Fast computation of determinants of Bézout matrices and application to curve implicitization

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Abstract

When using bivariate polynomial interpolation for computing the implicit equation of a rational plane algebraic curve given by its parametric equations, the generation of the interpolation data is the most costly of the two stages of the process. In this work a new way of generating those interpolation data with less computational cost is presented. The method is based on an efficient computation of the determinants of certain constant Bézout matrices.

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1 Introduction

In [4] an algorithm for finding the implicit equation of a rational plane algebraic curve given by its parametric equations is presented. The algorithm is based on an efficient computation of the resultant by means of classical bivariate polynomial interpolation.

In general, the interpolation data are computed by means of the computation of the determinants of the matrices obtained by evaluating the corresponding symbolic Bézout matrix at certain interpolation points.

In the present paper an alternative approach is used: instead of evaluating the symbolic Bézout matrix at the interpolation nodes we compute the determinants of the evaluated matrices starting from the polynomials obtained by evaluating at those interpolation points the polynomials arising from the parametric equations.

This method will reduce the computational cost of the implicitization algorithm given in [4] from $O(n^5)$ to $O(n^4)$ arithmetic operations.

The rest of the paper is organized as follows. In Section 2 we recall some basic facts about curve implicitization by means of the Bézout matrix and about the use of bivariate interpolation for finding the implicit equation. In Section 3 an algorithm for computing the determinants of Bézout matrices of evaluated polynomials is presented, while Section 4 is devoted to present an example which tries to make the whole process clear.

2 Curve implicitization and Bézout determinants

Let $P(t) = (u_1(t)/v_1(t), u_2(t)/v_2(t))$ be a proper rational parametrization of a plane curve C with $\gcd(u_1(t), v_1(t)) = \gcd(u_2(t), v_2(t)) = 1$. A parametrization $P(t) = (x(t), y(t))$ is said to be *proper* if every point on C except a finite number of exceptional points is generated by exactly one value of the parameter t. Since every rational curve has a proper parametrization [5] we can assume that the considered parametrization is proper. In this situation the following theorem holds [7]:

Theorem 1. *Let $P(t) = (u_1(t)/v_1(t), u_2(t)/v_2(t))$ be a proper rational parametrization of a plane curve C with $\gcd(u_1(t), v_1(t)) = \gcd(u_2(t), v_2(t)) = 1$. Then the polynomial defining the implicit equation of C is*

$$F(x, y) = \mathrm{Res}_t(u_1(t) - xv_1(t), u_2(t) - yv_2(t)),$$

that is, the resultant with respect to t of the polynomials $u_1(t) - xv_1(t)$ and

$u_2(t) - yv_2(t)$, *and*

$$\deg_x(F) = \max\{\deg_t(u_2), \deg_t(v_2)\}, \quad \deg_y(F) = \max\{\deg_t(u_1), \deg_t(v_1)\}.$$

This theorem tells us that the polynomial $F(x, y)$ defining the implicit equation of C is a polynomial with degree $m = \max\{\deg_t(u_2), \deg_t(v_2)\}$ in the variable x, and degree $n = \max\{\deg_t(u_1), \deg_t(v_1)\}$ in the variable y. So,

$$F(x, y) = \mathrm{Res}_t(u_1(t) - xv_1(t), u_2(t) - yv_2(t)) = \sum_{i=0}^{m}\sum_{j=0}^{n} c_{i,j}x^i y^j \in \Pi_{m,n}(x, y),$$

that is, $F(x, y)$ belongs to the space of the polynomials in the variables x and y with degree less than or equal to m in x and degree less than or equal to n in y. We will compute $F(x, y)$ by using bivariate Lagrange polynomial interpolation.

We consider the interpolation space $\Pi_{m,n}(x, y)$ with the basis

$$\begin{aligned} &\{x^i y^j : \quad i = 0, \ldots, m; j = 0, \ldots, n\} \\ &= \{1, y, \ldots, y^n, x, xy, \ldots, xy^n, \ldots, x^m, x^m y, \ldots, x^m y^n\} \end{aligned}$$

in that precise order. We choose the $(m + 1)(n + 1)$ interpolation nodes in the corresponding order

$$\begin{aligned} &\{(x_i, y_j) \in \mathbb{R}^2 : \quad i = 0, \ldots, m; j = 0, \ldots, n\} \\ &\quad = \{(x_0, y_0), (x_0, y_1), \ldots, (x_0, y_n), \\ &\quad (x_1, y_0), (x_1, y_1), \ldots, (x_1, y_n), \ldots, (x_m, y_0), (x_m, y_1), \ldots, (x_m, y_n)\}, \end{aligned}$$

where $x_{i_1} \neq x_{i_2}$ when $i_1 \neq i_2$, and $y_{j_1} \neq y_{j_2}$ when $j_1 \neq j_2$. The specific selection of the nodes will be detailed below.

In this situation, if we choose the interpolation data f_{ij} as the values $F(x_i, y_j)$ of the resultant at the points (x_i, y_j) and we order them in the same way as the basis and the nodes, then the unique solution of the interpolation problem, whose associated linear system has Kronecker product structure (see [4]), will give us the coefficients of the resultant.

We will compute the interpolation data by using the Bézout resultant. We start by giving the definition of the Bézout matrix of two polynomials.

Definition 2. *Let* $p(t) = \sum_{k=0}^{r} p_k t^k$ *and* $q(t) = \sum_{l=0}^{s} q_l t^l$ *be two polynomials of degrees* r *and* s*, with* $r \geq s$*. The Bézout matrix of* $p(t)$ *and* $q(t)$ *is defined* *as the r by r matrix B whose (k,l) entry is the coefficient corresponding to $t^{k-1}z^{l-1}$ in the bivariate polynomial*

$$B(t,z) = \frac{p(t)q(z) - p(z)q(t)}{t-z}. \quad (1)$$

Remark 3. *[6, 2].*

- *If $r = s$, the resultant of $p(t)$ and $q(t)$ is the determinant of the Bézout matrix of $p(t)$ and $q(t)$.*
- *If $r > s$, the resultant of $p(t)$ and $q(t)$ is the determinant of the Bézout matrix of $p(t)$ and $q(t)$ divided by p_r^{r-s}.*

Remark 4. *[6, 2]. Let us observe that in the case $r > s$ we can obtain a matrix whose determinant is exactly the resultant of $p(t)$ and $q(t)$ by a suitable modification of the last $r-s$ rows of the Bézout matrix. This "modified" Bézout matrix is also called the Bézout matrix of $p(t)$ and $q(t)$. The computer algebra system Maple uses a "modified" Bézout matrix for computing the Bézout matrix of two polynomials with different degrees, and this is the one used in [4] in order to avoid divisions by p_r^{r-s} when computing the interpolation data.*

Remark 5. *In this paper we will always consider the Bézout matrix presented in the definition above. The reason for this choice when $r > s$ is that the "modified" Bézout matrix does not preserve the structure of the Bézout matrix which we will need in order to use the techniques described in the following section. Let us observe that, for example, the Bézout matrix of the definition is symmetric while the "modified" Bézout matrix is not.*

In the sequel we will compute the resultant of $u_1(t) - xv_1(t)$ and $u_2(t) - yv_2(t)$ by using the Bézout matrix of these two polynomials, choosing as $p(t)$ (the first polynomial) the one with greatest degree. This choice of ordering could produce a change in the sign of the resultant, but the sign is irrelevant when considering the implicit equation.

Now we detail the specific selection of the interpolation nodes and the computation of the corresponding interpolation data. The interpolation nodes will be chosen in such a way that each matrix whose determinant has to be computed in order to obtain the interpolation datum is the Bézout matrix of two polynomials. We have to distinguish four different situations:

1. *The curve C is given by a polynomial parametrization $P(t) = (u_1(t), u_2(t))$ with $n = \deg_t(u_1(t)) = \deg_t(u_2(t))$.*

   Taking into account that in this situation the resultant of $p(t) = u_1(t)-x$ and $q(t) = u_2(t)-y$ is the determinant of the Bézout matrix of $p(t)$ and $q(t)$, we can compute the interpolation data $f_{ij} = F(x_i, y_j)$ by constructing the symbolic Bézout matrix of these polynomials, evaluating it at each interpolation node (x_i, y_j) and computing the determinant of the corresponding constant matrix. It is easy to check that although the Bézout matrix depends on the degree of the polynomials, in this case, as the degrees of $p(t)$ and $q(t)$ are the same as the degrees of the polynomials $u_1(t)-x_i$ and $u_2(t)-y_j$ (respectively), the symbolic Bézout matrix evaluated at (x_i, y_j) corresponds to the Bézout matrix of the evaluated polynomials $u_1(t)-x_i$ and $u_2(t)-y_j$. In this way, we can select any interpolation nodes satisfying $x_{i_1} \neq x_{i_2}$ when $i_1 \neq i_2$, and $y_{j_1} \neq y_{j_2}$ when $j_1 \neq j_2$. For simplicity we choose $(x_i, y_j) = (i, j)$ for $i = 0, \ldots, n$ and $j = 0, \ldots, n$.

2. *The curve C is given by a polynomial parametrization $P(t) = (u_1(t), u_2(t))$ with $n = \deg_t(u_1(t))$, $m = \deg_t(u_2(t))$ and $n > m$.*

   In this case we can also compute the interpolation data $f_{ij} = F(x_i, y_j)$ by constructing the symbolic Bézout matrix of $p(t) = u_1(t) - x$ and $q(t) = u_2(t) - y$, evaluating it at each interpolation node (x_i, y_j) and finally dividing the number obtained from computing the determinant of the constant matrix by p_n^{n-m} (a number). This division is not strictly necessary for obtaining an implicit equation; it will be necessary for the precise expression of the resultant. An alternative way to obtain this expression is to make the division at the end: one must divide all the coefficients by that number.

   The same argument as in Case 1 let us to choose any interpolation nodes with $x_{i_1} \neq x_{i_2}$ when $i_1 \neq i_2$, and $y_{j_1} \neq y_{j_2}$ when $j_1 \neq j_2$. As in Case 1 we select $(x_i, y_j) = (i, j)$ for $i = 0, \ldots, m$ and $j = 0, \ldots, n$.

   Let us point out that if $n < m$ the process is analogous, but now $p(t) = u_2(t) - y$ and $q(t) = u_1(t) - x$, and we must divide by p_m^{m-n}.

3. *The curve C is given by a rational parametrization $P(t) = (u_1(t)/v_1(t), u_2(t)/v_2(t))$ with $n = \max\{\deg_t(u_1(t)), \deg_t(v_1(t))\} = \max\{\deg_t(u_2(t)), \deg_t(v_2(t))\}$.*

   The computation of the interpolation data can be developed in the same way as in Case 1, but now considering $p(t) = u_1(t) - xv_1(t)$ and

$q(t) = u_2(t) - yv_2(t)$. However, we cannot choose the same interpolation nodes because it may happen that for some (i, j) the symbolic Bézout matrix evaluated at (i, j) does not coincide with the Bézout matrix of the evaluated polynomials $u_1(t) - iv_1(t)$ and $u_2(t) - jv_2(t)$. In order to avoid this, we choose the smallest nonnegative integers x_i for $i = 0, \ldots, n$ such that the degree of $u_1(t)-x_iv_1(t)$ is the same as the degree of $p(t)$, and the smallest nonnegative integers y_j for $j = 0, \ldots, n$ such that the degree of $u_2(t)-y_jv_2(t)$ is the same as the degree of $q(t)$, that is, we must exclude the unique value of x such that $p_n(x) = 0$ and the unique value of y such that $q_n(y) = 0$ (if these values exist).

4. *The curve C is given by a rational parametrization $P(t) = (u_1(t)/v_1(t), u_2(t)/v_2(t))$ with $n = \max\{\deg_t(u_1(t)), \deg_t(v_1(t))\}$, $m = \max\{\deg_t(u_2(t)), \deg_t(v_2(t))\}$ and $n > m$.*

   The computation of the interpolation data can be carried out in a similar way to Case 2: by constructing the symbolic Bézout matrix of $p(t) = u_1(t) - xv_1(t)$ and $q(t) = u_2(t) - yv_2(t)$, evaluating it and p_n^{n-m} (it could be a polynomial in x) at each interpolation node (x_i, y_j) and finally dividing the number obtained from computing the determinant of the constant matrix by the number obtained from the evaluation of p_n^{n-m} at x_i. As for the selection of the interpolation nodes, we must choose them in the same way as in Case 3.

   It must be observed that this selection of the nodes avoids also divisions by zero in the computation of the interpolation data because $(p_n(x_i))^{n-m} \neq 0$.

   Let us point out that when $n < m$ the process is completely analogous. In this case $p(t) = u_2(t) - yv_2(t)$, $q(t) = u_1(t) - xv_1(t)$ and we must divide by $(p_m(y_j))^{m-n}$.

In this way, we have guaranteed that each one of the matrices whose determinant we have to compute for obtaining the interpolation data is the Bézout matrix of two polynomials (the evaluated polynomials) with coefficients in $\mathbb{R}$. In the following section we present a method for computing such determinants with a computational complexity of $O(n^2)$ arithmetic operations. Taking this into account, the computational complexity of the complete implicitization algorithm will be of $O(n^4)$ arithmetic operations instead of $O(n^5)$ (see [4]).

For the construction of the symbolic Bézout matrix the algorithm presented in [1] can be used.

Remark 6. *The process of constructing the symbolic Bézout matrix and evaluating it at the corresponding interpolation nodes to compute the interpolation data is equivalent to compute the matrices evaluated at* (x_i, y_j) *as Bézout matrices of the evaluated polynomials:* $u_1(t) - xv_1(t)$ *evaluated at* x_i *and* $u_2(t) - yv_2(t)$ *evaluated at* y_j*. This idea will be used in the following section, where in addition it is seen that the evaluated Bézout matrices are not constructed explicitly.*

3 Bézout determinant computation

In the previous section it was explained how the computation of a symbolic Bézout determinant, and hence the curve implicitization problem, can be reduced to the computation of a number of *constant* Bézout determinants. This is the problem to which we turn now.

The following notation will be used intensively throughout this section.

Definition 7. *Let* $\mathbf{x} := (x_k)_{k=1}^r$ *and* $\mathbf{y} := (y_k)_{k=1}^r$ *be arbitrary vectors of* data points*. Then we denote by* $D_\mathbf{x}$ *and* $D_\mathbf{y}$ *the corresponding diagonal matrices, by* $V_\mathbf{x} = (x_k^{l-1})$ *and* $V_\mathbf{y} = (y_k^{l-1})$*,* $k = 1, \ldots, r$*,* $l = 1, \ldots, r$ *the corresponding square Vandermonde matrices, by* $\tilde{V}_\mathbf{x}$ *and* $\tilde{V}_\mathbf{y}$ *the rectangular Vandermonde matrices having an extra column* $l = r + 1$ *in their Vandermonde structure.*

Note that the above definition introduced some auxiliary quantities which we called the vectors of data points $\mathbf{x}$ and $\mathbf{y}$. These should not be confused with the so-called interpolation points of the previous section (whose role is of no importance anymore, since we assume in this section just a given, *constant* Bézout matrix).

We have the following transformation theorem.

Theorem 8. *Let* $p(t) = \sum_{k=0}^r p_k t^k$ *and* $q(t) = \sum_{l=0}^r q_l t^l$ *be two polynomials of degree at most* r*, let* B *be the corresponding Bézout matrix in Definition 2, and let* $\mathbf{x}$ *and* $\mathbf{y}$ *be arbitrary vectors of data points. Then the matrix*

$$C := V_\mathbf{x} B V_\mathbf{y}^T \tag{2}$$

is a Cauchy-like matrix *w.r.t.* $D_\mathbf{x}$ *and* $D_\mathbf{y}$*, i.e. it satisfies the matrix equation*

$$D_\mathbf{x} C - C D_\mathbf{y} = \mathrm{Rk}\ 2, \tag{3}$$

with Rk 2 *a matrix of rank at most 2. More specifically*

$$\mathrm{Rk}\ 2 = \tilde{V}_\mathbf{x} \begin{bmatrix} \tilde{\mathbf{p}} & \tilde{\mathbf{q}} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{q}}^{\,T} \\ -\tilde{\mathbf{p}}^{\,T} \end{bmatrix} \tilde{V}_\mathbf{y}^T, \tag{4}$$

where $\tilde{\mathbf{p}} = (p_{k-1})_{k=1}^{r+1}$ *and* $\tilde{\mathbf{q}} = (q_{k-1})_{k=1}^{r+1}$ *are the so-called* stacking vectors *of* $p(t)$, $q(t)$.

PROOF. The (k, l) element of the matrix on the left hand side of (3) is given by

$$\begin{aligned}(D_{\mathbf{x}}C - CD_{\mathbf{y}})_{k,l} &= (x_k - y_l)C_{k,l} \\ &= (x_k - y_l)(\frac{p(x_k)q(y_l) - q(x_k)p(y_l)}{x_k - y_l}) \\ &= p(x_k)q(y_l) - q(x_k)p(y_l),\end{aligned}$$

where the second transition follows by the defining equations (2) and (1). On the other hand, the (k, l) element of the matrix in (4) is given by

$$\begin{aligned}(\mathrm{Rk}\ 2)_{k,l} &= \begin{bmatrix} p(x_k) & q(x_k) \end{bmatrix} \begin{bmatrix} q(y_l) \\ -p(y_l) \end{bmatrix} \\ &= p(x_k)q(y_l) - q(x_k)p(y_l).\end{aligned}$$

Thus we have $(D_{\mathbf{x}}C - CD_{\mathbf{y}})_{k,l} = (\mathrm{Rk}\ 2)_{k,l}$ for all k and l, hence proving the theorem. □

The previous theorem provided a transformation between the classes of Bézoutian and Cauchy-like matrices. Moreover, assuming from now on that

$$x_k - x_l \neq 0, \quad y_k - y_l \neq 0 \quad (\text{all } k \neq l), \quad x_k - y_l \neq 0 \quad (\text{all } k \text{ and } l), \quad (5)$$

then the definition $C = V_{\mathbf{x}} B V_{\mathbf{y}}^T$ implies that

$$\det B = \frac{\det C}{\prod_{k,l,k>l}(x_k - x_l) \prod_{k,l,k>l}(y_k - y_l)}. \quad (6)$$

Thus we obtain that any Bézout determinant can be expressed in terms of a suitable Cauchy-like determinant. Therefore we will concentrate now on the class of Cauchy-like matrices.

The following definition will be helpful for doing this.

Definition 9. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be arbitrary vectors of data points satisfying (5). We define the* Cauchy matrix *w.r.t.* $\mathbf{x}$ *and* $\mathbf{y}$ *to be the matrix*

$$\tilde{C} = \left[\frac{1}{x_k - y_l}\right]_{k,l}. \quad (7)$$

Moreover, we define the polynomial of degree r

$$f_{\mathbf{y}}(t) := \prod_k (t - y_k), \tag{8}$$

and the polynomials of degree $r-1$

$$f_{\mathbf{y},l}(t) := \prod_{k,k\neq l} (t - y_k), \tag{9}$$

for any $1 \leq l \leq r$.

The Cauchy matrix satisfies the following identity.

Lemma 10. *We have*

$$\tilde{C} = (\operatorname{diag}(f_{\mathbf{y}}(x_k))_{k=1}^r)^{-1} V_{\mathbf{x}} V_{\mathbf{y}}^{-1} \operatorname{diag}(f_{\mathbf{y},l}(y_l))_{l=1}^r. \tag{10}$$

PROOF. See for example [3, Proposition 3]. □

The Cauchy matrix $\tilde{C}$ should not be confused with the Cauchy-*like* matrix C that we introduced earlier. The Cauchy matrix is special in the sense that we can express any other Cauchy-like matrix in terms of it. This can be seen by rewriting (3) as

$$D_{\mathbf{x}} C - C D_{\mathbf{y}} = \text{Rk } 2 =: \begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix} \begin{bmatrix} \mathbf{d}^T \\ -\mathbf{c}^T \end{bmatrix}, \tag{11}$$

with $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ in $\mathbb{C}^{r\times 1}$, from which it follows that $(x_k - y_l)C_{k,l} = a_k d_l - b_k c_l$ and hence

$$C = \left[\frac{a_k d_l - b_k c_l}{x_k - y_l}\right]_{k,l} = D_{\mathbf{a}} \tilde{C} D_{\mathbf{d}} - D_{\mathbf{b}} \tilde{C} D_{\mathbf{c}}. \tag{12}$$

Now we introduce a final class of structured matrices.

Definition 11. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be arbitrary vectors of data points satisfying (5), let* C *be a Cauchy-like matrix w.r.t.* $D_{\mathbf{x}}$ *and* $D_{\mathbf{y}}$*, and let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *and* $\mathbf{d}$ *be fixed vectors satisfying (11). We define*

$$V_C := \begin{bmatrix} D_{\mathbf{a}} V_{\mathbf{x}} & D_{\mathbf{b}} V_{\mathbf{x}} \\ D_{\mathbf{c}} V_{\mathbf{y}} & D_{\mathbf{d}} V_{\mathbf{y}} \end{bmatrix} \tag{13}$$

to be the coupled Vandermonde matrix *of* C.

Theorem 12. *It holds that*

$$\det C = (-1)^{\binom{r}{2}} \frac{\det V_C}{\prod_{k,l}(x_k - y_l)}. \tag{14}$$

PROOF. Assume for the moment that $D_{\mathbf{d}}$ is nonsingular. Then by the Schur complement formula, the determinant of V_C can be expanded as

$$\begin{aligned} \det V_C &= \det D_{\mathbf{d}} \det V_{\mathbf{y}} \det(D_{\mathbf{a}} V_{\mathbf{x}} - D_{\mathbf{b}} V_{\mathbf{x}} (D_{\mathbf{d}} V_{\mathbf{y}})^{-1} D_{\mathbf{c}} V_{\mathbf{y}}) \\ &= \det D_{\mathbf{d}} \det V_{\mathbf{y}}^2 \det(D_{\mathbf{a}} V_{\mathbf{x}} V_{\mathbf{y}}^{-1} - D_{\mathbf{b}} V_{\mathbf{x}} V_{\mathbf{y}}^{-1} D_{\mathbf{d}}^{-1} D_{\mathbf{c}}) \\ &= \det V_{\mathbf{y}}^2 \det(D_{\mathbf{a}} V_{\mathbf{x}} V_{\mathbf{y}}^{-1} D_{\mathbf{d}} - D_{\mathbf{b}} V_{\mathbf{x}} V_{\mathbf{y}}^{-1} D_{\mathbf{c}}), \end{aligned} \tag{15}$$

where we used the fact that any two diagonal matrices commute with each other. Moreover, by continuity it follows that (15) must hold even if we remove the condition that D_d is nonsingular. Now because of (10), (12), and using again the commuting of diagonal matrices, this equation can be rewritten as

$$\det V_C = \det V_{\mathbf{y}}^2 \det(\operatorname{diag}(f_{\mathbf{y}}(x_k))_{k=1}^r) \det C \det(\operatorname{diag}(f_{\mathbf{y},l}(y_l))_{l=1}^r)^{-1}. \tag{16}$$

Now

$$\det(\operatorname{diag}(f_{\mathbf{y}}(x_k))_{k=1}^r) = \prod_{k,l}(x_k - y_l),$$

$$\det(\operatorname{diag}(f_{\mathbf{y},l}(y_l))_{l=1}^r)^{-1} = (\prod_{k,l,k\neq l}(y_k - y_l))^{-1},$$

$$\det V_{\mathbf{y}}^2 = (-1)^{\binom{r}{2}} \prod_{k,l,k\neq l}(y_k - y_l).$$

Substituting these expressions in (16), it follows that

$$\det V_C = (-1)^{\binom{r}{2}} \prod_{k,l}(x_k - y_l) \det C,$$

which is equivalent to the required equality (14). □

The previous theorem allows to rewrite a given Bézout determinant in terms of a coupled Vandermonde determinant. This follows by combining (14) and (6) into the formula

$$\det B = \frac{(-1)^{\binom{r}{2}} \det V_C}{\prod_{k,l}(x_k - y_l) \prod_{k,l,k>l}(x_k - x_l) \prod_{k,l,k>l}(y_k - y_l)}. \tag{17}$$

Since the products of data points in the above formula can be computed in $O(r^2)$ work, we would obtain a fast way for computing the Bézout determinant, provided we have a fast algorithm to compute the coupled Vandermonde determinant.

Now the latter computation can be done by the $O(r^2)$ algorithm described in [3, Section 3]. Let us note that this algorithm does not explicitly form the coupled Vandermonde matrix, but instead computes everything in terms of its defining vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, $\mathbf{x}$ and $\mathbf{y}$ in (13). Moreover, note that the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ can be expressed in their turn in terms of $\mathbf{x}$, $\mathbf{y}$ and the coefficients of $p(t)$ and $q(t)$, in a way made precise by relations (11) and (4).

A final remark is due here to the choice of the vectors of data points $\mathbf{x}$ and $\mathbf{y}$. In principle these vectors may be chosen completely ad random. But of course it makes more sense to choose them to consist of *rational* numbers, since then all computations can be performed in infinite precision arithmetic. This is illustrated in the example of the next section, where the data points are taken to be small integers.

Let us add a final note concerning *finite* precision computations, where the complexity of the algorithm can be improved from $O(r^2)$ to $O(r\log(r))$.

Remark 13. *The complexity of the algorithm can be improved from fast to superfast by choosing the data points as*

$$x_k = \omega_r^{k-1}, \quad y_l = \omega_{2r}\omega_r^{l-1}, \tag{18}$$

$k = 1, \ldots, r$, $l = 1, \ldots, r$, *with* $\omega_p = \exp(\frac{2\pi i}{p})$ *denoting the* k*th root of unity.*

Indeed, let us first show how to simplify (17) in this case. Note that $f_{\mathbf{y}}(t) := \prod_k (t - y_k) = t^r + 1$, *by the special choice of the* y_l. *Hence*

$$\prod_{k,l}(x_k - y_l) = \prod_k (f_{\mathbf{y}}(x_k)) = \prod_k (x_k^r + 1) = 2^r, \tag{19}$$

by the special choice of the x_k. *Thus we obtain a simple expression for the first factor in the denominator of (17). Next, we may note that*

$$\prod_{k,l,k>l}(y_k - y_l) = \omega_{2r}^{\binom{r}{2}} \prod_{k,l,k>l}(x_k - x_l),$$

by the special choice of the x_k, y_l. Hence

$$
\begin{aligned}
(-1)^{\binom{r}{2}} \prod_{k,l,k>l} (y_k - y_l) \prod_{k,l,k>l} (x_k - x_l) &= \omega_{2r}^{\binom{r}{2}} \prod_{k,l,k\neq l} (x_k - x_l) \\
&= \omega_{2r}^{\binom{r}{2}} \prod_{k} \frac{r}{x_k} \\
&= \omega_{2r}^{\binom{r}{2}} r^r \omega_r^{-\binom{r}{2}} \\
&= r^r \omega_{2r}^{-\binom{r}{2}},
\end{aligned}
$$

where the second transition follows since for each fixed k, we have that

$$
\prod_{l,l\neq k} (x_k - x_l) = \lim_{x\to x_k} \frac{x^r - 1}{x - x_k} = r x_k^{r-1} = \frac{r}{x_k},
$$

by the l'Hôpital rule. Since moreover

$$
\begin{aligned}
\omega_{2r}^{-\binom{r}{2}} &= (\exp(i\frac{\pi}{r}))^{-\binom{r}{2}} \\
&= \exp(-i\frac{\pi r(r-1)}{2r}) \\
&= (\exp(-i\frac{\pi}{2}))^{r-1} \\
&= i^{-(r-1)},
\end{aligned}
$$

we can rewrite

$$
(-1)^{\binom{r}{2}} \prod_{k,l,k>l} (y_k - y_l) \prod_{k,l,k>l} (x_k - x_l) = r^r i^{-(r-1)}. \tag{20}
$$

Substituting (19) and (20), we obtain from (17) the final formula

$$
\det B = \frac{i^{r-1}}{(2r)^r} \det V_C. \tag{21}
$$

This is the desired simplification of (17).

Under the same assumptions, note that also the vectors **a**, **b**, **c** *and* **d** *can be obtained in a superfast way in terms of the input data, namely the coefficients of $p(t)$ and $q(t)$. This follows since the definitions in (11) and (4) can be expressed in terms of FFT's.*

Finally, also the computation of the coupled Vandermonde determinant can be performed then in a superfast way, as described in [3]. However, a main drawback is that in this case, one has to be cautious about the stability of this last algorithm: see [3] and the references therein.

4 A detailed example

We illustrate our approach with the following example. Let $P(t) = (x(t), y(t))$ be a rational parametrization of a curve C where

$$x(t) = \frac{t^5 + t^4 - 2t^3 + 3t^2 - t + 4}{t^5 + 3t^4 - 2t^3 + t^2 + t - 1},$$
$$y(t) = \frac{t^3 + t^2 - 5t + 3}{t^3 - 4t^2 - t + 2}.$$

Our aim is to compute by means of interpolation the polynomial $F(x, y)$ defining the implicit equation of the curve C, that is, the resultant $\mathrm{Res}_t(p(t), q(t))$ where

$$p(t) = (t^5 + t^4 - 2t^3 + 3t^2 - t + 4) - x(t^5 + 3t^4 - 2t^3 + t^2 + t - 1),$$
$$q(t) = (t^3 + t^2 - 5t + 3) - y(t^3 - 4t^2 - t + 2).$$

Taking Theorem 1 into account, we know that $m = \deg_x(F(x, y)) = 3$ and $n = \deg_y(F(x, y)) = 5$. Therefore, we consider the interpolation space $\Pi_{3,5}(x, y)$ with the basis

$$\{1, y, \cdots, y^5, x, xy, \cdots, xy^5, x^2, x^2y, \cdots, x^2y^5, x^3, x^3y, \cdots, x^3y^5\}.$$

The adequate interpolation nodes are in this case

$$\{(x_i, y_j) : \quad i = 0, \cdots, 3; j = 0, \cdots, 5\}$$

where $x_0 = 0$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$ and $y_0 = 0$, $y_1 = 2$, $y_2 = 3$, $y_3 = 4$, $y_4 = 5$, $y_5 = 6$. We order them in the same way as the interpolation basis. The vector with the interpolation data $\{F(x_i, y_j) : \quad i = 0, \cdots, 3; j = 0, \cdots, 5\}$ in the corresponding order is

$$\begin{aligned} f = (&2664, 959512, 7606776, 34094600, 109174984, 281657400, 0, -2185616, \\ &- 16894908, -71707072, -219049700, -544282704, 2259, -4866071, \\ &- 35464578, -144475729, -428258306, -1039906299, 11160, -8350232, \\ &- 58319784, -230217864, -666143000, -1586928312)^T. \end{aligned}$$

In this way, the unique solution of the interpolation problem is the solution of the linear system of order 24

$$Ac = f$$

whose coefficient matrix is $A = \hat{V}_x \otimes \hat{V}_y$, with $\hat{V}_x$ being the Vandermonde matrix generated by x_0, x_1, x_2, x_3 and $\hat{V}_y$ being the Vandermonde matrix generated by $y_0, y_1, y_2, y_3, y_4, y_5$. c is the vector with the coefficients in the adequate order. After solving the system by using the algorithm presented in [4] we obtain that

$$\begin{aligned}F(x,y) = &-109x^3y^5 + 2226x^3y^4 - 8809x^3y^3 + 9953x^3y^2 - 5069x^3y + \\ &531x^3 + 5694x^2y^5 - 76721x^2y^4 + 133061x^2y^3 - 100545x^2y^2 + \\ &27938x^2y - 1458x^2 - 66848xy^5 + 158504xy^4 - 204376xy^3 + \\ &99060xy^2 - 20212xy - 540x + 45088y^5 - 66032y^4 + 80448y^3 - \\ &22072y^2 + 7624y + 2664.\end{aligned}$$

Now we detail the computation of one interpolation datum, for example, $F(3,2)$. Taking into account the results presented in Section 2, $F(3,2)$ is the determinant of the Bézout matrix B of the evaluated polynomials

$$\begin{aligned}p3(t) &= (t^5 + t^4 - 2t^3 + 3t^2 - t + 4) - 3(t^5 + 3t^4 - 2t^3 + t^2 + t - 1), \\ q2(t) &= (t^3 + t^2 - 5t + 3) - 2(t^3 - 4t^2 - t + 2),\end{aligned}$$

divided by $(p_5(3))^{5-3} = (1-3)^2 = 4$. This Bézout matrix is a square matrix of order 5 whose determinant we compute following the results presented in Section 3. We consider the following vectors of data points

$$\mathbf{x} = (1,2,3,4,5)^T \quad \text{and} \quad \mathbf{y} = (-1,-2,-3,-4,-5)^T.$$

D_x and D_y are the corresponding diagonal matrices,V_x and V_y are the Vandermonde matrices generated by the components of $\mathbf{x}$ and $\mathbf{y}$ respectively, and $\tilde{V}_x$ and $\tilde{V}_y$ are the rectangular Vandermonde matrices having an extra column $l = 6$. The vectors with the coefficients of the polynomials $p3$ and $q2$ are respectively

$$\tilde{p} = (7,-4,0,4,-8,-2)^T \quad \text{and} \quad \tilde{q} = (-1,-3,9,-1,0,0)^T.$$

The matrices D_a, D_b, D_c and D_d are the diagonal matrices corresponding to the vectors a, b, c and d, where

$$\begin{aligned}a^T &= \tilde{V}_x\, \tilde{p}^T = (-3,-161,-1031,-3849,-10763)^T, \\ b^T &= \tilde{V}_x\, \tilde{q}^T = (4,21,44,67,84)^T, \\ c^T &= \tilde{V}_y\, \tilde{p}^T = (1,-81,-251,-233,777)^T, \\ d^T &= \tilde{V}_y\, \tilde{q}^T = (12,49,116,219,364)^T.\end{aligned}$$

Therefore, the coupled Vandermonde matrix V_C whose determinant we compute for obtaining the determinant of the Bézout matrix of $p3(t)$ and $q2(t)$ is:

$$\begin{pmatrix}
-3 & -3 & -3 & -3 & -3 & 4 & 4 & 4 & 4 & 4 \\
-161 & -322 & -644 & -1288 & -2576 & 21 & 42 & 84 & 168 & 336 \\
-1031 & -3093 & -9279 & -27837 & -83511 & 44 & 132 & 396 & 1188 & 3564 \\
-3849 & -15396 & -61584 & -246336 & -985344 & 67 & 268 & 1072 & 4288 & 17152 \\
-10763 & -53815 & -269075 & -1345375 & -6726875 & 84 & 420 & 2100 & 10500 & 52500 \\
1 & -1 & 1 & -1 & 1 & 12 & -12 & 12 & -12 & 12 \\
-81 & 162 & -324 & 648 & -1296 & 49 & -98 & 196 & -392 & 784 \\
-251 & 753 & -2259 & 6777 & -20331 & 116 & -348 & 1044 & -3132 & 9396 \\
-233 & 932 & -3728 & 14912 & -59648 & 219 & -876 & 3504 & -14016 & 56064 \\
777 & -3885 & 19425 & -97125 & 485625 & 364 & -1820 & 9100 & -45500 & 227500
\end{pmatrix}.$$

We compute the determinant of V_C by using the algorithm presented in Section 3, and so the explicit construction of V_C is not needed. Let us observe that also the construction of the Bézout matrix of $p(t)$ and $q(t)$, and of the Bézout matrix of $p3(t)$ and $q3(t)$ are not needed.

So, the determinant of the Bézout matrix B is

$$\begin{aligned}
\det(B) &= \frac{(-1)^{\binom{5}{2}} \det(V_C)}{\prod_{k,l}(x_k - y_l) \prod_{k,l,k>l}(x_k - x_l) \prod_{k,l,k>l}(y_k - y_l)} \\
&= \frac{-900034788870031093153136640000}{46240323500727336960000} = -19464284
\end{aligned}$$

and therefore, the interpolation datum is

$$F(3,2) = \frac{-19464284}{4} = -4866071.$$

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