

**Pseudospectra and stability radii for  
analytic matrix functions with  
application to time-delay systems**

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*Report TW425, March 2005*



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## **Abstract**

Definitions for pseudospectra and stability radii of an analytic matrix function are given, where the structure of the function is exploited. Various perturbation measures are considered and computationally tractable formulae are derived. The results are applied to a class of retarded delay differential equations. Special properties of the pseudospectra of such equations are determined and illustrated.

**Keywords :** delay equations, pseudospectrum, robustness, stability

# Pseudospectra and stability radii for analytic matrix functions with application to time-delay systems

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## Abstract

Definitions for pseudospectra and stability radii of an analytic matrix function are given, where the structure of the function is exploited. Various perturbation measures are considered and computationally tractable formulae are derived. The results are applied to a class of retarded delay differential equations. Special properties of the pseudospectra of such equations are determined and illustrated.

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## 1 Introduction

Closeness to instability is a key issue in understanding the behaviour of physical systems subject to perturbation. The computation of pseudospectra has become an established tool in analysing and gaining insight into this phenomenon (see, for instance, Trefethen [1], and the references therein). More explicitly, pseudospectra of a system are sets in the complex plane to which its

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eigenvalues can be shifted under a perturbation of a given size. In the simplest case of a matrix (or linear operator)  $A$ , the  $\epsilon$ -pseudospectrum  $\Lambda_\epsilon$  is defined as

$$\Lambda_\epsilon(A) := \{\lambda \in \mathbb{C} : \lambda \in \Lambda(A + P), \text{ for some } P \text{ with } \|P\| \leq \epsilon\}, \quad (1)$$

where  $\Lambda$  denotes the spectrum and  $\|\cdot\|$  denotes an arbitrary matrix (or operator) norm. Equation (1) is known to be equivalent to the following

$$\Lambda_\epsilon(A) = \{\lambda \in \mathbb{C} : \|R(\lambda, A)\| \geq 1/\epsilon\},$$

where  $R(\lambda, A) = (\lambda I - A)^{-1}$  denotes the corresponding resolvent operator.

Although most systems can be written in a first-order form, it is often advantageous to exploit the underlying structure of an equation in its analysis, for example, one may wish to compute pseudospectra of higher-order or delay differential equations (DDEs). In particular, this can be of importance in sensitivity investigations, where it is desirable to respect the structure of the governing system. For example, many physical problems involving vibration of structural systems and vibro-acoustics are modelled by second-order differential equations of the form  $A_2\ddot{x} + A_1\dot{x} + A_0x = 0$ , where  $A_2$ ,  $A_1$ , and  $A_0$  represent mass, damping and stiffness matrices, respectively. Stability is inferred from the eigenvalues, found as solutions of  $\det(A_2\lambda^2 + A_1\lambda + A_0) = 0$ . To understand the sensitivity of these eigenvalues with respect to complex perturbations with weights  $\alpha_i$  applied to  $A_i$ ,  $i = 0, 1, 2$ , the  $\epsilon$ -pseudospectrum of the matrix polynomial  $P(\lambda) = A_2\lambda^2 + A_1\lambda + A_0 \in \mathbb{C}^{n \times n}$  can be defined as

$$\begin{aligned} \Lambda_\epsilon(P) &:= \{\lambda \in \mathbb{C} : (P(\lambda) + \Delta P(\lambda))x = 0 \text{ for some } x \neq 0 \\ &\text{and } \Delta P(\lambda) = \delta A_2\lambda^2 + \delta A_1\lambda + \delta A_0 \\ &\text{with } \delta A_i \in \mathbb{C}^{n \times n} \text{ and } \|\delta A_i\| \leq \epsilon\alpha_i, \ i = 0, 1, 2\}, \end{aligned} \quad (2)$$

See [2] for a survey on the quadratic eigenvalue problem, including numerical solutions and applications, and [3] for pseudospectra of polynomial matrices. More recently, pseudospectra for matrix functions that arise as characteristic equations in DDEs have been defined and analysed [4]. In its simplest form of one, fixed, discrete delay  $\tau \in \mathbb{R}^+$ , the delayed characteristic is of the form  $Q(\lambda) = \lambda I - A_0 - A_1 \exp(-\lambda\tau)$ . Similar to (2) the associated pseudospectra is defined in [4] as

$$\begin{aligned} \Lambda_\epsilon(Q) &:= \{\lambda \in \mathbb{C} : (Q(\lambda) + \Delta Q(\lambda))x = 0 \text{ for some } x \neq 0 \\ &\text{and } \Delta Q(\lambda) = \delta A_0 + \delta A_1 \exp(-\lambda\tau) \\ &\text{with } \delta A_i \in \mathbb{C}^{n \times n} \text{ and } \|\delta A_i\| \leq \epsilon\alpha_i, \ i = 0, 1\}. \end{aligned} \quad (3)$$

The aim of this paper is twofold: first, to present a *unified* theory for the

definition and computation of pseudospectra of *general* matrix functions of the form

$$\det \left\{ \sum_{i=0}^m A_i p_i(\lambda) \right\} = 0, \quad (4)$$

where  $p_i$  is an entire function. It is easy to see that all the cases described above are in this class of matrix functions. Various perturbation measures are discussed, of which the above is only a particular case (Section 2). The second aim is to emphasise some special properties in the case of time-delay systems (Section 3). In this sense, we discuss the effects of weighting factors on the sensitivity of the eigenvalues in  $\mathbb{C}^+$ , and  $\mathbb{C}^-$ , respectively. Next, special attention is devoted to the asymptotic behaviour of pseudospectra and its relationship with root chains coming from infinity. To the best of the authors' knowledge, there does not exist any similar analysis in the literature.

It is important to mention that one of the practical applications of our results concerns the *stability radius*  $r_{\mathbb{C}}$  of (4), that is, a measure of the distance of the matrix function to instability, see also [5–7]. Specifically, if we decompose  $\mathbb{C}$  into two disjoint regions, a desired region  $\mathbb{C}_d$  and an undesired region  $\mathbb{C}_u$ , the complex stability radius is defined as

$$r_{\mathbb{C}}(\mathbb{C}_d, \|\cdot\|_{\text{glob}}) := \inf_{\lambda \in \mathbb{C}_u} \inf_{\Delta} \left\{ \|\Delta\|_{\text{glob}} : \det \left( \sum_{i=0}^m (A_i + \delta A_i) p_i(\lambda) \right) = 0 \right\}, \quad (5)$$

where  $\|\Delta\|_{\text{glob}}$  is a global measure of the perturbation  $\Delta$ , a combination of the complex perturbations  $\delta A_i$ ; this is discussed in detail in Section 2. In other words,  $r_{\mathbb{C}}$  defines the norm of the smallest perturbation that destroys the  $\mathbb{C}_d$ -stability, that is, having all the roots confined to  $\mathbb{C}_d$ . Furthermore,  $r_{\mathbb{C}}$  corresponds to the smallest  $\epsilon$  value at which the  $\epsilon$ -pseudospectrum has a non-empty intersection with  $\mathbb{C}_u$ . Note that for a system with continuous time, for example the DDEs discussed in Section 3 onwards,  $\mathbb{C}_d = \mathbb{C}^-$ ; for discrete time systems,  $\mathbb{C}_d = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ .

The paper is organised as follows. In Section 2 we first develop a theory of pseudospectra for general matrix functions, leading to computable formulae for arbitrary norms measuring the size of perturbations of the matrices  $A_i$  in (4). Next, the application to stability radii, as well as computational issues are discussed. It is also shown how particular results from the literature on stability radii arise as special cases within our unifying framework. In Section 3 we apply the theory to the specific case of DDEs and in Section 4 we present some numerical examples. Finally, in Section 5 we draw conclusions. The notations are standard.

## 2 Generalised pseudospectra for matrix functions

In this section, definitions and expressions for pseudospectra of Eq. (4) are presented. Furthermore, the connection with stability radii is clarified and computational issues are discussed.

### 2.1 Definition and expressions

We study the roots of the generalised matrix function given by (4), where  $A_i \in \mathbb{C}^{n \times n}$ ,  $i = 0, \dots, m$  and the functions  $p_i : \mathbb{C} \rightarrow \mathbb{C}$ ,  $i = 0, \dots, m$  are entire. In particular, we are interested in the effect of bounded perturbations of the matrices  $A_i$  on the position of the roots. For this, we analyze the perturbed equation,

$$\det \left\{ \sum_{i=0}^m (A_i + \delta A_i) p_i(\lambda) \right\} = 0. \quad (6)$$

The first step in our robustness analysis is to define the class of perturbations under consideration, as well as a measure of the combined perturbation

$$\Delta := (\delta A_0, \dots, \delta A_m).$$

In this work we assume that the allowable perturbations  $\delta A_i$ ,  $i = 0, \dots, m$ , are *complex* matrices, that is,

$$\Delta \in \mathbb{C}^{n \times n \times (m+1)}.$$

Introducing weights  $w_i \in \overline{\mathbb{R}}_0^+$ ,  $i = 0, \dots, m$ , where  $\overline{\mathbb{R}}_0^+ = \mathbb{R}^+ \setminus \{0\} \cup \{\infty\}$ , we define three global measures of the perturbations:

$$\|\Delta\|_{\text{glob}} := \|[w_0 \delta A_0 \dots w_m \delta A_m]\|_p, \quad (7)$$

or

$$\|\Delta\|_{\text{glob}} := \left\| \left\| \begin{bmatrix} w_0 \delta A_0 \\ \vdots \\ w_m \delta A_m \end{bmatrix} \right\|_p \right\|_p, \quad (8)$$

where  $\|M\|_p$  is the induced matrix norm given by  $\|M\|_p = \sup_{\|x\|_p=1} \|Mx\|_p$ ,  $p \in \overline{\mathbb{N}}$ . Notice that  $w_j = \infty$  for some  $j$  means that no perturbation on  $A_j$  is allowed when the combined perturbation  $\Delta$  is required to be bounded, that is  $w_j = \infty \implies \delta A_j = 0$ , for some  $j$ . Finally, we also consider a measure of

mixed-type:

$$\|\Delta\|_{\text{glob}} := \left\| \begin{bmatrix} w_0 \|\delta A_0\|_{p_1} \\ \vdots \\ w_m \|\delta A_m\|_{p_1} \end{bmatrix} \right\|_{p_2}, \quad p_1, p_2 \in \overline{\mathbb{N}}_0. \quad (9)$$

For instance, when  $p_2 = \infty$  and all weights are equal to one, the condition  $\|\Delta\|_{\text{glob}} \leq \epsilon$  corresponds to the natural assumptions of taking perturbations satisfying  $\|\delta A_i\|_{p_1} \leq \epsilon$ ,  $i = 0, \dots, m$ . In this special case, (9) is also equal to the  $p_1$ -norm of the *block diagonal* perturbation matrix  $\text{diag}(\delta A_0, \dots, \delta A_m)$ , considered in [6,7] for polynomial matrices.

Notice that, if all weights are finite, then the measures given by (7)-(9) are norms.

For any of the above definitions of  $\|\Delta\|_{\text{glob}}$ , we define the  $\epsilon$ -pseudospectrum of (4) as the set

$$\Lambda_\epsilon := \left\{ \lambda \in \mathbb{C} : \det \left( \sum_{i=0}^m (A_i + \delta A_i) p_i(\lambda) \right) = 0 \text{ for some } \Delta \text{ with } \|\Delta\|_{\text{glob}} \leq \epsilon \right\}. \quad (10)$$

We define the function  $f : \mathbb{C} \rightarrow \overline{\mathbb{R}}^+$  as the inverse of the size of the smallest perturbation which shifts a root to  $\lambda$  if such perturbations exist, and zero otherwise, more precisely,

$$f(\lambda) = \begin{cases} 0, & \text{when } \det(\sum_{i=0}^m (A_i + \delta A_i) p_i(\lambda)) \neq 0, \forall \Delta \in \mathbb{C}^{n \times n \times (m+1)}, \\ +\infty, & \text{when } \det(\sum_{i=0}^m A_i p_i(\lambda)) = 0, \\ \left( \inf \{ \|\Delta\|_{\text{glob}} : \det(\sum_{i=0}^m (A_i + \delta A_i) p_i(\lambda)) = 0 \} \right)^{-1}, & \text{otherwise.} \end{cases} \quad (11)$$

Therefore, we can also define the  $\epsilon$ -pseudospectra as

$$\Lambda_\epsilon = \left\{ \lambda \in \mathbb{C} : f(\lambda) \geq \epsilon^{-1} \right\}. \quad (12)$$

The boundary of pseudospectra is thus formed by the level sets of the function  $f$ , which can be written in a computational form as follows:

**Theorem 1** *For the perturbation measures (7)-(9) the function (11) satisfies*

$$f(\lambda) = \begin{cases} \left\| \left( \sum_{i=0}^m A_i p_i(\lambda) \right)^{-1} \right\|_\alpha \cdot \|w(\lambda)\|_\beta, & \det(\sum_{i=0}^m A_i p_i(\lambda)) \neq 0, \\ +\infty, & \det(\sum_{i=0}^m A_i p_i(\lambda)) = 0, \end{cases}$$

where

$$w(\lambda) = \begin{bmatrix} \frac{p_0(\lambda)}{w_0} \\ \vdots \\ \frac{p_m(\lambda)}{w_m} \end{bmatrix} \quad (13)$$

and

$$\begin{array}{l} \alpha = p, \beta = p, \\ \alpha = p, \beta = q, \quad \frac{1}{p} + \frac{1}{q} = 1, \\ \alpha = p_1, \beta = q_2, \quad \frac{1}{p_1} + \frac{1}{q_2} = 1 \end{array} \left| \begin{array}{l} \text{perturbation measure (7),} \\ \text{perturbation measure (8),} \\ \text{perturbation measure (9).} \end{array} \right.$$

**PROOF.** For the perturbation measure (7) one can directly generalise the proof of Theorem 13 of [7] for polynomial matrices. Therefore, we restrict ourselves to a proof for measures (8) and (9). For sake of conciseness we also assume that  $p, p_1, p_2$  and all weights  $w_i$  are finite. The proof for the other cases follows the same lines.

*Perturbation measure (8).*

For any  $\lambda \in \mathbb{C}$ , which is not a root of (4), the perturbed equation (6) can be written in the form

$$\det \left\{ I - M(\lambda) \begin{bmatrix} w_0 \delta A_0 \\ \vdots \\ w_m \delta A_m \end{bmatrix} \right\} = 0, \quad (14)$$

where

$$M(\lambda) = \left( \sum_{i=0}^m A_i p_i(\lambda) \right)^{-1} \begin{bmatrix} \frac{p_0(\lambda)}{w_0} I & \dots & \frac{p_m(\lambda)}{w_m} I \end{bmatrix}. \quad (15)$$

A standard result from robust control (see, for example, [8], [7, Lemma 2]) yields that (14) has a root  $\lambda^*$  for some perturbation  $\Delta$  if and only if  $M(\lambda^*) \neq 0$ . Furthermore, if  $M(\lambda^*) \neq 0$ , one has

$$\inf \left\{ \|\Delta\|_{\text{glob}} : \det \left( \sum_{i=0}^m (A_i + \delta A_i) p_i(\lambda^*) \right) = 0 \right\} = \frac{1}{\|M(\lambda^*)\|_p}.$$

Therefore, it has become sufficient to prove that

$$\|M(\lambda)\|_p = \left\| \left( \sum_{i=0}^m A_i p_i(\lambda) \right)^{-1} \right\|_p \|w(\lambda)\|_q.$$

This is trivial when  $\|w(\lambda)\|_q = 0$ . In the other case, note first that for any  $x = [x_0^T \dots x_m^T]^T \in \mathbb{C}^{(m+1)n \times 1}$ , one derives using Hölder's inequality:

$$\begin{aligned}
\frac{\|M(\lambda)x\|_p}{\|x\|_p} &\leq \frac{\|(\sum_{i=0}^m A_i p_i(\lambda))^{-1}\|_p \left\| \sum_{i=0}^m x_i \frac{p_i(\lambda)}{w_i} \right\|_p}{\|x\|_p} \\
&\leq \frac{\|(\sum_{i=0}^m A_i p_i(\lambda))^{-1}\|_p \left( \|x_0\|_p, \dots, \|x_m\|_p \right) \cdot \left( \frac{|p_0(\lambda)|}{w_0}, \dots, \frac{|p_m(\lambda)|}{w_m} \right)}{\|x\|_p} \\
&\leq \frac{1}{\|x\|_p} \left\| \left( \sum_{i=0}^m A_i p_i(\lambda) \right)^{-1} \right\|_p \cdot \left\| \begin{bmatrix} \|x_0\|_p \\ \vdots \\ \|x_m\|_p \end{bmatrix} \right\|_p \|w(\lambda)\|_q \\
&= \left\| \left( \sum_{i=0}^m A_i p_i(\lambda) \right)^{-1} \right\|_p \cdot \|w(\lambda)\|_q, \tag{16}
\end{aligned}$$

thus,

$$\|M(\lambda)\|_p := \sup \frac{\|M(\lambda)x\|_p}{\|x\|_p} \leq \left\| \left( \sum_{i=0}^m A_i p_i(\lambda) \right)^{-1} \right\|_p \cdot \|w(\lambda)\|_q. \tag{17}$$

Second, one readily verifies that the equality in (16) is attained for

$$\begin{aligned}
x &= \begin{bmatrix} |p_0(\lambda)|^{\frac{2-p}{p-1}} \overline{p_0(\lambda)} w_0^{1-q} \\ \vdots \\ |p_m(\lambda)|^{\frac{2-p}{p-1}} \overline{p_m(\lambda)} w_m^{1-q} \end{bmatrix} \otimes x^*, \quad p \neq 1, \\
x &= v(\lambda) \otimes x^*, \quad p = 1,
\end{aligned}$$

where  $x^*$  is chosen such that

$$\|x^*\|_p = 1, \quad \left\| \left( \sum_{i=0}^m A_i p_i(\lambda) \right)^{-1} x^* \right\|_p = \left\| \left( \sum_{i=0}^m A_i p_i(\lambda) \right)^{-1} \right\|_p$$

and  $v(\lambda) \in \mathbb{C}^{(m+1) \times 1}$  is given by

$$v_i(\lambda) = \begin{cases} \frac{\overline{p_i(\lambda)}}{|p_i(\lambda)|}, & \text{if } i \in \Upsilon(\lambda) \\ 0, & \text{otherwise} \end{cases},$$

with

$$\Upsilon(\lambda) = \left\{ i \in \{0, \dots, m\} : \left| \frac{p_i(\lambda)}{w_i} \right| = \max_{k \in \{0, \dots, m\}} \left| \frac{p_k(\lambda)}{w_k} \right| \right\}. \tag{18}$$

*Perturbation measure (9).*

A perturbation  $\Delta$  shifts a root to  $\lambda \in \mathbb{C}$ , again  $\lambda$  is not a root of (4), if and

only if,

$$\det \left( I - \left( \sum_{i=0}^m A_i p_i(\lambda) \right)^{-1} \sum_{i=0}^m \delta A_i p_i(\lambda) \right) = 0;$$

see (14) and (15). Necessary conditions on such perturbations are given by

$$\left\| \left( \sum_{i=0}^m A_i p_i(\lambda) \right)^{-1} \sum_{i=0}^m \delta A_i p_i(\lambda) \right\|_{p_1} \geq 1,$$

$$\sum_{i=0}^m \|\delta A_i\|_{p_1} |p_i(\lambda)| \geq \frac{1}{\|(\sum_{i=0}^m A_i p_i(\lambda))^{-1}\|_{p_1}},$$

$$\left| (w_0 \|\delta A_0\|_{p_1}, \dots, w_m \|\delta A_m\|_{p_1}) \cdot \left( \frac{|p_0(\lambda)|}{w_0}, \dots, \frac{|p_m(\lambda)|}{w_m} \right) \right| \geq \frac{1}{\|(\sum_{i=0}^m A_i p_i(\lambda))^{-1}\|_{p_1}},$$

and, when using Hölder's inequality, by

$$\left\| \begin{bmatrix} w_0 \|\delta A_0\|_{p_1} \\ \vdots \\ w_m \|\delta A_m\|_{p_1} \end{bmatrix} \right\|_{p_2} \cdot \|w(\lambda)\|_{q_2} \geq \frac{1}{\|(\sum_{i=0}^m A_i p_i(\lambda))^{-1}\|_{p_1}}.$$

Consequently, in the case  $\|w(\lambda)\|_q = 0$ , there does not exist a perturbation shifting a root to  $\lambda$ , which implies  $f = 0$  by definition, and the assertion of the theorem holds. In the other case, a perturbation  $\Delta$  can only shift a root to  $\lambda$  when it satisfies,

$$\|\Delta\|_{\text{glob}} \geq \frac{1}{\|(\sum_{i=0}^m A_i p_i(\lambda))^{-1}\|_{p_1} \|w(\lambda)\|_{q_2}}. \quad (19)$$

We now prove by construction that such perturbations do exist, and, furthermore, there are perturbations for which the equality in (19) holds. Firstly, we define  $U$  as a square matrix such that

$$\det \left\{ I - \left( \sum_{i=0}^m A_i p_i(\lambda) \right)^{-1} U \right\} = 0 \text{ and } \|U\|_{p_1} = \frac{1}{\|(\sum_{i=0}^m A_i p_i(\lambda))^{-1}\|_{p_1}}.$$

In the case  $p_2 \neq 1$  we define

$$\Delta_c = (\delta A_{0c} \dots \delta A_{mc}),$$

where

$$\delta A_{kc} = \frac{w_k^{-q_2} |p_k(\lambda)|^{\frac{2-p_2}{p_2-1}} \overline{p_k(\lambda)}}{\sum_{i=0}^m w_i^{-q_2} |p_i(\lambda)|^{q_2}} U, \quad k = 0, \dots, m.$$

It follows that

$$\begin{aligned}
\|\Delta_c\|_{\text{glob}} &= \frac{\|U\|_{p_1}}{\sum_{i=0}^m w_i^{-q_2} |p_i(\lambda)|^{q_2}} \left( \sum_{i=0}^m w_i^{p_2(1-q_2)} |p_i(\lambda)|^{\frac{p_2}{p_2-1}} \right)^{\frac{1}{p_2}} \\
&= \frac{\|U\|_{p_1}}{\left( \sum_{i=0}^m w_i^{-q_2} |p_i(\lambda)|^{q_2} \right)^{1-1/p_2}} \\
&= \frac{1}{\|(\sum_{i=0}^m A_i p_i(\lambda))^{-1}\|_{p_1} \left( \sum_{i=0}^m w_i^{-q_2} |p_i(\lambda)|^{q_2} \right)^{1/q_2}} \\
&= \frac{1}{\|(\sum_{i=0}^m A_i p_i(\lambda))^{-1}\|_{p_1} \|w(\lambda)\|_{q_2}}, \tag{20}
\end{aligned}$$

and the perturbed characteristic function satisfies

$$\begin{aligned}
&\det \left\{ I - \left( \sum_{i=0}^m A_i p_i(\lambda) \right)^{-1} \left( \sum_{i=0}^m \delta A_{ic} p_i(\lambda) \right) \right\} \\
&= \det \left\{ I - \left( \sum_{i=0}^m A_i p_i(\lambda) \right)^{-1} U \frac{\sum_{i=0}^m w_i^{-q_2} |p_i(\lambda)|^{\frac{2-p_2}{p_2-1}} |p_i(\lambda)|^2}{\sum_{i=0}^m w_i^{-q_2} |p_i(\lambda)|^{q_2}} \right\} \\
&= \det \left\{ I - \left( \sum_{i=0}^m A_i p_i(\lambda) \right)^{-1} U \right\} = 0. \tag{21}
\end{aligned}$$

From (19)–(21) and the definition (11) the statement of the theorem follows.

In the case  $p_2 = 1$  one comes to the same conclusion when taking

$$\Delta_c := \xi(\lambda)^T \otimes U,$$

where  $\xi(\lambda) \in \mathbb{C}^{(m+1) \times 1}$  is given by

$$\xi_i(\lambda) = \begin{cases} \frac{\overline{p_i(\lambda)}}{\text{card}\Upsilon(\lambda) |p_i(\lambda)|^2}, & \text{if } i \in \Upsilon(\lambda) \\ 0, & \text{otherwise} \end{cases},$$

with  $\Upsilon$  defined in (18).

□

## 2.2 Connection with stability radii

As outlined in the introduction the concept of stability radii given by (5) is closely related to pseudospectra. To further clarify this relation and to arrive at a computable formula, we need the following continuity property of the individual roots of (4) with respect to changes of matrices  $A_i$ :

**Proposition 2** For all  $\mu > 0$  and  $\lambda_0 \in \mathbb{C}$ , there exists a  $\nu > 0$  such that for all  $\Delta = (\delta A_0, \dots, \delta A_m) \in \mathbb{C}^{n \times n \times (m+1)}$  with  $\|\Delta\|_{\text{glob}} < \nu$ , (6) has the same number of roots<sup>1</sup> as (4) in the disc  $\{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \mu\}$ .

**PROOF.** Based on the implication  $\|\Delta\|_{\text{glob}} \rightarrow 0 \Rightarrow \delta A_i \rightarrow 0, i = 0, \dots, m$  and an application of [9, Theorem A1].

Assume that all the roots of (4) are in  $\mathbb{C}_d$ . Let  $\Delta_c$  be an arbitrary perturbation with  $\|\Delta_c\|_{\text{glob}}$  finite, for which there is at least one root in  $\mathbb{C}_u$  (such perturbations always exist by Theorem 1). Next, apply the perturbation  $\Delta := \epsilon \Delta_c$ , where  $\epsilon \geq 0$  is a parameter. Clearly the function

$$\epsilon \in [0, 1] \rightarrow \epsilon \Delta_c$$

is continuous with respect to the measure  $\|\cdot\|_{\text{glob}}$ . Consequently, by Proposition 2 one of the following phenomena must happen to the roots of the perturbed system when  $\epsilon$  is continuously varied from zero to one:

- (1) some roots move from  $\mathbb{C}_d$  to  $\mathbb{C}_u$ ;
- (2) roots coming from infinity appear in  $\mathbb{C}_u$  (only for unbounded  $\mathbb{C}_u$ ).

If the second case can be excluded, a loss of stability is always associated with roots on the boundary of  $\mathbb{C}_d$  and it becomes sufficient to scan this boundary in the outer optimization of (5). In other words, the stability radius is the smallest value of  $\epsilon$  for which an  $\epsilon$ -pseudospectrum contour reaches the boundary of  $\mathbb{C}_d$ . Formally, using (11) one has:

**Corollary 3** Assume that all the roots of (4) are in  $\mathbb{C}_d$ . Then

$$r_{\mathbb{C}}(\mathbb{C}_d, \|\cdot\|_{\text{glob}}) = \inf_{\lambda \in \Gamma_{\mathbb{C}_d}} \frac{1}{f(\lambda)} = \frac{1}{\sup_{\lambda \in \Gamma_{\mathbb{C}_d}} f(\lambda)},$$

where  $\Gamma_{\mathbb{C}_d}$  is the boundary of the set  $\mathbb{C}_d$ .

The following example demonstrates that Corollary 3 does not hold if perturbations create roots coming from infinity in  $\mathbb{C}_u$ .

**Example 4** The equation  $p(\lambda) = 0$ , with

$$p(\lambda) = \lambda + 1 + \delta a e^\lambda,$$

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<sup>1</sup> multiplicity taken into account

is  $\mathbb{C}^-$ -stable for  $\delta a = 0$ . With  $\|\Delta\|_{\text{glob}} = |\delta a|$ , we have

$$\inf_{\lambda \in \Gamma_{\mathbb{C}^-}} \frac{1}{f(\lambda)} = \inf_{\omega \geq 0} \frac{|1 + j\omega|}{|e^{j\omega}|} = 1,$$

that is, shifting roots to the imaginary axis requires  $|\delta a| \geq 1$ . However, the stability radius is zero because for any real  $\delta a \neq 0$ , there are infinitely many roots in the open right half plane, whose real parts move off to plus infinity as  $|\delta a| \rightarrow 0+$ . To see this, note that  $p(-\lambda)$  can be interpreted as the characteristic function of the DDE  $\dot{x}(t) = x(t) + \delta a x(t-1)$ , which has infinitely many eigenvalues located in a logarithmic section of the left half plane [10].

### 2.3 Computational issues

From (12) and Theorem 1 pseudospectra of (4) can be computed by evaluating

$$\left\| \left( \sum_{i=0}^m A_i p_i(\lambda) \right)^{-1} \right\|_{\alpha} \|w(\lambda)\|_{\beta}$$

for  $\lambda$  on a grid over a region of the complex plane. By using a contour plotter to view the results, the boundaries of  $\epsilon$ -pseudospectra are then identified. Notice that for  $\alpha = 2$ , the left term can be computed as the inverse of the smallest singular value of  $\sum_{i=0}^m A_i p_i(\lambda)$ . Analogously, from Corollary 3 the complex stability radius can be computed using a grid, laid on the boundary of the stability region. Such an approach is taken for the numerical examples of Section 4.

It is important to mention that for the computation of pseudospectra of specific problems (for example large matrices with a special structure) and, in particular, for optimization problems related to pseudospectra, the efficiency can often be improved by exploiting properties of the problem under consideration. See, for instance, [7, Section 6] and the references therein for an efficient algorithm to compute stability radii of polynomial matrices and [11] for the efficient computation and optimization of so-called pseudospectral abscissa of matrices. However, this is beyond the scope of this paper, where generality is the main concern.

To conclude this section we give in Table 1 an overview of publications, where results from Theorem 1 or Corollary 3 were obtained for special cases.

reference	problem	perturbation measure	weights
[12]	matrix pencil	(7)	/
[7]	polynomial matrices	(7),(9) with $p_2 = \infty$	/
[6]	polynomial matrices	(7),(9) with $p_2 = \infty$	yes
[3]	polynomial matrices	(7),(9) with $p_2 = \infty$	yes
[4]	delay systems	(9) with $p_1 = 2$ and $p_2 = \infty$	yes

Table 1

Special cases of Theorem 1/ Corollary 3, treated in the literature.

### 3 Pseudospectra of delay differential equations

We apply the results of Section 2 to linear DDEs of the form

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^m A_i x(t - \tau_i), \quad (22)$$

where we assume that  $0 < \tau_1 < \dots < \tau_m$  and that the system matrices  $A_i \in \mathbb{R}^{n \times n}$ ,  $i = 0, \dots, m$  are uncertain. In what follows we denote the spectrum of (22) by  $\Lambda$ , that is

$$\Lambda := \left\{ \lambda \in \mathbb{C} : \det \left( \lambda I - A_0 - \sum_{i=1}^m A_i e^{-\lambda \tau_i} \right) = 0 \right\}.$$

#### 3.1 Expressions

Pseudospectra and stability radii of (22), following the general definitions (5) and (10), can be computed as follows:

**Proposition 5** *For perturbations  $\delta A_i \in \mathbb{C}^{n \times n}$ ,  $i = 0, \dots, m$ , measured by (7)-(9), the pseudospectrum  $\Lambda_\epsilon$  of (22) satisfies*

$$\Lambda_\epsilon = \Lambda \cup \left\{ \lambda \in \mathbb{C} : \left\| \left( \lambda I - A_0 - \sum_{i=1}^m A_i e^{-\lambda \tau_i} \right)^{-1} \right\|_\alpha \cdot \|w(\lambda)\|_\beta \geq \epsilon^{-1} \right\} \quad (23)$$

and if the zero solution of (22) is asymptotically stable, the associated stability radius satisfies

$$r_{\mathbb{C}^-, \|\cdot\|_{\text{glob}}} = \frac{1}{\left( \sum_{i=0}^m w_i^{-\beta} \right)^{\frac{1}{\beta}} \sup_{\omega \geq 0} \left\| (j\omega I - A_0 - \sum_{i=1}^m A_i e^{-j\omega \tau_i})^{-1} \right\|_\alpha}, \quad (24)$$

where  $w(\lambda) = \left[ \frac{1}{w_0} \frac{e^{-\lambda \tau_1}}{w_1} \dots \frac{e^{-\lambda \tau_m}}{w_m} \right]^T$  and  $\alpha$  and  $\beta$  are defined as in Theorem 1.

**PROOF.** The characteristic matrix of (22),

$$\lambda I - A_0 - \sum_{i=1}^m A_i e^{-\lambda \tau_i},$$

is affine in the system matrices  $(I, A_0, \dots, A_m)$  and thus the characteristic equation is of the form (4). The first term  $\lambda I$  is however assumed to be unperturbed, which one can cope with in the general framework of Section 2 by assigning an infinite weight to a perturbation of this term in the definition of  $\|\cdot\|_{\text{glob}}$ . In this way a direct application of Theorem 1 yields (23).

Expression (24) relies on the fact that for any  $M > 0$ , there exists a  $R > 0$ , such that all the eigenvalues in the closed right half-plane have modulus smaller than  $R$  whenever  $\|\Delta\|_{\text{glob}} < M$ , hence roots coming from infinity are excluded. For this, note that

$$\det \left\{ \lambda I - (A_0 + \delta A_0) - \sum_{i=1}^m (A_i + \delta A_i) e^{-\lambda \tau_i} \right\} = 0$$

implies that

$$|\lambda| \leq \|A_0 + \delta A_0\| + \sum_{i=1}^m \|A_i + \delta A_i\| |e^{-\lambda \tau_i}|,$$

and for  $\Re(\lambda) \geq 0$ ,

$$|\lambda| \leq \sum_{i=0}^m \|A_i + \delta A_i\|,$$

with  $\|\cdot\|$  any matrix norm. The bound on each  $\delta A_i$  and thus the bound  $R$  on  $|\lambda|$  is implied by the bound on  $\|\Delta\|_{\text{glob}}$ .

As a consequence, Corollary 3 is applicable. Using  $|e^{-j\omega \tau_i}| = 1$ ,  $i = 0, \dots, m$  one arrives at (24).  $\square$

**Remark 6** *For the system*

$$\dot{x}(t) = (A + \delta A)x(t), \tag{25}$$

with  $\|\Delta\|_{\text{glob}} = \|\delta A\|_2$ , expression (23) simplifies to

$$\Lambda_\epsilon = \Lambda \cup \left\{ \lambda \in \mathbb{C} : \|\mathcal{R}(\lambda, A)\|_2 \geq \epsilon^{-1} \right\}, \tag{26}$$

where  $\mathcal{R}(\lambda, A) = (\lambda I - A)^{-1}$  is the resolvent of  $A$ . As mentioned in the introduction, the right-hand side of (26) can also be considered as a definition for the  $\epsilon$ -pseudospectrum of (25).

In general, one can formulate (22) as an abstract evolution equation over the Hilbert space  $X := \mathbb{C}^n \times \mathcal{L}^2([-\tau_m, 0], \mathbb{C}^n)$ , equipped with the usual inner product

$$\langle (y_0, y_1), (z_0, z_1) \rangle_X = \langle y_0, z_0 \rangle_{\mathbb{C}^n} + \langle y_1, z_1 \rangle_{\mathcal{L}^2},$$

namely:

$$\frac{d}{dt} z(t) = \mathcal{A} z(t), \quad (27)$$

where

$$D(\mathcal{A}) = \left\{ z = (z_0, z_1) \in X : z_1 \text{ is absolutely continuous on } [-\tau_m, 0], \right. \\ \left. \frac{dz_1}{d\theta}(\cdot) \in \mathcal{L}^2([-\tau_m, 0], \mathbb{C}^n), z_0 = z_1(0) \right\}, \\ \mathcal{A} z = \begin{pmatrix} A_0 z_0 + \sum_{i=1}^m A_i z_1(-\tau_i) \\ \frac{dz_1}{d\theta}(\cdot) \end{pmatrix}, \quad z \in D(\mathcal{A})$$

and the solutions of (27) and (22) are connected by the relation  $z_0(t) \equiv x(t)$ ,  $z_1(t) \equiv x(t + \theta)$ ,  $\theta \in [-\tau_m, 0]$ . In this way, one can alternatively define the  $\epsilon$ -pseudospectrum of (22) as the set

$$\Lambda \cup \left\{ \lambda \in \mathbb{C} : \|\mathcal{R}(\lambda, \mathcal{A})\| \geq \epsilon^{-1} \right\}. \quad (28)$$

See for example [13] for further discussions and remarks related to the abstract formulation of differential delay equations and [14] for pseudospectra of closed linear operators on complex Banach spaces.

Definition (28) is related with the effect of unstructured perturbations of the operator  $\mathcal{A}$  on stability. In this paper we have chosen a more practical definition, by directly relating pseudospectra to concrete perturbations on the system matrices. Notice that such a practical definition is typically used also for polynomial equations [3, 6, 7].

### 3.2 Effect of weighting

Applying different weights to the system matrices  $A_i$  of (22),  $i = 1, \dots, m$ , leads to changes in the pseudospectra. This can be understood by investigating the weighting function  $w(\lambda) = w(\sigma + j\omega)$ , where

$$\|w(\sigma + j\omega)\|_\beta = \left\| \left[ \frac{1}{w_0}, \frac{e^{-\sigma\tau_1}}{w_1}, \dots, \frac{e^{-\sigma\tau_m}}{w_m} \right]^T \right\|_\beta, \quad \forall \sigma, \omega \in \mathbb{R}. \quad (29)$$

Note that  $w(\lambda)$  only depends on the real part  $\sigma$ , that is,  $w(\lambda) \equiv w(\sigma)$ . From (29) the following conclusions can be drawn:

- (1) Eigenvalues in the right half-plane are more sensitive to perturbations of the non-delayed term  $A_0$ ;
- (2) Eigenvalues in the left half-plane are more sensitive to perturbations of the delayed terms  $A_i$ ,  $i = 1, \dots, m$ ;

- (3) Furthermore, the intersection of an  $\epsilon$ -pseudospectrum contour with the imaginary axis is independent of the weights, provided that the  $\beta$ -norm of  $w(\lambda) = w(0)$  is constant. As a consequence, under this condition also the stability radius is independent of the weights.

### 3.3 Asymptotic properties.

In order to characterise boundedness properties of pseudospectra, we investigate the behaviour of

$$f(\lambda) := \begin{cases} \left\| \left( \lambda I - A_0 - \sum_{i=1}^m A_i e^{-\lambda \tau_i} \right)^{-1} \right\|_{\alpha} \|w(\lambda)\|_{\beta}, & \lambda \notin \Lambda \\ +\infty & \lambda \in \Lambda, \end{cases}$$

as  $|\lambda| \rightarrow \infty$ . The results follow:

**Proposition 7** For all  $\mu \in \mathbb{R}$ ,

$$\lim_{R \rightarrow \infty} \inf \left\{ f(\lambda)^{-1} : \Re(\lambda) > \mu, |\lambda| > R \right\} = \infty. \quad (30)$$

**PROOF.** Follows from

$$\sup \left\{ |e^{-\lambda \tau_i}| : \Re(\lambda) > \mu \right\} = e^{-\mu \tau_i}, \quad i = 1, \dots, m.$$

□

As a consequence the cross-section between any pseudospectrum  $\Lambda_{\epsilon}$ ,  $\epsilon > 0$ , and any right half-plane is bounded.

**Proposition 8** Assume that  $w_m$  is finite. For all  $\gamma \in \mathbb{R}^+$ , let the set  $\Psi_{\gamma} \subseteq \mathbb{C}$  be defined as

$$\Psi_{\gamma} := \left\{ \lambda \in \mathbb{C} : \Re(\lambda) < -\gamma, |\lambda| < e^{-(\Re(\lambda) + \gamma)\tau_m} \right\}. \quad (31)$$

Furthermore, let

$$l = \begin{cases} \frac{w_m}{\|A_m^{-1}\|_{\alpha}}, & A_m \text{ regular,} \\ 0, & A_m \text{ singular.} \end{cases}$$

Then the following convergence property holds:

$$\forall \kappa > 0, \exists \gamma > 0 \text{ such that } |f(\lambda)^{-1} - l| < \kappa, \forall \lambda \in \Psi_{\gamma}. \quad (32)$$

**PROOF.** We can write:

$$f(\lambda) = \begin{cases} \|(A_m - R_1(\lambda))^{-1}\|_\alpha \cdot \frac{1+R_2(\lambda)}{w_m}, & \det(A_m - R_1(\lambda)) \neq 0 \\ +\infty, & \text{otherwise,} \end{cases} \quad (33)$$

where

$$R_1(\lambda) = (\lambda I - A_0)e^{\lambda\tau_m} - \sum_{i=1}^{m-1} A_i e^{\lambda(\tau_m - \tau_i)}$$

and

$$R_2(\lambda) = w_m \|w(\lambda)e^{\lambda\tau_m}\|_\beta - 1.$$

Notice that we have by (31):

$$\lim_{\gamma \rightarrow +\infty} \sup_{\lambda \in \Psi_\gamma} \|R_i(\lambda)\|_\alpha = 0, \quad i = 1, 2. \quad (34)$$

*Case 1 -  $A_m$  regular:*

From

$$f(\lambda) = \begin{cases} \|A_m^{-1}(I - A_m^{-1}R_1(\lambda))^{-1}\|_\alpha \frac{1+R_2(\lambda)}{w_m}, & \det(I - A_m^{-1}R_1(\lambda)) \neq 0 \\ +\infty, & \text{otherwise,} \end{cases} \quad (35)$$

we obtain under the condition  $\|A_m^{-1}R_1(\lambda)\|_\alpha < 1$ :

$$\frac{\|A_m^{-1}\|_\alpha}{w_m} \left( 1 - \frac{\|A_m^{-1}R_1(\lambda)\|_\alpha}{1 - \|A_m^{-1}R_1(\lambda)\|_\alpha} \right) (1 + R_2(\lambda)) \leq f(\lambda) \leq \frac{\|A_m^{-1}\|_\alpha}{w_m} \left( 1 + \frac{\|A_m^{-1}R_1(\lambda)\|_\alpha}{1 - \|A_m^{-1}R_1(\lambda)\|_\alpha} \right) (1 + R_2(\lambda)). \quad (36)$$

Combining (34) and (36) yields the statement of the proposition.

*Case 2 -  $A_m$  singular:*

From (33) we have when  $\det(A_m - R_1(\lambda)) \neq 0$ :

$$\begin{aligned} f(\lambda) &\geq r_\sigma((A_m - R_1(\lambda))^{-1}) \frac{1+R_2(\lambda)}{w_m}, \\ &= \frac{1}{|\lambda_{\min}(A_m - R_1(\lambda))|} \frac{1+R_2(\lambda)}{w_m}, \end{aligned}$$

with  $r_\sigma(\cdot)$ , and  $\lambda_{\min}(\cdot)$  denoting the spectral radius and the eigenvalue with the smallest modulus. Hence,

$$\begin{aligned} |f(\lambda)^{-1}| &\leq \frac{w_m |\lambda_{\min}(A_m - R_1(\lambda))|}{1+R_2(\lambda)} \\ &\leq \frac{w_m k (\|R_1(\lambda)\|_2)^{1/n}}{1+R_2(\lambda)}, \end{aligned} \quad (37)$$

for some constant  $k$ , provided that  $\|R_1(\lambda)\|_2$  is sufficiently small, see [15, p.343]. The assertion of the proposition follows from (34) and (37).  $\square$

Notice that for any  $\gamma > 0$  the set  $\Psi_\gamma$  is a *logarithmic sector* stretching out into the left half-plane. Furthermore, the collection  $\{\Psi_\gamma\}_{\gamma \geq 0}$  is nested in the sense

$$\gamma_1 \leq \gamma_2 \Rightarrow \Psi_{\gamma_2} \subseteq \Psi_{\gamma_1}.$$

Restating the proposition in terms of pseudospectra yields:

**Corollary 9** *Let  $\Psi_\gamma$  be defined as in Proposition 8. If  $A_m$  is regular, then*

$$\begin{aligned} \forall \epsilon \in \left(0, \frac{w_m}{\|A_m^{-1}\|_\alpha}\right), \exists \gamma > 0 \text{ such that } \Psi_\gamma \cap \Lambda_\epsilon = \emptyset, \\ \forall \epsilon > \frac{w_m}{\|A_m^{-1}\|_\alpha}, \exists \gamma > 0 \text{ such that } \Psi_\gamma \subset \Lambda_\epsilon. \end{aligned} \quad (38)$$

*If  $A_m$  is singular, then*

$$\forall \epsilon > 0, \exists \gamma > 0 \text{ such that } \Psi_\gamma \subset \Lambda_\epsilon. \quad (39)$$

In the case of singular  $A_m$ , the pseudospectrum  $\Lambda_\epsilon$  thus stretches out along the negative real axis, for *any* value of  $\epsilon > 0$ . Conversely, for the case of regular  $A_m$ , this only happens for  $\epsilon > w_m/\|A_m^{-1}\|_\alpha$ . As a consequence, *infinitesimal* perturbations may result in the introduction of eigenvalues with *small* imaginary parts (but large negative real parts).

The two cases are connected as follows: when the matrix  $A_m$  is regular, we have

$$\inf_{\delta A_m \in \mathbb{C}^{n \times n}} \{\|\delta A_m\|_\alpha : \det(A_m + \delta A_m) = 0\} = 1/\|A_m^{-1}\|_\alpha,$$

that is, the smallest rank reducing perturbation has size  $1/\|A_m^{-1}\|_\alpha$ . The smallest perturbation  $\Delta = (\delta A_0, \dots, \delta A_m)$  on the delay equation (22), which introduces an eigenvalue with a predetermined very large negative real part but small imaginary part, can be decomposed into a minimal size perturbation  $\Delta_c = (0, \dots, 0, \delta A_m)$  which makes  $A_m$  singular (due to the weights we have  $\|\Delta_c\|_{\text{glob}} = w_m/\|A_m^{-1}\|_\alpha$ ), together with a very small perturbation to place the eigenvalue, according to (39).

## 4 Illustrative examples

To demonstrate the above results we first consider the following DDE,

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-1) \quad (40)$$

where

$$A_0 = \begin{bmatrix} -5 & 1 \\ 2 & -6 \end{bmatrix} \quad \text{and} \quad A_1 = \begin{bmatrix} -2 & 1 \\ 4 & -1 \end{bmatrix}. \quad (41)$$

Figure 1(a) shows the spectrum of (40) computed using `DDE-BIFTOOL`, a `Matlab` package for the bifurcation analyses of DDEs [16]. The system is shown to be stable with all eigenvalues confined to the left-half plane. To investigate how this stability may change under perturbations of the matrices  $A_0$  and  $A_1$  we need to compute the corresponding pseudospectra.

To this end, we consider perturbations of  $A_0$  and  $A_1$  using the global measure (9) with  $p_1 = 2$  and  $p_2 = \infty$ . Pseudospectra can then be computed using Theorem 1 with  $\alpha = 2$  and  $\beta = 1$ . Specifically (for  $\lambda \notin \Lambda$ ),

$$f(\lambda) = \left\| \left( \lambda I - A_0 - A_1 e^{-\lambda} \right)^{-1} \right\|_2 \left( \frac{1}{w_0} + \frac{|e^{-\lambda}|}{w_1} \right). \quad (42)$$

By evaluating  $f$  on a grid over a region of the complex plane, and by using a contour plotter, we have identified the boundaries of  $\epsilon$ -pseudospectra.

Figures 1(b)–(d) show the  $\epsilon$ -pseudospectra of (40) where different weights have been applied to  $A_0$  and  $A_1$ . Specifically,  $(w_0, w_1) = (\infty, 1)$  (b),  $(w_0, w_1) = (2, 2)$  (c), and  $(w_0, w_1) = (1, \infty)$  (d). In each panel, from outermost to innermost (or rightmost to leftmost if the curve is not closed), the curves correspond to boundaries of  $\epsilon$ -pseudospectra with  $\epsilon = 10^{1.25}, 10^{1.0}, 10^{0.75}, 10^{0.5}, 10^{0.25}, 10^0$ , and  $10^{-0.5}$ . It can be seen that the conclusions drawn in Section 3.2 hold, that is, perturbations of  $A_0$  stretch pseudospectra lying in the right half-plane (d). While perturbations applied to  $A_1$  stretch the pseudospectra lying in the left half-plane (b). Furthermore, Fig. 2 shows the intersection of  $\epsilon$ -pseudospectrum curves with the imaginary axis. In each panel, the darkest curve corresponds to an  $\epsilon$ -pseudospectrum curve of Fig. 1(a), the next to a curve of Fig. 1(b), and the lightest curve corresponds to an  $\epsilon$ -pseudospectrum curve of Fig. 1(c). Specifically, Fig. 2(a) shows the intersection of the three curves for  $\epsilon = 10^{1.25}$ , Fig. 2(b) for  $\epsilon = 10^{1.0}$ , and Fig. 2(c) for  $\epsilon = 10^{0.75}$ . For a given  $\epsilon$ , these curves are seen to intersect the imaginary axis at the same point, independent of the weighting applied to the system matrices, thus, demonstrating the third conclusion of Section 3.2.

Figure 3 shows which  $\epsilon$ -pseudospectrum curve intersects the imaginary axis at  $\lambda = j\omega$ , that is  $f^{-1}(j\omega)$ , for each  $\omega \in [-50, 50]$ . The minimum of this curve represents the stability radius of the system,

$$r_{\mathbb{C}^-}(\|\cdot\|_{\text{glob}}) \approx 3.28011.$$

Since the minimum is reached for  $\omega = 0$  the smallest destabilizing perturba-

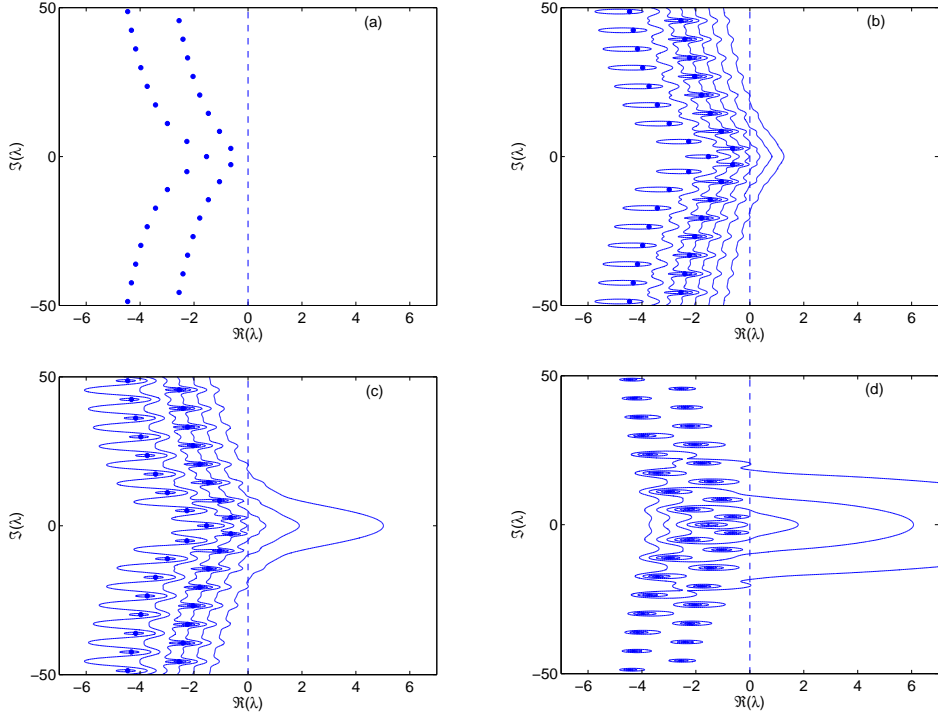


Fig. 1. Weighted pseudospectra of the DDE (40). Panel (a) shows the spectrum of the unperturbed problem computed using DDE-BIFTOOL. In all other panels, from rightmost to leftmost, the contours correspond to  $\epsilon = 10^{1.25}, 10^{1.0}, 10^{0.75}, 10^{0.5}, 10^{0.25}, 10^0$ , and  $10^{-0.5}$ . From (b) to (d), the weights  $w_0$  and  $w_1$  applied to the  $A_0$  and  $A_1$  matrices were  $(w_0, w_1) = (\infty, 1)$ ,  $(w_0, w_1) = (2, 2)$ , and  $(w_0, w_1) = (1, \infty)$ , respectively.

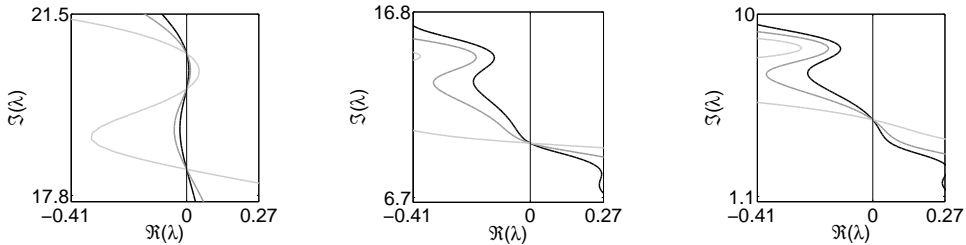


Fig. 2. Crossings of  $\epsilon$ -pseudospectrum curves with the imaginary axis, for  $\epsilon = 10^{1.25}$  (left),  $\epsilon = 10$  (middle) and  $\epsilon = 10^{0.75}$  (right). In the three cases the darkest contour corresponds to the weights  $(w_0, w_1) = (\infty, 1)$ , the middle curve to  $(2, 2)$  and the lightest curve to  $(1, \infty)$ .

tions shift an eigenvalue to the origin.

Proposition 8 applies to this problem with

$$l = \frac{w_1}{\|A_1^{-1}\|_2} \approx 0.4282 w_1. \quad (43)$$

In Figure 4(a) we show  $\epsilon$ -pseudospectra for the weights  $(w_0, w_1) = (\infty, 1)$

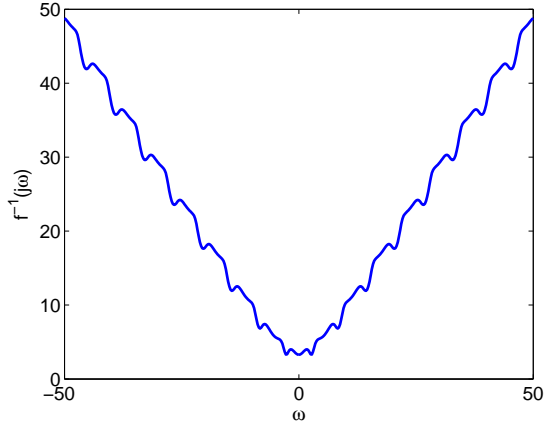


Fig. 3. The function  $\omega \rightarrow f^{-1}(j\omega)$  for the system (40). The minimum is the complex stability radius.

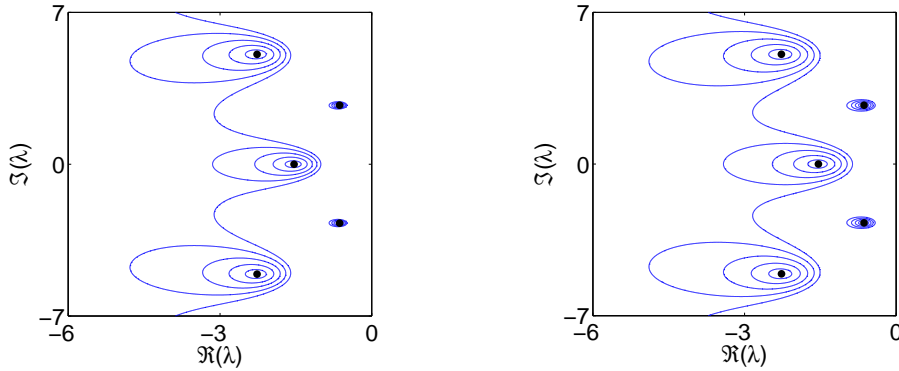


Fig. 4. (left)-  $\epsilon$ -pseudospectrum curves for  $(w_0, w_1) = (\infty, 1)$  and  $\epsilon = 0.1, 0.2, 0.3, 0.4, 0.5$ . (right)-  $\epsilon$ -pseudospectrum curves for  $(w_0, w_1) = (2, 2)$  and  $\epsilon = 0.2, 0.4, 0.6, 0.8, 1$ .

and  $\epsilon = 0.1, 0.2, 0.3, 0.4, 0.5$ . In Figure 4(b) we take  $(w_0, w_1) = (2, 2)$  and  $\epsilon = 0.2, 0.4, 0.6, 0.8, 1$ . In both cases only the  $\epsilon$ -pseudospectrum for the largest value of  $\epsilon$  stretches out infinitely far along the negative real axis, as follows from (43).

As a second example we analyze the system

$$\dot{x}(t) = Ax(t-1), \quad (44)$$

where

$$A = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}$$

and  $\|\Delta\|_{\text{glob}} = \|\delta A\|_2$ . The *singularity* of  $A$  implies by Proposition 8 and

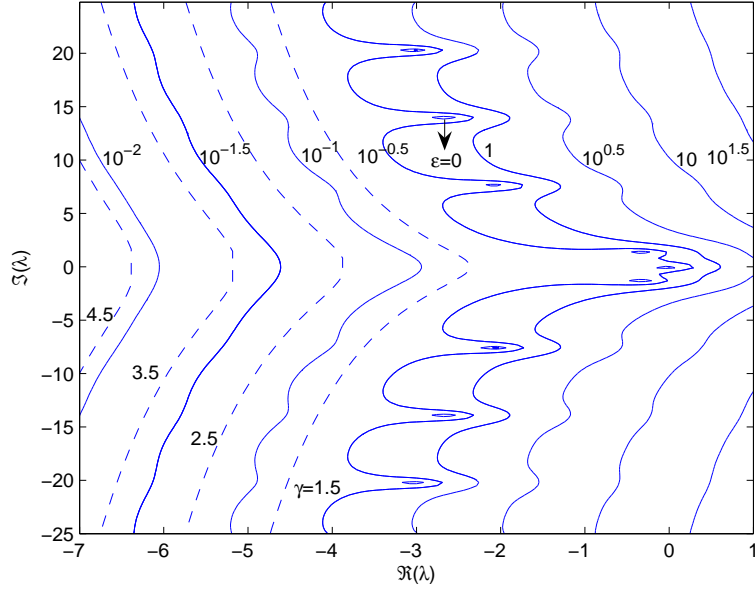


Fig. 5. Boundaries of pseudospectra of (44) for different values of  $\epsilon$  (solid lines), as well as the boundary of the sets  $\Psi_\gamma$ , for various values of  $\gamma$  (dashed lines).

Corollary 9 that all pseudospectra stretch out along the negative real axis, in contrast to the previous example. Figure 5 shows the boundaries of the pseudospectrum  $\Lambda_\epsilon$  for  $\epsilon = 10^{-2}, 10^{-1.5}, \dots, 10^{1.5}$ . Also displayed are the sets  $\Psi_\gamma$ , defined in Proposition 8, for  $\gamma = 1.5, 2.5, 3.5, 4.5$ . The remarkable correspondence indicates that the sets  $\Psi_\gamma$ , as defined in Proposition 8, are in some sense as large as possible with respect to the validity of the convergence property (32). The latter is for instance not valid anymore on the sets,

$$\left\{ \Re(\lambda) < \gamma, |\lambda| < e^{-(\Re(\lambda)+\gamma)\tau} \right\}_{\gamma>0},$$

if  $\tau > \tau_m$ .

From the pseudospectra it follows that even *arbitrarily small* perturbations may lead to the introduction of eigenvalues in the vicinity of the real axis. To illustrate this, we take the perturbation

$$\delta A = \begin{bmatrix} 0 & 0 \\ \mu & -\mu \end{bmatrix}, \quad \mu \in \mathbb{C}_0, \quad (45)$$

which results in the characteristic equation:

$$(\lambda + e^{-\lambda})(\lambda + \mu e^{-\lambda}) = 0. \quad (46)$$

Whereas the non-zero eigenvalues of the unperturbed system lie on a single

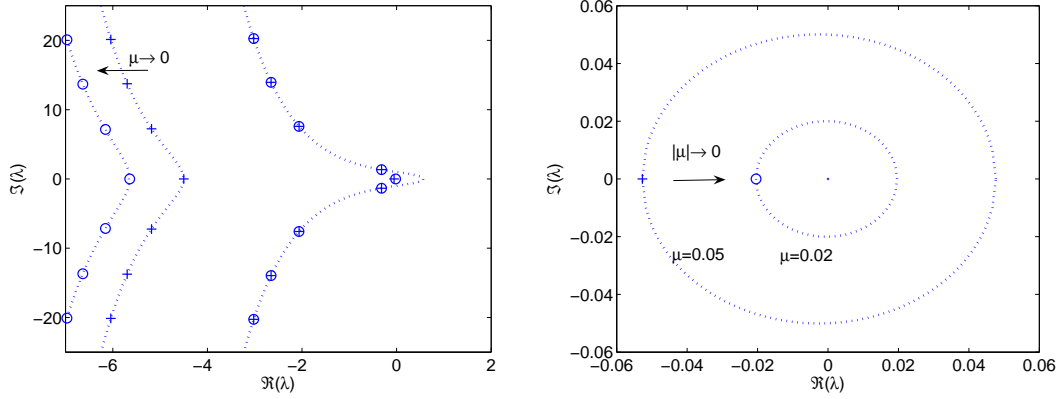


Fig. 6. (left)-Rightmost roots of (46) for  $\mu = 0.05$  ('+') and  $\mu = 0.02$  ('o'). The dotted lines correspond to the curves defined by (47) and (48). (right)-Close-up around  $\lambda = 0$ .

curve defined by

$$|\lambda| = e^{-\Re(\lambda)}, \quad (47)$$

the perturbed system exhibits an additional tail of infinitely many eigenvalues. The latter are lying on the curve

$$|\lambda| = |\mu|e^{-\Re(\lambda)} \quad \left( = e^{-\Re(\lambda) + \log |\mu|} \right), \quad (48)$$

whose left part shifts along the real axis towards  $-\infty$  as  $|\mu| \rightarrow 0$ . This is shown in Figure 6, where the rightmost eigenvalues are displayed for  $\mu = 0.05$  and  $\mu = 0.02$ .

One easily shows that any second order system of the form (44) has at most one asymptotic tail of eigenvalues if  $A$  is singular. Hence, a perturbation which destroys this topological property necessarily destroys the singularity of the matrix.

It is worthwhile to mention that the mechanism displayed in Figure 6 is in general not the only possible way in which infinitesimal perturbations of a system (22) with singular  $A_m$  can create additional eigenvalues along the negative real axis. For this, we conclude with the example

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \\ -\mu & 0 \end{bmatrix} x(t-1),$$

where  $\mu \in \mathbb{C}$  also represents a perturbation. The roots of the characteristic equation,

$$\lambda^2 + e^{-\lambda} + \mu e^{-2\lambda}, \quad (49)$$

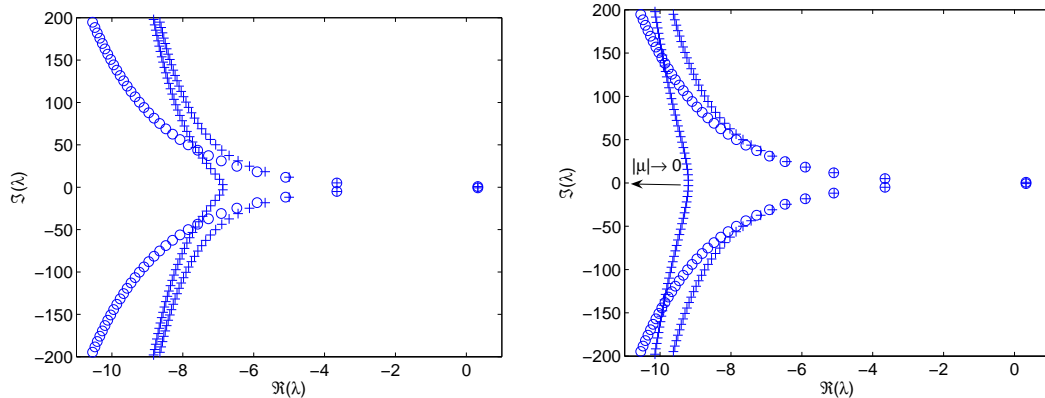


Fig. 7. (left)-Rightmost roots of (49) for  $\mu = 10^{-3}$  ('+') and  $\mu = 0$  ('o'). (right)-The rightmost roots for  $\mu = 10^{-4}$  ('+') and  $\mu = 0$  ('o').

are shown in Figure 7 for  $\mu = 0, 10^{-3}, 10^{-4}$ . For small  $\mu > 0$ , there are again two branches of eigenvalues. The right branch consists of eigenvalues, which uniformly converge on compact sets to eigenvalues of the unperturbed system, whereas the left branch contains eigenvalues whose real parts move off to minus infinity as  $|\mu| \rightarrow 0$ . However, for any fixed  $\mu \neq 0$  both branches converge to *one* asymptotic tail as  $|\lambda| \rightarrow \infty$ , having a larger slope than in the unperturbed case. This happens because the term  $\mu e^{-2\lambda}$  eventually dominates  $e^{-\lambda}$  in (49). This results in one asymptotic tail characterised by  $|\lambda|/(e^{-\Re(\lambda)}) \rightarrow \sqrt{|\mu|}$ , in contrast to the case  $\mu = 0$  where  $|\lambda|/(e^{-\Re(\lambda)/2}) = 1$ .

## 5 Concluding remarks

In the first part of this paper we presented a unifying treatment of pseudospectra and stability radii of analytic matrix functions. These may arise in the modelling and subsequent spectral analysis of systems described by higher-order differential equations, differential algebraic equations, delay differential (algebraic) equations. Various perturbation measures were considered and formulae for the computation of both pseudospectra and stability radii were derived.

In the second part we identified special properties of pseudospectra of a class of retarded delay differential equations and related these properties with the behaviour of the eigenvalues. The effect of weights in the perturbation measures on the pseudospectra was emphasised. It was shown that increased perturbations applied to the leading delay matrix stretched the pseudospectra to the left; whereas increased perturbations applied to the non-delayed matrix stretched the pseudospectra to the right. However, for all weighted perturbations, the intersections of the pseudospectrum contours with the imaginary axis and, as a consequence, the stability radius were shown to remain con-

stant. Furthermore, boundedness properties of a pseudospectrum contour were shown to be directly related to the weighting applied to the perturbations of the matrix corresponding to the largest delay and to the rank of this matrix. In the singular case, where all pseudospectra were shown to stretch out along the negative real axis, the asymptotic behaviour of the eigenvalues of the system, subjected to infinitesimal perturbations, was investigated.

One of the next steps is going beyond a theoretical analysis and incorporating the information obtained from pseudospectra in a physical application requiring control. This is motivated by issues such as pole placement, the optimizing of stability, robustness, and transient behaviour.

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