1}$ needed 1352 time steps to find the solution with acceptable precision. The stop criterion for the inner BiCGStab iterations was 0.1δ . This required a total of 29751 BiCGStab iterations for the Euler steps. Applying BiCGStab straightaway to $\tilde{Z}^{-1}Z$ required 5801 iterations. This is more than the number of time steps, but no inner BiCGStab solves are needed which makes the overall process much cheaper. Table III shows some results.

7.5. Mushroom : comparison with other iterative methods

A comparison with other methods developed for exterior acoustics is a hard task that should be performed in a separate study. We compared with techniques proposed by Magolu and coworkers [29] [27] [28], but we did not have satisfactory performance for the exterior acoustics problems.

We have compared the preconditioner with the preconditioner that is close to the preconditioners advocated by [15] and [16]. In order to do that, we did not use ILU(1) for

$$\tilde{Z} = K + 2i\omega^2 M + i\omega C + \omega C$$

but for

$$\begin{aligned}\tilde{Z}_1 &= K + 2i\omega^2 M + i\omega C \\ \tilde{Z}_2 &= K + i\omega^2 M + i\omega C .\end{aligned}$$

The ωC term is missing in \tilde{Z}_1 and \tilde{Z}_2 . The results are very comparable, although we found that convergence was slightly slower, especially for \tilde{Z}_2 . See Table IV.

8. Conclusions

We have presented a preconditioner for Krylov methods that finds its origin in time integration. This preconditioner corresponds to shifting the eigenvalues into the stable half plane, which can be considered as artificial damping. It is inspired by the frequency shift time integration method. The iterative solver is superior to the time integration method.

From the analysis and numerical examples, it appears that the preconditioner loses its effectiveness for larger frequencies. This is somehow discouraging, since meshes become finer for larger frequencies and iterative methods might be the methods of choice for larger frequencies.

Although the method does not converge for all problems we have tested, performance usually is far better than standard incomplete factorization for realistic problems. For simple geometries, the gain is low.

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