Differential geometry and multigrid for the div-grad, curl-curl and grad-div equations

Tim Boonen
Geoffrey Deliége
Stefan Vandewalle

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Katholieke Universiteit Leuven
Department of Computer Science
Celestijnenlaan 200A – B-3001 Heverlee (Belgium)

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Abstract

This paper is concerned with the application of principles of differential geometry in multigrid for the div-grad, curl-curl and grad-div equations. First, the discrete counterpart of the formulas for edge, face and volume elements are used to derive a sequence of a commuting edge, face and volume prolongator from an arbitrary partition of unity nodal prolongator. The implied coarse topology and the normalization of the prolongators are analyzed, and it is proved that they form a discrete de Rham sequence if they are normalized. Numerical results are presented for the resulting edge prolongator. It is shown that this edge prolongator is a generalization of the Reitzinger-Schöberl prolongator. Next, the partition of unity and commutation properties are used to prove that all matrices in a multigrid hierarchy for the considered equations can be factorized as a matrix product separating the metric and topological information. Finally, those properties are identified as requirements for the multigrid restriction to reflect the typical topological characteristics of the div-grad and curl-curl equations.

Keywords : multigrid, finite elements, differential geometry, de Rham complex, commutation
AMS(MOS) Classification : Primary : 65N55, Secondary : 65N30, 58J10.

Differential geometry and multigrid for the div-grad, curl-curl and grad-div equations

Tim Boonen[*], Geoffrey Deliège, Stefan Vandewalle

1 Introduction

This paper is concerned with the application and identification of principles from differential geometry in multigrid for finite element discretizations of the div-grad (diffusion), curl-curl and grad-div equations

$$div(\epsilon\, grad(V)) = \rho \quad \text{with } \epsilon > 0, \tag{1}$$

$$curl(\nu\, curl(\vec{A})) = \vec{J} \quad \text{with } \nu > 0, \tag{2}$$

$$grad(\alpha\, div(\vec{X})) = \vec{y} \quad \text{with } \alpha > 0 \tag{3}$$

on tetrahedral meshes with piecewise linear shape functions. Differential geometry offers a natural framework for the analysis of those equations and their discretizations [1, 4, 11]. In differential geometry, the study of the topological properties is separated from the metric properties (e.g. material parameters, distances). These topological properties are embodied in the topological operators gradient, curl and divergence, that are related in a so-called de Rham complex. With Ω the computational domain, the de Rham complex in the continuum describes the relation of the sequence of spaces

$$H^1(\Omega) \quad \xrightarrow{grad} \quad H(curl;\Omega) \quad \xrightarrow{curl} \quad H(div;\Omega) \quad \xrightarrow{div} \quad L^2(\Omega) \tag{4}$$

and the topological operators, with cst denoting a constant scalar function:

$$Im\,(cst) \subset Ker\,(grad)\,, \quad Im\,(grad) \subset Ker\,(curl)\,, \quad Im\,(curl) \subset Ker\,(div) \tag{5}$$

These relations induce the splittings $H^1(\Omega) = Ker(grad) \oplus Ker^{\perp}(grad)$, $H(curl;\Omega) = Ker(curl) \oplus Ker^{\perp}(curl)$ and $H(div;\Omega) = Ker(div) \oplus Ker^{\perp}(div)$. Typically, PDEs behave differently in different parts of such a splitting. For instance, the eddy current extension of the curl-curl equation $curl(\nu\, curl(\vec{A})) + \sigma\, \partial\vec{A}/\partial t = \vec{J}$ is only elliptic in $Ker^{\perp}(curl)$ [12].
For the sake of numerical stability, discrete versions of (4) and (5) should be satisfied in the discretization [1]. It is well known that this holds for the finite element spaces spanned by the usual lowest order nodal, edge, face and volume elements on a simplicial mesh [1, 4]. In this paper, it will be shown that it is possible as well to build a de Rham complex on the

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[*] Corresponding author: tim.boonen@cs.kuleuven.be. Research Assistent of the Fund for Scientific Research - Flanders (Belgium) (F.W.O.-Vlaanderen)

coarse levels of a multigrid hierarchy for the finite element discretizations of (1), (2) and (3) on a tetrahedral mesh with piecewise linear shape functions. For this sake, the formulas for edge, face and volume elements on tetrahedral meshes will be used to derive a commuting edge, face and volume prolongator from an arbitrary partition of unity nodal prolongator P_n. This procedure can be repeated recursively. It will be shown that the partition of unity and commutation properties can be used to factorize all matrices in the multigrid hierarchy as a product reflecting the structure of the underlying equation. This factorization is important, since it shows that all system matrices in the multigrid hierarchy are structurally identical, which is required for the recursive application of most multigrid algorithms to make sense. The partition of unity and the commutation properties are also requirements for an appropriate conservation of the righthand sides by the restriction in a multigrid algorithm.

This paper is organized as follows. In Section 2, the formulas for the edge, face and volume prolongator will be derived. The implied coarse topology and the impact of the normalization of the prolongators will be discussed. Next, the commutation properties and the existence of the coarse de Rham complex will be proved. Finally, it will be shown that on simplicial meshes, the proposed edge prolongator can be considered a generalization of the so-called Reitzinger-Schöberl prolongator, and some numerical results will be presented. In Section 3, the factorization of the matrices in a multigrid hierarchy for the div-grad, curl-curl and grad-div equations will be proved. In Section 4, it will be shown that the partition of unity and the edge-node commutation property are requirements for an appropriate restriction in multigrid for the diffusion and curl-curl equations.

The following notations will be used. Small letters and capital letters will denote entities on the fine and coarse level respectively. If this is not sufficient to indicate the level, superscripts '$...^f$' and '$...^c$' will be used to refer to the fine resp. coarse level. A node will be denoted as n_i. The oriented edge from n_i to n_j will be denoted as e_{ij}. The oriented triangle f_{ijk} containing the nodes n_i, n_j, n_k refers to the triangle with normal such that the rotation indicated by the sequence of the nodes is positive w.r.t that normal. The positive direction of the boundary surface of an oriented tetraeder v_{ijkl} is indicated by the direction of the normal of the face f_{ijk}. A nodal element basis function (or 'nodal element' for short) for n_i will be denoted N_i. An edge element for e_{ij} will be denoted $\vec{E}_{ij}$. A face element for f_{ijk} will be denoted $\vec{F}_{ijk}$. A volume element for v_{ijkl} will be denoted V_{ijkl}.

2 Prolongators derived from a partition of unity nodal prolongator

In this section, an edge, face and volume prolongator are derived from a partition of unity nodal prolongator for a tetrahedral fine mesh, and their properties are studied. An analysis of the implied coarse topology will show that this procedure can be repeated recursively. It is also shown that this procedure is in general impossible for non-simplicial meshes.

2.1 Edge prolongator

Suppose a partition of unity nodal prolongator P_n for a tetrahedral fine mesh is given. Such a prolongator is represented by a matrix with all of its row sums equal to one,

$$\sum_K P_n(i,K) = 1 \qquad \forall \text{ fine node } n_i. \tag{6}$$

The I'th column of P_n can be considered as a representation of a coarse nodal element:

$$N_I = \sum_k P_n(k,I)\, N_k. \tag{7}$$

Next, we will derive an edge prolongator P_e, by constructing the analogon of (7) for edge elements. For that purpose, we will use the formula for edge elements $\vec{E}_{ij}$, which reads for tetrahedral meshes (see [4, 8]):

$$\vec{E}_{ij} \quad = \quad N_i\, grad(N_j) - N_j\, grad(N_i). \tag{8}$$

Assuming that (8) also holds on the coarse mesh, and using (7), we have

$$\begin{aligned}\vec{E}_{IJ} &= \left(\sum_k P_n(k,I)N_k\right) grad\left(\sum_l P_n(l,J)N_l\right) - \left(\sum_l P_n(l,J)N_l\right) grad\left(\sum_k P_n(k,I)N_k\right)\\ &= \sum_k\sum_l P_n(k,I)P_n(l,J)\left(N_k grad(N_l) - N_l grad(N_k)\right)\\ &= \sum_k\sum_l P_n(k,I)P_n(l,J)\vec{E}_{kl},\end{aligned} \tag{9}$$

where in the last step, we took into account that the fine mesh is tetrahedral. In that case, the existence of the fine edge e_{kl} is guaranteed if $N_k grad(N_l) - N_l grad(N_k) \neq 0$. Considering that if $k = l$ then $\vec{E}_{kl} = 0$, and that $\vec{E}_{kl} = -\vec{E}_{lk}$, we can simplify (9):

$$\begin{aligned}\vec{E}_{IJ} &= \sum_{k<l}\left(P_n(k,I)P_n(l,J) - P_n(l,I)P_n(k,J)\right)\vec{E}_{kl}\\ &= \sum_{k<l}\begin{vmatrix} P_n(k,I) & P_n(k,J) \\ P_n(l,I) & P_n(l,J)\end{vmatrix}\vec{E}_{kl} = \sum_{k<l} P_e(kl,IJ)\vec{E}_{kl}.\end{aligned} \tag{10}$$

The latter equality allows to identify the elements of the edge prolongator P_e, and it shows that the derived coarse edge elements are linear combinations of fine edge elements. We propose to use formula (10) to derive an edge prolongator P_e from the nodal prolongator P_n, *regardless of the underlying coarse mesh topology.*

Remark 2.1 The derivation also holds for triangular fine meshes.

Remark 2.2 Note that the underlying mesh does not strictly need to be tetrahedral. For the validity of the last step in (9), it suffices that the following condition is satisfied: for each couple of overlapping fine nodal elements N_i and N_j, the fine edge e_{ij} must exist. This does not hold for instance on rectangular or hexahedral meshes.

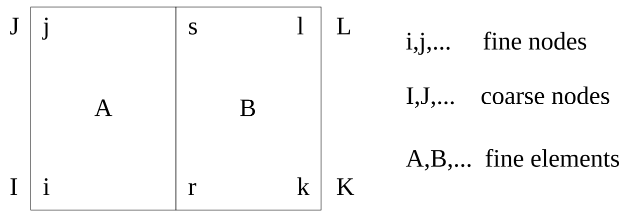

Figure 1: Coarsening on a 2D grid of quadrilaterals.

Remark 2.3 A similar procedure is impossible for the formula for edge elements on quadrilateral or hexahedral fine meshes. With $\mathcal{F}_{i\bar{j}}$ the faces containing node i and *not* containing node j, this formula reads (see [8]):

$$\vec{E}_{ij} \quad = \quad N_j \sum_{r\in\mathcal{F}_{\bar{i}j}} grad(N_r) - N_i \sum_{r\in\mathcal{F}_{i\bar{j}}} grad(N_r). \tag{11}$$

A counterexample is illustrated on Figure 1. Using (7) and the partition of unity nodal prolongator

$$P_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & 1-a & 0 \\ 0 & b & 0 & 1-b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

formula (11) applied to the coarse edge e_{IJ} of the coarse quadrilateral that is the union of A and B, becomes:

$$\begin{aligned} \vec{E}_{IJ} &= N_J\, grad(N_J + N_L) - N_I\, grad(N_I + N_K) \\ &= (N_j + b\, N_s)\, grad(N_j + N_s + N_l) - (N_i + a\, N_r)\, grad(N_i + N_r + N_k) \\ &= \vec{E}_{ij} + b\, N_s\, grad(N_j + N_s + N_l) - a\, N_r\, grad(N_i + N_r + N_k), \end{aligned}$$

where in the last step, we took into account that $N_j\, grad(N_l) = 0 = N_i\, grad(N_k)$. If $a = b$, the coarse edge element $\vec{E}_{IJ}$ can be written as a linear combination of fine edge elements $\vec{E}_{IJ} = \vec{E}_{ij} + a\,\vec{E}_{rs}$. If $a \neq b$, this is not possible anymore. Suppose fine element A is the unit square, and consider for instance the restriction of $\vec{E}_{IJ}$ to A:

$$\left(\vec{E}_{IJ}\right)_{|A} \quad = \quad \begin{bmatrix} 0 \\ 1-x \end{bmatrix} + b\,xy \begin{bmatrix} 0 \\ 1 \end{bmatrix} - a\,(1-y)x \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1-(1-a)x+(b-a)xy \end{bmatrix}.$$

Clearly, the coarse edge elements based on (11) will in general be piecewise polynomials of second order. So, it will be impossible to rewrite them as linear combinations of fine edge elements, which are piecewise polynomials of first order.

2.2 Face prolongator

We will derive a face prolongator P_f from a partition of unity nodal prolongator P_n for a tetrahedral fine mesh, by constructing the analogon of (7) for face elements. For that purpose, we will use the formula for face elements $\vec{F}_{ijk}$, which reads for tetrahedral meshes [4]:

$$\vec{F}_{ijk} = 2N_i grad(N_j) \times grad(N_k) + 2N_j grad(N_k) \times grad(N_i) + 2N_k grad(N_i) \times grad(N_j).$$

For piecewise linear nodal elements N_i on tetrahedral meshes, we can use (8) to find

$$\begin{aligned} curl(\vec{E}_{ij}) &= curl\left(N_i grad(N_j) - N_j grad(N_i)\right) \\ &= 2grad(N_i) \times grad(N_j). \end{aligned} \tag{12}$$

Hence, the formula for face elements can be rewritten as

$$\vec{F}_{ijk} = N_i curl(\vec{E}_{jk}) + N_j curl(\vec{E}_{ki}) + N_k curl(\vec{E}_{ij}). \tag{13}$$

In order to identify the face prolongator, we assume that (13) also holds on the coarse mesh. Using (7) and (9), we have

$$\begin{aligned} \vec{F}_{IJK} &= \left(\sum_r P_n(r,I)N_r\right) curl\left(\sum_s\sum_t P_n(s,J)P_n(t,K)\vec{E}_{st}\right) \\ &\quad + \left(\sum_s P_n(s,J)N_s\right) curl\left(\sum_t\sum_r P_n(t,K)P_n(r,I)\vec{E}_{tr}\right) \\ &\quad + \left(\sum_t P_n(t,K)N_t\right) curl\left(\sum_r\sum_s P_n(r,I)P_n(s,J)\vec{E}_{rs}\right) \\ &= \sum_r\sum_s\sum_t P_n(r,I)P_n(s,J)P_n(t,K)\left(N_r\, curl(\vec{E}_{st}) + N_s\, curl(\vec{E}_{tr}) + N_t\, curl(\vec{E}_{rs})\right) \\ &= \sum_r\sum_s\sum_t P_n(r,I)P_n(s,J)P_n(t,K)\vec{F}_{rst}. \end{aligned} \tag{14}$$

where in the last step, we took into account that the fine mesh is tetrahedral. In that case, the fine face f_{rst} is guaranteed to exist if $N_r\, curl(\vec{E}_{st}) + N_s\, curl(\vec{E}_{tr}) + N_t\, curl(\vec{E}_{rs}) \neq 0$. Considering that $\vec{F}_{rst} = 0$ if two of the indices r, s, t are equal and taking into account that

$$\vec{F}_{rst} = \vec{F}_{str} = \vec{F}_{trs} = -\vec{F}_{srt} = -\vec{F}_{tsr} = -\vec{F}_{rts},$$

we can rewrite summation (14) as:

$$\begin{aligned} \vec{F}_{IJK} &= \sum_{r<s<t}\left(\begin{array}{rl} & P_n(r,I)P_n(s,J)P_n(t,K) - P_n(r,I)P_n(t,J)P_n(s,K) \\ - & P_n(s,I)P_n(r,J)P_n(t,K) + P_n(s,I)P_n(t,J)P_n(r,K) \\ + & P_n(t,I)P_n(r,J)P_n(s,K) - P_n(t,I)P_n(s,J)P_n(r,K) \end{array}\right)\vec{F}_{rst} \\ &= \sum_{r<s<t}\begin{vmatrix} P_n(r,I) & P_n(r,J) & P_n(r,K) \\ P_n(s,I) & P_n(s,J) & P_n(s,K) \\ P_n(t,I) & P_n(t,J) & P_n(t,K) \end{vmatrix}\vec{F}_{rst}. \end{aligned} \tag{15}$$

This shows that the derived coarse face elements are linear combinations of fine face elements and it allows to identify the elements of the corresponding face prolongator P_f. We propose to use formula (15) to derive a face prolongator P_f from the nodal prolongator P_n, *regardless of the underlying coarse mesh topology.*

Remark 2.4 Formula (13) is not applicable to triangular meshes. In fact, faces degenerate to edges in the 2D case, and the 2D face elements only depend on two nodal elements instead of three. So, the above derivation does not hold for triangular fine meshes.

Remark 2.5 Note that the underlying mesh does not strictly need to be tetrahedral. For the validity of the last step in (14), it suffices that the following condition is satisfied: for each triplet of overlapping fine nodal elements N_i, N_j and N_k, the fine face f_{ijk} must exist. This does not hold for instance on hexahedral meshes.

Remark 2.6 A similar procedure for face elements on hexahedral meshes is impossible for the same reason as in Remark 2.3.

2.3 Volume prolongator

In an analogue way, coarse volume elements can be derived from the formula for volume elements V_{ijkl} on tetrahedral meshes:

$$V_{ijkl} \quad = \quad N_i\, div(\vec{F}_{jkl}) + N_j\, div(\vec{F}_{kli}) + N_k\, div(\vec{F}_{lij}) + N_l\, div(\vec{F}_{ijk}). \tag{16}$$

The resulting coarse volume elements are linear combinations of fine volume elements, which allows to identify the elements of the corresponding volume prolongator P_v:

$$V_{IJKL} \quad = \quad \sum_{r<s<t<u} \begin{vmatrix} P_n(r,I) & P_n(r,J) & P_n(r,K) & P_n(r,L) \\ P_n(s,I) & P_n(s,J) & P_n(s,K) & P_n(s,L) \\ P_n(t,I) & P_n(t,J) & P_n(t,K) & P_n(t,L) \\ P_n(u,I) & P_n(u,J) & P_n(u,K) & P_n(u,L) \end{vmatrix} V_{rstu}. \tag{17}$$

We propose to use formula (17) to derive a volume prolongator P_v from the nodal prolongator P_n, *regardless of the underlying coarse mesh topology.* A similar procedure is impossible on triangular and hexahedral meshes for the same reasons as in Remark 2.4 and 2.3 respectively.

2.4 Implied coarse topology

Although formulas (8), (13) and (16) hold in principle only for triangular or tetrahedral meshes, it is *formally* possible to apply them to any couple, triplet and quadruplet of nodal elements on arbitrary meshes. In fact, the proposed edge, face and volume prolongators of (10), (15) and (17) are based on this approximation.

The application of these formulas implies a topology. Indeed, a couple, triplet or quadruplet of nodes can be considered connected by an implied edge, face or volume, if the application of the appropriate formula to their associated nodal elements is different from zero. This result can only be different from zero if the nodal elements involved have an overlapping support. For instance, two nodes n_i and n_j can be considered connected by an implied edge e_{ij}, if $\vec{E}_{ij}$ computed from (8) is nonzero, i.e. their corresponding nodal elements N_i and N_j have an

overlapping support.
Because of the nature of the formulas, this implied topology will consist of triangular faces and tetrahedral volumes, regardless of the real underlying topology. If the real topology is tetrahedral, the real and the implied topology will be identical. However, if the real topology is not tetrahedral, the implied topology will contain crossing edges, crossing and overlapping triangular faces, and overlapping tetrahedral volumes, see Figure 2.
In the same way, formulas (10), (15) and (17) imply a coarse topology when applied to the same nodal prolongator P_n. For instance, the coarse nodes n_I and n_J can be considered connected by an implied coarse edge e_{IJ} if $\vec{E}_{IJ}$ computed by (10) is different from zero, i.e. if the coarse nodal elements N_I and N_J have overlapping support. The implied coarse topology will consist of (logically) triangular faces and (logically) tetrahedral volumes, and it will contain in general crossing edges, crossing and overlapping faces, and overlapping volumes. This is illustrated in the right panel of Figure 2.
A discrete topology is described totally by its discrete gradient G, its discrete curl C and its discrete divergence D [7]. These topological operators describe respectively the edges, faces and volumes of the topology:

$$\forall e_{ij} : G(ij,k) = \begin{cases} -1 & \text{if} \quad n_i = n_k \\ 1 & \text{if} \quad n_j = n_k \\ 0 & \text{else} \end{cases} \tag{18}$$

$$\forall f_{ijk} : \, C(ijk,rs) = \begin{cases} 1 & \text{if} \quad e_{rs} = e_{ij}, e_{jk} \text{ or } e_{ki} \\ -1 & \text{if} \quad e_{rs} = e_{ji}, e_{kj} \text{ or } e_{ik} \\ 0 & \text{else} \end{cases} \tag{19}$$

$$\forall v_{ijkl} : \, D(ijkl,rst) = \begin{cases} 1 & \text{if} \quad f_{rst} = f_{ijk}, f_{lkj}, f_{ikl} \text{ or } f_{jil} \\ -1 & \text{if} \quad f_{rst} = f_{jik}, f_{klj}, f_{kil} \text{ or } f_{ijl} \\ 0 & \text{else} \end{cases} \tag{20}$$

The coarse topological operators G^c, C^c and D^c, describing the implied coarse edges, faces and volumes, can be built using the following algorithms, extending the algorithm for building the coarse gradient in [16]:

$$\exists\, e_{kl} : \left\{ \begin{array}{lcl} P_n(k,I) & \neq & 0 \\ P_n(l,J) & \neq & 0 \end{array} \right. \iff \exists\, p : \left\{ \begin{array}{lcl} G^c(IJ,I) & = & (-1)^p \\ G^c(IJ,J) & = & (-1)^{p+1} \end{array} \right. \tag{21}$$

$$\exists\, f_{rst} : \left\{ \begin{array}{lcl} P_n(r,I) & \neq & 0 \\ P_n(s,J) & \neq & 0 \\ P_n(t,K) & \neq & 0 \end{array} \right. \iff \exists\, p : \left\{ \begin{array}{lcl} C^c(IJK,IJ) & = & (-1)^p \\ C^c(IJK,JK) & = & (-1)^p \\ C^c(IJK,KI) & = & (-1)^p \end{array} \right. \tag{22}$$

$$\exists\, v_{rstu} : \left\{ \begin{array}{lcl} P_n(r,I) & \neq & 0 \\ P_n(s,J) & \neq & 0 \\ P_n(t,K) & \neq & 0 \\ P_n(u,L) & \neq & 0 \end{array} \right. \iff \exists\, p : \left\{ \begin{array}{lcl} D^c(IJKL,IJK) & = & (-1)^p \\ D^c(IJKL,KLI) & = & (-1)^p \\ D^c(IJKL,JKL) & = & (-1)^{p+1} \\ D^c(IJKL,LIJ) & = & (-1)^{p+1} \end{array} \right. \tag{23}$$

The orientations are determined by p and can be chosen arbitrarily. For normalization purposes (see Remark 2.9), the coarse volume elements should be oriented outward. The corresponding dual coarse differential operators are defined as usual [7]:

$$\tilde{G}^c = -D^{cT}, \quad \tilde{C}^c = C^{cT}, \quad \tilde{D}^c = -G^{cT}. \tag{24}$$

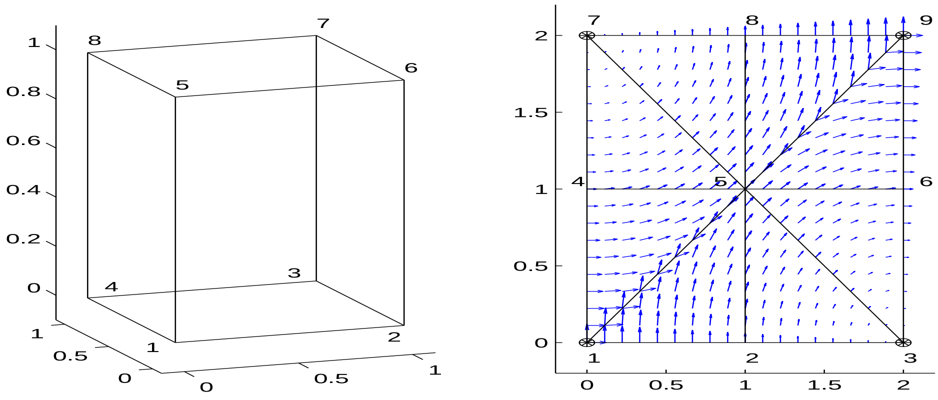

Figure 2: Illustration of the possible difference between the real topology and the topology implied by formulas (8), (13) and (16). For a fine hexaeder (left), all eight nodal elements have an overlapping support. Hence all couples, triplets and non-coplanar quadruplets of nodes are connected by an implied edge, face and volume respectively. The implied edges can be crossing (e.g. e_{13} and e_{24}). The implied faces are triangular and can be overlapping (e.g. f_{123} and f_{124}) or crossing (e.g. f_{135} and f_{246}). The implied volumes are tetrahedral and can be overlapping (e.g. v_{1235} and v_{1236}). For a coarse 2D example (right) with coarse nodes n_1^c, n_3^c, n_7^c and n_9^c, the situation is similar if the coarse nodal elements N_1^c, N_3^c, N_7^c and N_9^c all prolongate to the fine center node n_5. In this case, the implied coarse topology will for instance contain crossing edges (e_{19}^c and e_{37}^c). The figure shows a plot of the coarse edge element $\vec{E}_{19}^c$ derived by (10) from a linear nodal prolongator.

Remark 2.7 The derivation of the formulas (10), (15) and (17) still holds if a mesh topology with the characteristics of an implied topology (as described above) is used as the fine mesh. Indeed, the only requirement related to the fine mesh topology is that for any couple, triplet or quadruplet of overlapping fine nodal elements, a fine edge, face or volume exists (see Remarks 2.2 and 2.5). This requirement is satisfied for the implied topology described by (21), (22) and (23), which allows to use formulas (10), (15) and (17) in a recursive way.

Remark 2.8 Formulas (10), (15) and (17) only allow to identify which coarse nodes are connected by a coarse edge, face or volume. They *don't* allow to trace a coarse edge, face or volume in the fine mesh as an oriented set of fine edges, faces or volumes. This will only be possible if the coarse nodal, edge, face and volume elements are normalized, see Section 2.5.

2.5 Normalization

The usual lowest order nodal, edge, face and volume elements on tetrahedral meshes are normalized as

$$N_i(n_j) = \int_{e_j} \vec{E}_i \, d\vec{l} = \iint_{f_j} \vec{F}_i \, d\vec{\Omega} = \iiint_{v_j} V_i \, dV \quad = \quad \delta_{ij}.$$

Thanks to this normalization, the corresponding degrees of freedom (DoF) have a clear meaning with respect to the underlying problem: a nodal DoF represents the value of the solution in its associated node, an edge DoF represents the line integral of the solution along its associated edge, a face DoF represents the surface integral of the solution over its associated face, and a volume DoF represents the volume integral of the solution over its associated volume:

$$\begin{array}{llllll} S = \sum_i \alpha_i \, N_i & \Longleftrightarrow & S(n_i) = \alpha_i & \vec{S} = \sum_i \alpha_i \, \vec{E}_i & \Longleftrightarrow & \int_{e_i} \vec{S} \, d\vec{l} = \alpha_i \\ S = \sum_i \alpha_i \, V_i & \Longleftrightarrow & \iiint_{v_i} S \, dV = \alpha_i & \vec{S} = \sum_i \alpha_i \, \vec{F}_i & \Longleftrightarrow & \iint_{f_i} \vec{S} \, d\vec{\Omega} = \alpha_i \end{array}$$

From this point of view, it would be desirable to work with normalized shape functions on all levels of a multigrid hierarchy as well. In the context of multigrid algorithms, normalization can even be considered a *requirement* for the efficiency of the solver. Indeed, if the coarse shape functions are not normalized, the resulting coarse system matrix is scaled by left and right multiplication with a diagonal matrix. It is well known that multigrid methods can be very sensitive to such scaling. Hence, normalization of the coarse shape functions is an important issue from the point of view of multigrid algorithms. In this section, the requirements for normalization will be derived. This will reveal that normalization of the nodal, edge, face and volume elements of formulas (10), (15) and (17) implies the absence of crossing edges, crossing or overlapping faces and overlapping volumes.

2.5.1 Normalization of the coarse edge elements.

We are interested in the value of the line integral $\int_{e_{KL}} \vec{E}_{IJ} \, d\vec{l}$ of an implied coarse edge element $\vec{E}_{IJ}$ along a coarse edge e_{KL}. Since the formula for coarse edge elements (10) does not allow to trace a coarse edge on the fine mesh, we consider an arbitrary path p_{KL} from the coarse node n_K to the coarse node n_L. Suppose the coarse nodal elements are nonnegative and normalized, i.e. $N_I(n_J) = \delta_{IJ}$. Let $X(p)$ denote the defect of the partition of unity of the coarse nodal elements N_I and N_J along p_{KL}:

$$\forall p \in p_{KL} : \; N_I(p) + N_J(p) + X(p) = 1, \; \text{with } X(p) \geq 0.$$

The line integral of $\vec{E}_{IJ}$ along an arbitrary path p_{KL} reads:

$$\begin{aligned} \int_{p_{KL}} \vec{E}_{IJ} \, d\vec{l} &= \int_{p_{KL}} (N_I \, grad(N_J) - N_J \, grad(N_I)) \, d\vec{l} \qquad (25) \\ &= \int_{p_{KL}} (N_I \, grad(1 - N_I - X) - (1 - N_I - X) \, grad(N_I)) \, d\vec{l} \\ &= \int_{p_{KL}} -grad(N_I) \, d\vec{l} - \int_{p_{KL}} (N_I \, grad(X) - X \, grad(N_I)) \, d\vec{l}. \qquad (26) \end{aligned}$$

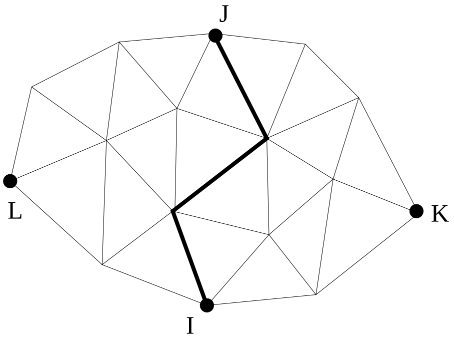

Figure 3: A zero defect path connecting 2 coarse nodes cannot be crossed by any other coarse edge.

For an associated path p_{IJ}, this becomes, thanks to the normalization of N_I:

$$\int_{p_{IJ}} \vec{E}_{IJ}\, d\vec{l} \;=\; 1 - \int_{p_{IJ}} (N_I\, grad(X) - X\, grad(N_I))\, d\vec{l}. \tag{27}$$

In general, this line integral will only be equal to 1 for a path with zero defect $X(p)$ for all $p \in p_{IJ}$. This implies that all points along p_{IJ} can only receive prolongation contributions from N_I and N_J. If such a path is found as a sequence of fine edges, it is a valid candidate to be identified as the coarse edge e_{IJ}. In 2D, a path with zero defect cannot be crossed by coarse edges. Indeed, the supports of any two coarse nodal elements at different sides of such path are separated by this path. So, their associated coarse nodes are not connected by an implied coarse edge, see Figure 3. For all other paths p_{KL}, line integral (25) will only become zero in general if $N_I(p) = 0$ or $N_J(p) = 0$ for all $p \in p_{KL}$. This would imply that the support of N_I and/or N_J does not cross p_{KL}, and hence that p_{KL} is not crossed by a coarse edge connecting n_I and n_J.

So, the normalization of the coarse nodal elements is a first requirement for the normalization of the coarse edge elements. Additionally, a coarse edge element can only be normalized in general if it is not crossed by another coarse edge: if the coarse edge e_{IJ} is crossed by some other coarse edge e_{KL}, there will be no path p_{IJ} along which the line integral of $\vec{E}_{IJ}$ is 1, and their will be no path p_{KL} along which the line integral of $\vec{E}_{IJ}$ is 0.

2.5.2 Normalization of the coarse face elements.

We are interested in the value of the surface integral $\iint_{f_{RST}} \vec{F}_{IJK}\, d\vec{\Omega}$ of a coarse face element $\vec{F}_{IJK}$ over a coarse face f_{RST}. Assume the coarse nodal elements N_R, N_S and N_T and the coarse edge elements $\vec{E}_{RS}, \vec{E}_{ST}$ and $\vec{E}_{TR}$ are normalized. This implies that the coarse edges e_{RS}, e_{ST} and e_{TR} can be traced on the fine mesh. Since the formula for coarse face elements (15) does not allow to trace a coarse face on the fine mesh, we consider an arbitrary oriented surface s_{RST} with outer boundary defined by the coarse edges e_{RS}, e_{ST} and e_{TR}. Let X denote the defect of the partition of unity of the coarse nodal elements N_I, N_J and N_K over s_{RST}:

$$\forall s \in s_{RST} :\; N_I(s) + N_J(s) + N_K(s) + X(s) = 1, \text{ with } X(s) \geq 0.$$

The surface integral of $\vec{F}_{IJK}$ over s_{RST} can be computed as:

$$\iint_{s_{RST}} \vec{F}_{IJK}\, d\vec{\Omega} = \iint_{s_{RST}} \begin{pmatrix} 2N_I\, grad(N_J) \times grad(N_K) \\ +\ 2N_J\, grad(N_K) \times grad(N_I) \\ +\ 2N_K\, grad(N_I) \times grad(N_J) \end{pmatrix} d\vec{\Omega} \tag{28}$$

$$= \iint_{s_{RST}} \begin{pmatrix} -\ 2N_I\, grad(N_J) \times grad(N_I + N_J + X) \\ -\ 2N_J\, grad(N_I + N_J + X) \times grad(N_I) \\ +\ 2(1 - N_I - N_J - X)\, grad(N_I) \times grad(N_J) \end{pmatrix} d\vec{\Omega}$$

$$= \iint_{s_{RST}} curl(\vec{E}_{IJ})\, d\vec{\Omega} - \int_{s_{RST}} \begin{pmatrix} 2N_I\, grad(N_J) \times grad(X) \\ +\ 2N_J\, grad(X) \times grad(N_I) \\ +\ 2X\, grad(N_I) \times grad(N_J) \end{pmatrix} d\vec{\Omega}, \tag{29}$$

where identity (12) is used to arrive at the last equality. For an associated oriented surface s_{IJK}, this becomes, thanks to the normalization of $\vec{E}_{IJ}$:

$$\iint_{s_{IJK}} \vec{F}_{IJK}\, d\vec{\Omega} \;=\; 1 - \int_{s_{RST}} \begin{pmatrix} 2N_I\, grad(N_J) \times grad(X) \\ +\ 2N_J\, grad(X) \times grad(N_I) \\ +\ 2X\, grad(N_I) \times grad(N_J) \end{pmatrix} d\vec{\Omega}. \tag{30}$$

In general, this surface integral will only be equal to 1 for a surface with zero defect $X(s)$ for all $s \in s_{IJK}$. This implies that all points in s_{IJK} can only receive prolongation contributions from N_I, N_J and N_K. If such a surface is found as a set of fine surfaces, it is a valid candidate to be identified as the coarse face f_{IJK}. In 3D, a surface with zero defect cannot be crossed by coarse edges or faces. Indeed, the supports of any two coarse nodal elements at different sides of such a surface are separated by this surface, so their associated coarse nodes cannot be connected by a an implied coarse edge or face. For all other surfaces s_{RST}, the line integral will only become zero in general if $N_I(s) = 0$, $N_J(s) = 0$ or $N_K(s) = 0$ for all $s \in s_{RST}$, as can be seen directly from (28). This would imply that the support of N_I, N_J and/or N_K does not cross s_{RST}, and hence that s_{RST} is not crossed by a coarse face connecting n_I, n_J and n_K. So, the normalization of the coarse edge elements is a first requirement for the normalization of the coarse face elements. Additionally, a coarse face element can only be normalized in general if it is not crossed by a coarse edge or by another coarse face.

Remark 2.9 In an analogous way, it can be shown that the normalization of the coarse face elements is a requirement for the normalization of the coarse volume elements, and that a coarse volume element can only be normalized if it does not have an overlap with other coarse volumes. Additionally, the volume integral of a normalized coarse volume element over its associated coarse volume will depend on the orientation of that volume: if oriented outward, the volume integral is 1, else -1.

Remark 2.10 The previous analysis shows that the constraints for the normalization of the coarse nodal, edge, face and volume elements are increasingly demanding. Indeed, the normalization of the shape functions associated with d-dimensional geometric entities requires the normalization of all shape functions associated with geometric entities of lower dimension. The normalization of the coarse nodal elements however is trivial, and is not troubled by crossing edges, crossing or overlapping faces, or overlapping volumes. Still, the implied coarse topology will have an impact on the fill-in of the matrices of the multigrid hierarchy. This observation suggests that the fill-in might be controlled by controlling the implied coarse topology.

Remark 2.11 In [15], the coarse topology is built first, and constraints in a form equivalent with the zero defect approach are imposed on the nodal prolongator to guarantee compatibility with that coarse topology. This implies that the coarse topology has no crossing edges, crossing or overlapping faces or overlapping volumes. However, the coarse faces and volumes are not guaranteed to be logically triangular and tetrahedral. This approach can be considered the inverse of the approach of Section 2.4, where the coarse topology is derived from a nodal prolongator.

2.6 Coarse de Rham complex

It is well known that the finite element spaces spanned by the usual lowest order nodal, edge, face and volume elements on some given tetrahedral mesh and the discrete gradient, curl and divergence form a de Rham complex [1, 4, 11]. In this section, we will prove that the range spaces of the nodal, edge, face and volume prolongators P_n, P_e, P_f and P_v derived from a partition of unity nodal prolongator P_n by the formulas (10), (15) and (17) and the coarse topological operators (21), (22) and (23) implied by P_n, form a de Rham complex as well if all prolongators have full rank.

First, we will prove several commutation properties relating the prolongators P_n, P_e, P_f, P_v and the fine and coarse topological operators. These commutation properties are useful on themselves, as they are often needed in a multigrid context. For instance, most multigrid algorithms for the curl-curl equation (2) are based on the node-edge commutation property [12, 16, 3, 13]. Next, the commutation properties will be used to prove the existence of the coarse de Rham complex if all prolongators have full rank. Finally, it will be shown that normalization of the coarse shape functions is a sufficient condition for the prolongators to have full rank.

In this section, it will be assumed that the formulas (10), (15), (17) and (21), (22), (23) are applicable. This amounts to requiring that the fine mesh topology is tetrahedral or that it has the characteristics of an implied topology (see Section 2.4).

Lemma 2.1 **(Edge-node commutation)** *If an edge prolongator P_e can be derived from a partition of unity nodal prolongator P_n by (10), then P_n and P_e satisfy the edge-node commutation property*

$$P_e\, G^c = G^f\, P_n. \tag{31}$$

PROOF.

Consider the righthand side of (31). Using (18), we have

$$(G^f\, P_n)(ij, K) = G^f(ij, :)\, P_n(:, K) = P_n(j, K) - P_n(i, K).$$

The same entry of the lefthand side reads, using (21),

$$\begin{aligned}(P_e\, G^c)(ij, K) &= P_e(ij, :)\, G^c(:, K) = \sum_{L \neq K} - \begin{vmatrix} P_n(i,K) & P_n(i,L) \\ P_n(j,K) & P_n(j,L) \end{vmatrix} \\ &= \begin{vmatrix} P_n(j,K) & \sum_L P_n(j,L) \\ P_n(i,K) & \sum_L P_n(i,L) \end{vmatrix} \overset{(6)}{=} P_n(j,K) - P_n(i,K).\end{aligned}$$

Since the equality holds for all entries, the commutation property is proved.

Lemma 2.2 (Face-edge commutation) *If an edge and face prolongator P_e and P_f can be derived from a partition of unity nodal prolongator P_n using (10) and (15), then P_e and P_f satisfy the face-edge commutation property*

$$P_f\, C^c = C^f\, P_e. \tag{32}$$

Proof.
Consider the righthand side of (32). Using (19), we have

$$(C^f\, P_e)(rst, IJ) \quad = \quad C^f(rst, :)\, P_e(:, IJ) = P_e(rs, IJ) + P_e(st, IJ) + P_e(tr, IJ).$$

The same entry of the lefthand side reads, using (22):

$$\begin{aligned}
(P_f\, C^c)(rst, IJ) \quad &= \quad \sum_K P_f(rst, IJK) \\
&= \quad \sum_K \begin{vmatrix} P_n(r,I) & P_n(r,J) & P_n(r,K) \\ P_n(s,I) & P_n(s,J) & P_n(s,K) \\ P_n(t,I) & P_n(t,J) & P_n(t,K) \end{vmatrix} \\
&= \quad \begin{vmatrix} P_n(r,I) & P_n(r,J) & \sum_K P_n(r,K) \\ P_n(s,I) & P_n(s,J) & \sum_K P_n(s,K) \\ P_n(t,I) & P_n(t,J) & \sum_K P_n(t,K) \end{vmatrix} \\
&\overset{(6)}{=} \quad \begin{vmatrix} P_n(s,I) & P_n(s,J) \\ P_n(t,I) & P_n(t,J) \end{vmatrix} - \begin{vmatrix} P_n(r,I) & P_n(r,J) \\ P_n(t,I) & P_n(t,J) \end{vmatrix} + \begin{vmatrix} P_n(r,I) & P_n(r,J) \\ P_n(s,I) & P_n(s,J) \end{vmatrix} \\
&\overset{(10)}{=} \quad P_e(st, IJ) + P_e(tr, IJ) + P_e(rs, IJ).
\end{aligned}$$

Since the equality holds for all entries, the commutation property is proved.

Lemma 2.3 (Volume-face commutation) *If a face and volume prolongator P_f and P_v can be derived from a partition of unity nodal prolongator P_n using (15) and (17), then P_f and P_v satisfy the volume-face commutation property*

$$P_v\, D^c = D^f\, P_f. \tag{33}$$

Proof: analogous to the proofs of Lemmas 2.1 and 2.2

Theorem 2.1 (Coarse De Rham sequence) *Suppose the fine discrete nodal, edge, face and volume spaces $\mathcal{N}^f, \mathcal{E}^f, \mathcal{F}^f$ and $\mathcal{V}^f$ and the fine discrete gradient, curl and divergence G^f, C^f and D^f form a de Rham complex. If an edge, face and volume prolongator P_e, P_f and P_v can be derived from a partition of unity nodal prolongator P_n by (10), (15) and (17), then their range spaces $\mathcal{N}^c, \mathcal{E}^c, \mathcal{F}^c$ and $\mathcal{V}^c$ and the coarse differential operators (21), (22) and (23) implied by P_n form a de Rham complex if P_e, P_f and P_v have full rank.*

Proof.
The total sequence of commutation properties (31), (32) and (33) can be represented as:

$$\begin{tikzcd}
\mathcal{N}^f \arrow[r, "G^f"] & \mathcal{E}^f \arrow[r, "C^f"] & \mathcal{F}^f \arrow[r, "D^f"] & \mathcal{V}^f \\
\mathcal{N}^c \arrow[r, "G^c"] \arrow[u, "P_n"] & \mathcal{E}^c \arrow[r, "C^c"] \arrow[u, "P_e"] & \mathcal{F}^c \arrow[r, "D^c"] \arrow[u, "P_f"] & \mathcal{V}^c \arrow[u, "P_v"]
\end{tikzcd} \tag{34}$$

Hence, the coarse spaces have the correct dimensions to form a de Rham complex. It remains to prove the properties (5). With P_n being partition of unity, P_e, P_f and P_v having full rank, and using the commutation properties (31), (32) and (33) and the properties (5) of the fine de Rham complex, we have:

$$\begin{array}{lllllllll}
P_e\, G^c\, 1^c & = & G^f\, P_n\, 1^c & = & G^f\, 1^f & \equiv & 0 & \Longrightarrow & G^c\, 1^c \equiv 0 \\
P_f\, C^c\, G^c & = & C^f\, P_e\, G^c & = & C^f\, G^f\, P_n & \equiv & 0 & \Longrightarrow & C^c\, G^c \equiv 0 \\
P_v\, D^c\, C^c & = & D^f\, P_f\, C^c & = & D^f\, C^f\, P_e & \equiv & 0 & \Longrightarrow & D^c\, C^c \equiv 0.
\end{array}$$

Lemma 2.4 *If all fine edge/face/volume elements and all coarse edge/face/volume elements derived from a full rank partition of unity nodal prolongator P_n using (10)/(15)/(17) are normalized, the associated prolongator $P_e/P_f/P_v$ has full rank.*

Proof.
We only prove the result for P_e. The proofs for P_f and P_v are analogous.
Because of the normalization, it is possible to trace all coarse edges on the fine mesh as an oriented sequence of fine edges. Using the normalization of the fine and coarse edge elements, the line integral of $\vec{E}_{IJ}$ along a coarse edge e_{KL} is:

$$\int_{e_{KL}} \vec{E}_{IJ}\, d\vec{l} = \int_{e_{KL}} \sum_{e_{rs}} P_e(rs, IJ)\, \vec{E}_{rs}\, d\vec{l} = \sum_{e_{rs} \in e_{KL}} P_e(rs, IJ) = \delta_{IJ,KL}.$$

Consider the oriented subsum corresponding to e_{IJ} of any linear combination of the columns of P_e. Using the previous results, this becomes:

$$\sum_{e_{rs} \in e_{IJ}} \sum_{e_{UV}} \alpha_{UV}\, P_e(rs, UV) = \sum_{e_{UV}} \alpha_{UV} \sum_{e_{rs} \in e_{IJ}} P_e(rs, UV) = \sum_{e_{UV}} \alpha_{UV}\, \delta_{UV,IJ} = \alpha_{IJ}.$$

Hence, the IJ^{th} column of P_e is linearly independent of the other columns of P_e. Since this holds for any column IJ, P_e has full rank.

2.7 Relation to the Reitzinger-Schöberl edge prolongator

Lemma 2.5 (Piecewise constant edge prolongator) *When applying (10) on a fine simplicial mesh to a piecewise constant nodal prolongator P_n*

$$\forall\, \textit{fine node } n_i, \exists!\, \textit{coarse node } n_K : P_n(i, L) = \left\{ \begin{array}{lll} 1 & \textit{if} & K = L \\ 0 & \textit{if} & K \neq L \end{array} \right. ,$$

the piecewise constant edge prolongator of [16] derived from P_n is obtained:

$$P_e(kl, IJ) = \begin{cases} \;\;1 & \textit{if} \quad P_n(k,I) = 1 = P_n(l,J), \\ -1 & \textit{if} \quad P_n(k,J) = 1 = P_n(l,I), \\ \;\;0 & \textit{else} \quad (e_{kl} \textit{ is a fine edge internal to a coarse node}). \end{cases}$$

PROOF.
Comparing the prolongation contributions of the two schemes for all possible combinations of fine nodes n_k, n_l and coarse nodes n_I, n_J proves the Lemma:

$P_n(k,I)$	$P_n(k,J)$	$P_n(l,I)$	$P_n(l,J)$	(10)	[16]
1	0	1	0	$\det\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$	0
1	0	0	1	$\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1
0	1	1	0	$\det\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	-1
0	1	0	1	$\det\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$	0
else				0	0

Remark 2.12 Lemma 2.5 shows that on simplicial meshes, the new edge prolongator (10) contains the prolongator of [16] as the special case for a piecewise constant nodal prolongator. Hence, on simplicial meshes, (10) can be considered a generalization of [16] for *arbitrary* partition of unity nodal prolongators.

Remark 2.13 Since on simplicial meshes, the edge prolongator of [16] is a special case of (10), the foundation in differential geometry of (10) carries over to [16]. On simplicial meshes, formula (10) can even be applied to a piecewise constant nodal prolongator without any restriction, because in this case, the coarse topology is simplicial as well.
This suggests that the coarse edge elements represented by the columns of the edge prolongator of [16] can be considered piecewise constant. This is somewhat counterintuitive, since the prolongation to all fine edges internal to a coarse node is always identical to 0. However, from the point of view of the restriction, this zero-prolongation is normal and can be explained from the underlying physics. For instance in the curl-curl equation for magnetostatics, the absence of some righthand side components in the restriction is similar to the far field annihilation of local current loops, see Figure 4.

Remark 2.14 It is immediately clear that the edge prolongator from [16] is normalized. Indeed, a coarse edge can be traced on the fine mesh as any of the fine edges to which it prolongates. Each of those fine edges forms a zero defect path.

2.8 Numerical results

In combination with an appropriate smoother, the edge prolongator of (10) can be applied in an AMG solver for the curl-curl equation. As an illustration, consider the discretization using first order edge elements of $curl(curl(\vec{A})) + 10^{-4}\,\vec{A} = \vec{J}$ on a triangle meshed by recursively

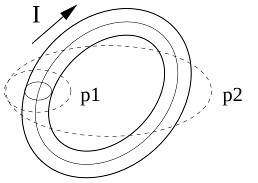

Figure 4: With respect to its far field magnetic effect, a current loop is self-annihilating. Consider the integral form of Ampères law $\oint_p \vec{H}\, d\vec{l} = I$. For the path p_1, the righthand side is I, but for any 'far field' path p_2, the righthand side is 0.

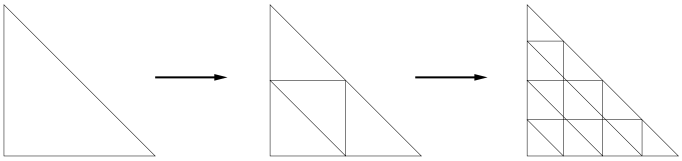

Figure 5: Uniform refinement of a triangle by adding new nodes in the middle of existing edges.

	standalone	with CG			
nbEdges	half(1,1)	half(1,1)	half(2,2)	full(1,1)	full(2,2)
1584	0.326(25)	0.104(13)	0.012(7)	0.054(10)	0.023(8)
6240	0.326(25)	0.105(13)	0.013(7)	0.054(10)	0.023(8)
24768	0.327(25)	0.105(13)	0.013(7)	0.054(10)	0.023(8)
98688	0.327(25)	0.105(13)	0.013(7)	0.054(10)	0.023(8)

Table 1: Geometric mean convergence factors and number of iterations for convergence to relative tolerance 10^{-12} of AMG with a V-cycle using the edge prolongator (10) and the hybrid smoother of [12] for the curl-curl equation discretized on a uniformly refined triangle (Figure 5). The column headers refer to the configuration of the hybrid smoother and the number of pre- and postsmoothing steps. For an explanation of the half and full hybrid smoother, see [17].

subdividing it as indicated in Figure 5. The edge prolongator is built by applying formula (10) to the nodal prolongator resulting from the so-called *direct interpolation* scheme (see [18]) for the nodal auxiliary matrix $G^T G$. For this particular problem, this amounts to simple linear interpolation. By this procedure, the implied coarse topologies are guaranteed to remain triangular, so (10) can be applied without any restriction. This problem is tailored towards (10), and the resulting convergence is as expected, see Table 1. The solver scales perfectly with the problem size: it has linear complexity $\mathcal{O}(N)$.

On unstructured simplicial meshes, the performance suffers from the problematic implied coarse topologies (see Sections 2.4 and 2.5). For 2D problems with up to 50.000 unknowns, the average convergence factors are in the range 0.3 - 0.45, comparable to the results in [3].

3 Factorization of the matrices of a multigrid hierarchy

In this section, it will be shown that all matrices of a multigrid hierarchy for the finite element discretization of the div-grad (diffusion), curl-curl and grad-div equation on a tetrahedral mesh can be factorized as a product reflecting the structure of the underlying differential operator, if the appropriate commutation properties are satisfied. For this purpose, the results of Section 2 are interpreted as a constructive proof that it is possible to satisfy those commutation properties if the fine mesh is tetrahedral. This factorization is important, since it shows that all system matrices in the multigrid hierarchy have the same structure with respect to the discrete de Rham complexes. The same kind of factorizations are figuring in multigrid for FIT [7, 6].

The derivations in Section 3.1 will hold for simplicial meshes, but the derivations in Sections 3.2 and 3.3 only for tetrahedral meshes. This is due to the fact that the derivations are based on the results of Section 2, which focus on tetrahedral meshes and only partially carry over to triangular meshes. However, this does *not* preclude the existence of similar factorizations for the curl-curl and grad-div equations on triangular meshes.

3.1 Diffusion equation

We will show that all matrices of a multigrid hierarchy for the finite element discretization of the diffusion equation (1) on a simplicial mesh using piecewise linear nodal elements, can be written as a product of the discrete dual divergence $\tilde{D}$, the edge element mass matrix M_ϵ^{edge} for the material parameter ϵ, and the discrete gradient G

$$K_{\text{diffusion}} = \tilde{D}\, M_\epsilon^{edge}\, G. \tag{35}$$

Lemma 3.1 *Piecewise linear nodal elements on simplicial meshes and the corresponding edge elements (8) satisfy*

$$\forall\, n_i : \; grad(N_i) = \sum_j \vec{E}_{ji}. \tag{36}$$

Proof.
Using (8) and the partition of unity property for the nodal elements $\sum_i N_i = 1$, we have:

$$\sum_j \vec{E}_{ji} \;\; = \;\; \sum_j N_j\, grad(N_i) - N_i\, grad(\sum_j N_j) \;\; = \;\; grad(N_i).$$

Theorem 3.1 *If the boundary conditions are not taken into account, the finite element discretization matrix of the diffusion equation (1) on a simplicial mesh using piecewise linear nodal elements can be written as the product of the discrete dual divergence $\tilde{D} = -G^T$, the edge element mass matrix M_ϵ^{edge}*

$$M_\epsilon^{edge}(ij, kl) = \iiint_V \epsilon\, \vec{E}_{ij} \vec{E}_{kl}\, dV, \tag{37}$$

and the discrete gradient G (18).

Proof.
With the signs accounted for by the order of the indices of the edge elements, we have

$$\begin{aligned}(-G^T\,M_\epsilon^{edge}\,G)(i,j) &= -G^T(:,i)\,M_\epsilon^{edge}\,G(:,j)\\ &\overset{(18)}{=} -\iiint_V \epsilon\left(\sum_k \vec{E}_{ki}\right)\left(\sum_l \vec{E}_{lj}\right)\,dV.\end{aligned}$$

Using (36) and $\tilde{D} = -G^T$ [7], this becomes:

$$(\tilde{D}\,M_\epsilon^{edge}\,G)(i,j) = -\iiint_V \epsilon\,grad(N_i)\,grad(N_j)\,dV \equiv K_{\text{diffusion}}(i,j).$$

Theorem 3.2 *The Galerkin coarse system matrix* $P_n^T\,K^f_{\mathit{diffusion}}\,P_n$ *can be factorized in the form (35), if* P_n *is a partition of unity nodal prolongator on which formula (10) can be applied and if* $K^f_{\mathit{diffusion}}$ *can be factorized as (35).*

Proof.
The coarse system matrix is given by:

$$K^c_{\text{diffusion}} = P_n^T\,K^f_{\text{diffusion}}\,P_n = -P_n^T\,G^{fT}\,M^f_{edge}\,G^f\,P_n.$$

Consider the edge prolongator P_e derived from P_n by (10) and the coarse gradient G^c (21) implied by P_n. Since commutation property (31) holds, this matrix can be rewritten as:

$$K^c_{\text{diffusion}} = -G^{cT}\,P_e^T\,M^f_{edge}\,P_e\,G^c.$$

Matrix $P_e^T\,M^f_{edge}\,P_e$ is the mass matrix for the coarse edge elements (10):

$$\begin{aligned}(P_e^T\,M^f_{edge}\,P_e)(I,J) &= \iiint_V \epsilon\left(\sum_r P_e(r,I)\vec{E}_r\right)\left(\sum_s P_e(s,J)\vec{E}_s\right)\,dV\\ &= \iiint_V \epsilon\,\vec{E}_I\,\vec{E}_J\,dV = M^c_{edge}(I,J).\end{aligned}$$

Hence, using $\tilde{D}^c = -G^{cT}$, we have the result

$$K^c_{\text{diffusion}} = -G^{cT}\,M^c_{edge}\,G^c = \tilde{D}^c\,M^c_{edge}\,G^c. \tag{38}$$

Equation (38) shows that the coarse system matrix can be considered a discretization of the diffusion operator on the mesh topology represented by G^c and implied by P_n. In general, this implied coarse topology will contain crossing edges, and the coarse edges cannot be traced on the fine grid, as shown in Section 2.4.

Together, Theorems 3.1 and 3.2 prove that all matrices of a multigrid hierarchy for the finite element discretization of the diffusion equation on simplicial meshes can be factorized as (35) if all nodal prolongators are partition of unity. Hence, all matrices have the same structure with respect to the appropriate discrete de Rham complexes (see Section 2.6):

$$M_{\text{diffusion}} = -G^T\,M_\epsilon^{edge}\,G \quad : \quad \mathcal{N} \xrightarrow{G} \mathcal{E} \xrightarrow{M_\epsilon^{edge}} \mathcal{E} \xrightarrow{-G^T} \mathcal{N}$$

This implies that at each level l, the behaviour of the matrices is fundamentally different on the different parts of the splitting $\mathcal{N}_l = ker(G_l) \oplus ker^{\perp}(G_l)$ induced by G_l. The requirement in most multigrid algorithms for (1) that all constant vectors ($= ker(G)$) must be interpolated exactly, is a way of taking this splitting into account.

Since Theorem 3.2 is based on the partition of unity property (6) for the nodal prolongator, this property is a sufficient requirement for the factorization to exist on the coarse levels. So thanks to (6), the structural properties of the discretization matrix that are typical for the diffusion equation with respect to differential geometry, carry over to every level in the multigrid hierarchy. Since the splitting $\mathcal{N} = ker(G) \oplus ker^{\perp}(G)$ related to this matrix structure is reflected in the usual requirements for nodal prolongators, this also provides a foundation from differential geometry for the recursive application of the prolongation scheme.

3.2 Curl-curl equation

Next, we will show that all matrices of a multigrid hierarchy for the finite element discretization of the curl-curl equation (2) on a tetrahedral mesh using piecewise linear edge elements, can be written as the product of the discrete dual curl $\tilde{C}$, the face element mass matrix M_ν for the material parameter ν and the discrete curl C

$$K_{\text{curlcurl}} = \tilde{C}\, M_\nu^{face}\, C. \tag{39}$$

Lemma 3.2 *The edge and face elements (8) and (13) for tetrahedral meshes satisfy*

$$\forall\, e_{ij} : \, curl(\vec{E}_{ij}) = \sum_k \vec{F}_{ijk}. \tag{40}$$

Proof.

Using (13), the partition of unity property for the nodal elements $\sum_i N_i = 1$ and (36), we have

$$\begin{aligned}\sum_k \vec{F}_{ijk} &= \sum_k N_i\, curl(\vec{E}_{jk}) + \sum_k N_j\, curl(\vec{E}_{ki}) + \sum_k N_k\, curl(\vec{E}_{ij}) \\ &= N_i\, curl(-grad(N_j)) + N_j\, curl(grad(N_i)) + curl(\vec{E}_{ij}) \\ &= curl(\vec{E}_{ij})\end{aligned}$$

Theorem 3.3 *If the boundary conditions are not taken into account, the finite element discretization matrix of the curl-curl equation (2) on a tetrahedral mesh using piecewise linear edge elements can be written as the product of the discrete dual curl $\tilde{C} = C^T$, the face element mass matrix M_ν^{face}*

$$M_\nu^{face}(i,j) = \iiint_V \nu\, \vec{F}_i \vec{F}_j\, dV, \tag{41}$$

and the discrete gradient C (19).

Proof.

With the signs accounted for by the order of the indices, we have

$$\begin{aligned}(C^T\, M_\nu^{face}\, C)(ij, rs) &= C^T(:, ij)\, M_\nu^{face}\, C(:, rs) \\ &\overset{(19)}{=} \iiint_V \nu \left(\sum_k \vec{F}_{ijk}\right)\left(\sum_t \vec{F}_{rst}\right) dV.\end{aligned}$$

Using (40) and $\tilde{C} = C^T$ [7], this becomes:

$$(\tilde{C}\, M_\nu^{face}\, C)(ij, rs) \quad = \quad \iiint_V \nu\, curl(\vec{E}_{ij})\, curl(\vec{E}_{rs})\, dV = K_{\text{curlcurl}}(ij, rs).$$

Theorem 3.4 *The Galerkin coarse system matrix $P_e^T\, K^f_{curlcurl} P_e$ can be factorized in the form (39), if P_e is an edge prolongator derived from a partition of unity nodal prolongator P_n using (10) and if $K^f_{curlcurl}$ can be factorized as (39).*

PROOF.
The coarse system matrix is given by:

$$K^c_{\text{curlcurl}} \quad = \quad P_e^T\, K^f_{\text{curlcurl}}\, P_e = P_e^T\, C^{fT}\, M^f_{face}\, C^f\, P_e.$$

Consider the face prolongator P_f derived from P_n by (15) and the coarse curl C^c (21) implied by P_n. Using commutation property (32), this can be rewritten as:

$$K^c_{\text{curlcurl}} \quad = \quad C^{cT}\, P_f^T\, M^f_{face}\, P_f\, C^c.$$

Matrix $P_f^T\, M^f_{face}\, P_f$ is the mass matrix for the coarse face elements (15):

$$\begin{aligned}(P_f^T\, M^f_{face}\, P_f)(I, J) \quad &= \quad \iiint_V \nu \left(\sum_r P_f(r, I)\vec{F}_r\right) \left(\sum_s P_f(s, J)\vec{F}_s\right) dV \\ &= \quad \iiint_V \nu\, \vec{F}_I\, \vec{F}_J\, dV = M^c_{face}(I, J).\end{aligned}$$

Hence, using $\tilde{C}^c = C^{cT}$, we have the result

$$K^c_{\text{curlcurl}} \quad = \quad C^{cT}\, M^c_{face}\, C^c = \tilde{C}^c\, M^c_{face}\, C^c. \tag{42}$$

Equation (42) shows that the coarse system matrix can be considered a discretization of the curl-curl operator on the mesh topology represented by C^c and implied by P_n. In general, this implied coarse topology will contain crossing and overlapping faces, and the coarse faces cannot be traced on the fine grid, as shown in Section 2.4.

Together, Theorems 3.3 and 3.4 prove that all matrices of a multigrid hierarchy for the finite element discretization of the curl-curl equation on a tetrahedral mesh using piecewise linear edge elements, can be factorized as (39), if all edge prolongators can be related to a partition of unity nodal prolongator by (10). So, all matrices have the same structure with respect to the appropriate discrete de Rham complexes (see Section 2.6):

$$M_{\text{curlcurl}} = C^T\, M_\nu^{face}\, C \quad : \quad \mathcal{E} \xrightarrow{C} \mathcal{F} \xrightarrow{M_\nu^{face}} \mathcal{F} \xrightarrow{C^T} \mathcal{E}$$

The importance of the splitting $\mathcal{E} = ker(C) \oplus ker^\perp(C)$, which has been demonstrated in [12], is immediately clear from this structure. This splitting is taken into account in most multigrid algorithms for (2). For instance, the use on all levels of the special smoothers of [2, 12] in the multigrid algorithms of [3, 12, 13, 16] is based on it. So, the existence of this factorization on all levels of a multigrid hierarchy is required for the application of these smoothers on all levels to make sense. This confirms the importance of the partition of unity property (6) and the edge-node commutation property (31), since they are the requirements for the existence of this factorization on the coarse levels.

3.3 Grad-div equation

All matrices of a multigrid hierarchy for the finite element discretization of the grad-div equation (3) on a tetrahedral mesh using piecewise linear face elements, can be factorized as the product of the discrete dual gradient $\tilde{G}$, the volume element mass matrix M_α^{volume} and the discrete divergence D

$$K_{\text{graddiv}}(i,j) = \left[\iiint_V \alpha\, div(V_i)\, div(V_j)\, dV\right] = \left(\tilde{G}\, M_\alpha^{volume}\, D\right)(i,j). \tag{43}$$

The requirements, proofs and interpretation are analogous to those in Sections 3.1 and 3.2.

4 Constraints for the restriction in a multigrid hierarchy

A good interplay between smoothing and prolongation/restriction is a basic requirement for an efficient multigrid algorithm. Usually, this is translated into a heuristic that focuses on the prolongation and on the error approximation on the coarse level [9, 5]: error components that are eliminated slowly by the smoother, must be approximated accurately on the coarse level. The resulting constraint or guideline for the construction of a prolongator is the partition of unity property for prolongators for the diffusion equation [19], and the edge-node commutation property for prolongators for the curl-curl equation [12].
In this section, those constraints will be interpreted from the perspective of the residual and its restriction. It will be shown that they guarantee an appropriate conservation of the righthand side to the coarse levels. This will be illustrated by the examples of electrostatics (diffusion) and magnetostatics (curl-curl). For those problems, the partition of unity and the edge-node commutation property guarantee charge conservation.

4.1 Excitation conservation in multigrid for the diffusion equation

Consider the div-grad formulation (1) for electrostatics, with V the electric potential, ϵ the electric permittivity, and ρ the electric charge density. The righthand side q of the finite element discretization of (1) with first order nodal elements N_i, is a partitioning of the total charge present in the problem volume Ω. Indeed, since the usual lowest order nodal elements form a partition of unity, we have:

$$\sum_i q(i) = \sum_i \iiint_\Omega \rho\, N_i\, dV = \iiint_\Omega \rho \sum_i N_i\, dV = \iiint_\Omega \rho\, dV. \tag{44}$$

If all nodal prolongators in a multigrid hierarchy for the discretized div-grad equation are partition of unity, the total charge present in the system is conserved by the restriction with P_n^T, and is partitioned over the coarse nodes according to P_n:

$$\sum_k q_c(k) = \sum_k P_n(:,k)^T q_f = \sum_k \sum_i P_n(i,k) q_f(i) = \sum_i \sum_k P_n(i,k) q_f(i) = \sum_i q_f(i). \tag{45}$$

4.2 Excitation conservation in multigrid for the curl-curl equation

Next, we consider the curl-curl formulation (2) for magnetostatics and its finite element discretization using first order edge elements, with $\vec{A}$ the magnetic vector potential, $1/\nu$ the magnetic permeability and $\vec{J}$ the electric current density. Equation (2) implies $div(\vec{J}) = 0$, as can be seen by applying the divergence to both sides. This is the charge conservation equation for static problems or problems without free charges, like eddy current problems.

Ideally, charge conservation should be reflected on the discrete level as well. With j the righthand side of the finite element discretization of (2), it will be shown first that the individual components of $\tilde{D}\, j$ are local discrete versions of charge conservation, and that the sum of all components $\sum \tilde{D}\, j$ is zero, representing global charge conservation. Next, it will be shown that charge conservation holds on all levels of a multigrid hierarchy for finite element discretizations of (2), if all edge prolongators commute with a partition of unity nodal prolongator.

Consider the i^{th} component of $\tilde{D}\, j$:

$$\left(\tilde{D}\, j\right)(i) = \left(-G^T j\right)(i) = \sum_{k \in \mathcal{N}_i} j_{ik}. \tag{46}$$

In this formula, the discrete currents are associated with the edges (or equivalently, but more correct from the point of view of differential geometry, with the dual faces) of the mesh. So, the mesh can be considered as a circuit of current carrying conductors, and (46) represents the current flowing from node n_i. Local charge conservation in a circuit (Kirchhoff's current law) requires that the nett current flowing from a node must be zero. If (36) holds and with $div(\vec{J}) = 0$, this requirement is satisfied, except in boundary nodes where current enters the model:

$$\begin{aligned}
\Longrightarrow (-G^T j)(i) &= -\sum_e G(e,i)^T \iiint_\Omega \vec{J}\,\vec{E}_e\, dV \\
&= -\iiint_\Omega \vec{J} \sum_{k \in \mathcal{N}_i} \vec{E}_{ki}\, dV \\
&= -\iiint_\Omega \vec{J}\, grad(N_i)\, dV \\
&= -\iiint_\Omega div(N_i\, \vec{J})\, dV \\
&= -\iint_{\partial\Omega} N_i\, \vec{J}\, d\vec{\Omega} = 0
\end{aligned} \tag{47}$$

If $(-G^T j)(i) \neq 0$, current is injected into or removed from the circuit model via node n_i. This allows to consider the nodes as current sinks or sources. Global charge conservation requires that the nett current injection into the model is zero. This is satisfied on the discrete level as well, since the sum of all discrete current injections $(-G^T j)(i)$ is equal to the real nett current injection, reflecting global charge conservation:

$$\sum_i (-G^T j)(i) = -\sum_i \iint_{\delta\Omega} \vec{J}\, N_i\, d\vec{\Omega} = -\iint_{\delta\Omega} \vec{J}\, d\vec{\Omega} = 0. \tag{48}$$

Ideally, properties (47) and (48) should also hold on the coarse levels of a multigrid hierarchy. Commutation of all edge prolongators P_e with a partition of unity nodal prolongator P_n is a sufficient requirement for these properties to be carried over to the coarse levels. Indeed, thanks to the commutation, the current sources located in the fine nodes are partitioned over the coarse nodes according to P_n:

$$-G^{cT}\, j_c = -G^{cT}\, P_e^T\, j_f = -P_n^T\, G^{fT}\, j_f.$$

This implies that $(G^{cT}\, j_c)(I) = 0$ unless the coarse node n_I prolongates to a fine boundary node where current enters the model. In that case, n_I can be considered a coarse boundary node where current enters the model, so this is an appropriate coarsening of the local charge conservation (47). Global charge conservation is satisfied as well, since $-\sum G^c\, j_c$, representing the nett current entering the system, is zero if P_n is partition of unity:

$$-\sum_i \left(G^{cT}\, j_c\right)(i) = -\sum_i \left(\sum_j P_n(j,i) \sum_k G^f(k,j)\, j_f(k) \right) = -\sum_j \left(G^{fT}\, j_f\right)(j) \overset{(48)}{=} 0.$$

So, commutation of the edge prolongators with partition of unity nodal prolongators is a sufficient condition for a correct coarsening of the charge conservation implied by the topological structure of (2).

5 Conclusion

We have shown by a constructive proof that from each partition of unity nodal prolongator for a fine tetrahedral mesh, a commuting sequence of an edge, face and volume prolongator and a coarse topology can be derived, and that this procedure can be repeated recursively. Numerical results were presented for the resulting edge prolongator. A theoretical analysis revealed that the coarse topology can be problematic, and that this can affect the efficiency of the resulting multigrid algorithm and the fill-in of the matrices in the multigrid hierarchy. These observations indicate that controlling the coarse topology might be important.

We also presented new evidence that the partition of unity property and the commutation properties are very important in the context of multigrid algorithms. First, they are the key to a coarse de Rham complex. Second, they can be used to factorize all system matrices in a multigrid hierarchy as a product reflecting the structure of the differential operator. This factorization is a sufficient requirement for the recursive application of most multigrid algorithms to make sense, and it provides a direct link to the underlying equation. Third, they are required for the appropriate conservation of the excitation by the restriction in a multigrid algorithm. These theoretical observations are strong arguments that multigrid prolongators should satisfy the partition of unity property and/or the appropriate commutation properties.

References

[1] D. Arnold, Differential complexes and numerical stability, manuscript, 2003.

[2] D. Arnold and R. Falk and R. Winther, Multigrid in H(div) and H(curl), *Numerische Mathematik* **85** (2000) pp. 197-218.

[3] P. Bochev, C. Garasi, J. Hu, A. Robinson, R. Tuminaro, An improved algebraic multigrid method for solving Maxwell's equations, *SIAM J. Scient. Comp.* **25** (2003) pp. 623-642.

[4] A. Bossavit, *Computational electromagnetism: variational formulations, complementarity, edge elements*, Academic Press, San Diego, 1998.

[5] M. Brezina, A. Cleary, R. Falgout, V. Henson, J. Jones, T. Manteuffel, S. McCormick, J. Ruge, Algebraic multigrid based on element interpolation (AMGe), *SIAM J. Scient. Comp.* **22** (2001), pp. 1570-1592.

[6] M. Clemens, S. Feigh, T. Weiland, Geometric Multigrid Algorithms Using the Conformal Finite Integration Technique, *IEEE Trans. on Magnetics*, **40(2)** (2004), pp. 1065-1068.

[7] M. Clemens, T. Weiland, Discrete electromagnetism with the Finite Integration Technique, *PIER Monograph Series* **32** (2001) pp. 65-87.

[8] P. Dular, J.-Y. Hody, A. Nicolet, A. Genon, W. Legros, Mixed finite elements associated with a collection of tetrahedra, hexahedra and prisms, *IEEE Trans. on Magnetics* **30(5)** (1994) pp. 2980-2983.

[9] R.D. Falgout, P.S. Vassilevski, On Generalizing the AMG Framework, *SIAM J. Numer. Anal.* **42** (2004), pp. 1669-1693.

[10] F. Henrotte, *Modélisation des efforts électromagnétiques et leurs effets dans les structures quelconques*, PhD-thesis (feb. 2000), Université de Liège, Belgium.

[11] R. Hiptmair, Finite elements in computational electromagnetism, *Acta Numerica* **11** (2002) pp. 237-340.

[12] R. Hiptmair, Multigrid method for Maxwell's equations, *SIAM J. Numer. Anal.* **36(1)** (1999) pp. 204-255.

[13] J. Hu, R. Tuminaro, P. Bochev, C. Garasi, A. Robinson, Toward an h-independent algebraic multigrid method for Maxwell's equations, *SIAM J. Scient. Comp.* (to appear in 2005).

[14] J. Jones, B. Lee, A multigrid method for variable coefficient Maxwell's equations, *SIAM J. Scient. Comp.* (to appear in 2005).

[15] J. Jones, P. Vassilevski, AMGe Based on Element Agglomeration, *SIAM J. Sci. Comput.* **23** (2001), pp. 109-133.

[16] S. Reitzinger, J. Schöberl, An algebraic multigrid method for finite element discretizations with edge elements, *Numer. Lin. Alg. Appl.* **9** (2002) pp. 223-238.

[17] M. Sala, J. Hu, R. Tuminaro, ML 3.1 Smoothed Aggregation User's Guide, http://software.sandia.gov/trilinos/packages/ml.

[18] K. Stüben, A review of algebraic multigrid. In: D. Sloan, E. Süli, S. Vandewalle (eds.), *Partial Differential Equations*, Numerical Analysis 2000, Vol.7, Elsevier, 2001, Amsterdam, pp. 281-310.

[19] U. Trottenberg, C. Oosterlee, A. Schüller, *Multigrid*, Academic Press, London, 2001.