Determining stability of pulses for partial differential equations with time delays

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Report TW 405, September 2004

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Abstract

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Keywords : travelling waves, delay partial differential equations, stability

Determining stability of pulses for partial differential equations with time delays

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Abstract

Partial differential equations with time delays serve as models for systems where both spatial structure and memory effects are important. The asymptotic stability of travelling waves in such systems is still determined entirely by the spectrum of the linearization about the wave. We compare the spectra of localized waves on the unbounded real line with spectra computed on large intervals with appropriate boundary conditions applied at their end points. We show that the spectrum on large intervals approximates the spectrum on the real line when periodic boundary conditions are used. If separated boundary conditions are applied, it is the so-called absolute spectrum together with the extended point spectrum that is approximated; their union typically differs from the spectrum on the real line.

1 Introduction

Partial differential equations with time delays (DPDEs) are natural models for dynamical systems with spatial heterogeneity and memory effects. Examples of such systems arise mainly in biology, for instance, in population dynamics where the growing-up period from childhood to adulthood has a delayed effect and in epidemiology where incubation periods need to be taken into account [22]. In many of these systems, travelling waves may form which are states that evolve by moving with constant velocity through space while maintaining their shape. We are interested in finding numerical means to determine their stability.

We consider one-dimensional reaction-diffusion systems with a single time delay of the form

$$u_t(y,t) = Du_{yy}(y,t) + f(u(y,t), u(y,t-\tau)), \qquad y \in \mathbb{R} \tag{1.1}$$

where $u(y,t) \in \mathbb{R}^n$, D is a strictly positive, diagonal matrix, $\tau > 0$ is the fixed time delay, and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ denotes an analytic function. We remark that our results are true for more general partial differential equations

$$u_t(y,t) = \mathcal{A}(\partial_y)u(y,t) + f(u(y,t), u(y,t-\tau_1), \ldots, u(y,t-\tau_d))$$

with a finite number of discrete delays $\tau_j > 0$ for $j = 1, \ldots, d$, where $\mathcal{A}(z)$ is an appropriate matrix-valued polynomial in z so that $\mathcal{A}(\partial_y)$ is sectorial on a suitable function space. For the sake of clarity, however, we restrict ourselves to (1.1).

Travelling waves are solutions to (1.1) of the form

$$u(y,t) = q(y + ct)$$

for some $c \in \mathbb{R}$, where $q(x) \in \mathbb{R}^n$ with $x = y + ct$ satisfies the delay differential equation (DDE)

$$Dq''(x) - cq'(x) + f(q(x), q(x - c\tau)) = 0, \qquad x \in \mathbb{R}. \tag{1.2}$$

Alternatively, we may consider (1.1) in the moving coordinate frame $(x,t) = (y + ct, t)$ to get

$$u_t(x,t) = Du_{xx}(x,t) - cu_x(x,t) + f(u(x,t), u(x - c\tau, t - \tau)) \tag{1.3}$$

whose stationary solutions are travelling waves that satisfy (1.2).

In this work, we focus on pulses which are travelling waves whose profile $q(x)$ satisfies $q(x) \to q_\infty$ as $x \to \pm\infty$. Note that non-constant pulses come in one-parameter families: together with $q(x)$, any translate $q(x + x_0)$, for an arbitrary but fixed $x_0 \in \mathbb{R}$, is also a travelling wave.

Stability of pulses to DPDEs is determined by the spectrum of the linearization of (1.3) about the wave in question. In most applications, it appears to be hard to calculate these spectra analytically. One exception are the results in [20] where existence and stability of monotone fronts was proved for a class of scalar equations. Thus, in general, one has to resort to numerical computations. A natural strategy consists of truncating the spatial domain $\mathbb{R}$ to a large bounded interval, supplemented by appropriate boundary conditions. The resulting linear equation can then be discretized and its spectrum computed numerically.

The issue addressed in this paper is whether, and in what sense, the spectrum of the linearization posed on a large interval $(-\ell, \ell)$ approaches that on the unbounded domain as the interval length ℓ

tends to infinity. We shall see that the answer depends crucially on whether the boundary conditions are separated or not. For periodic boundary conditions, we indeed recover the spectrum on $\mathbb{R}$. If separated boundary conditions are used, however, we approximate not the original spectrum but a different spectral set, the so-called absolute spectrum together with the extended point spectrum, that will be described in §3 below.

These results are known for partial differential equations (PDEs) without delays [17, 19]. We show here merely how the spectral problem for DPDEs can be put into the form discussed in [19] and refer then to this work for the remaining parts of the proofs. Furthermore, extending the analysis in [2, §3.7], we prove that the Evans function in the infinite-dimensional context counts precisely[1] the algebraic multiplicity of eigenvalues. The main goal of this paper, however, is the proof of concept: as demonstrated in the example shown in §5, it is possible to compute spectra of pulses in DPDEs numerically.

We remark that pulses can be computed numerically as homoclinic solutions of the delay equation (1.2). A numerical algorithm for the computation of connecting orbits that utilizes projection boundary conditions has been derived in [16] and implemented in DDE-BIFTOOL, a MATLAB package for the bifurcation analysis of DDEs [5, 6].

This paper is organized as follows. In §2, we introduce notation, hypotheses, and the definition of spectra on $\mathbb{R}$, while the approximation of spectra on $\mathbb{R}$ by spectra on intervals $(-\ell, \ell)$ for $\ell \gg 1$ is discussed in §3. In §5, we outline the implementation of this approach for computing spectra and present the computational results for the FitzHugh–Nagumo equation with delays. Section 4 contains a discussion of the numerical computation of pulses and its theoretical justification. Lastly, we comment in §6 on extensions to fronts.

2 Spectra of pulses on the real line

2.1 Standing hypotheses

Throughout this paper, we assume that the nonlinearity $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is analytic and that $q(x)$ is a stationary solution of (1.3)

$$u_t(x,t) = Du_{xx}(x,t) - cu_x(x,t) + f(u(x,t), u(x - c\tau, t - \tau)) \tag{2.1}$$

for some strictly positive speed $c > 0$. We also assume that there are positive constants α and C and an element $q_\infty \in \mathbb{R}^n$ so that

$$|q(x) - q_\infty| \le C\mathrm{e}^{-\alpha|x|}, \qquad x \in \mathbb{R}.$$

Lastly, we assume that the matrix J_1^∞ defined below in (2.4) is invertible.

We remark that analyticity of f is used only to ensure backward uniqueness of solutions (see [10, Hypothesis 1.1 and §4.4]).

[1] In [2, Proposition 3.18], the authors were only able to prove that the algebraic multiplicity is less or equal to the order of roots of the Evans function.

2.2 Linearization and spectra

Stability of travelling waves is usually analysed by linearizing the equation around the wave and examining the spectrum of the resulting linear operator. The spectrum should then provide information about the stability of the wave with respect to the full nonlinear equation. Thus, we linearize (2.1) about $q(x)$ and obtain the equation

$$u_t(x,t) = Du_{xx}(x,t) - cu_x(x,t) + J_0(x)u(x,t) + J_1(x)u(x - c\tau, t - \tau) \tag{2.2}$$

where

$$J_0(x) = \partial_{u_0} f(q(x), q(x - c\tau)), \qquad J_1(x) = \partial_{u_1} f(q(x), q(x - c\tau)) \tag{2.3}$$

with $f = f(u_0, u_1)$. As shown in [20], equation (2.2) generates a $\mathcal{C}^0$-semigroup on the space

$$X_\infty := \mathcal{C}^0(\mathbb{R} \times [-\tau, 0])$$

with generator

$$[\mathcal{B}u](x,t) = \begin{cases} u_t(x,t) & -\tau < t < 0 \\ [Bu](x,0) & t = 0 \end{cases}$$

where $[Bu](x,t)$ denotes the right-hand side of (2.2). Seeking solutions of (2.2) via formal Laplace transform using the ansatz

$$u(x,t) = \mathrm{e}^{\lambda t} u(x),$$

we see that $u(x,t)$ satisfies (2.2) if, and only if, $u(x)$ satisfies $\mathcal{L}_\lambda u = 0$ where $\mathcal{L}_\lambda$ is the closed, densely defined operator

$$\begin{aligned} \mathcal{L}_\lambda \quad &: \quad L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}) \\ [\mathcal{L}_\lambda u](x) \quad &:= \quad Du_{xx}(x) - cu_x(x) + [J_0(x) - \lambda]u(x) + \mathrm{e}^{-\lambda\tau} J_1(x) u(x - c\tau) \end{aligned}$$

with domain $\mathrm{D}(\mathcal{L}) = H^2(\mathbb{R})$. The spectrum of the operator pencil $\mathcal{L}_\lambda$ is defined via $\Sigma = \Sigma_{\mathrm{pt}} \cup \Sigma_{\mathrm{ess}}$ where[2]

$$\begin{aligned} \Sigma_{\mathrm{pt}} \quad &= \quad \{\lambda \in \mathbb{C};\ \mathcal{L}_\lambda \text{ is Fredholm with index 0 and } \dim \mathrm{N}(\mathcal{L}_\lambda) > 0\} \\ \Sigma_{\mathrm{ess}} \quad &= \quad \{\lambda \in \mathbb{C};\ \mathcal{L}_\lambda \text{ is not Fredholm}\}. \end{aligned}$$

We remark that the Fredholm index for pulses is either zero or not defined. The algebraic multiplicity of eigenvalues of the pencil $\mathcal{L}_\lambda$ is defined as usual via Jordan chains

$$\mathcal{L}_\lambda u_j = (\partial_\lambda \mathcal{L}_\lambda) u_{j-1}.$$

Schaaf proved in [20, Lemma 4.1 and 4.2] that the spectra of $\mathcal{B}$ on X_∞ and the operator pencil $\mathcal{L}_\lambda$ on $L^2(\mathbb{R})$ coincide, and it therefore suffices to find Σ. We record that, due to translational invariance, $\lambda = 0$ is always contained in the spectrum with $u(x) = q_x(x)$.

The essential spectrum Σ_{ess} is determined as follows by the asymptotic rest state q_∞. If we replace the travelling wave $q(x)$ by the asymptotic rest state q_∞, we obtain the operator

$$\begin{aligned} \mathcal{L}_\lambda^\infty \quad &: \quad L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}) \\ [\mathcal{L}_\lambda^\infty u](x) \quad &:= \quad Du_{xx}(x) - cu_x(x) + [J_0^\infty - \lambda]u(x) + \mathrm{e}^{-\lambda\tau} J_1^\infty u(x - c\tau) \end{aligned}$$

[2]Recall that an operator $\mathcal{L}$ is said to be Fredholm if its range $\mathrm{R}(\mathcal{L})$ is closed, and both the dimension of its null space $\mathrm{N}(\mathcal{L})$ and the codimension of $\mathrm{R}(\mathcal{L})$ are finite.

where

$$J_0^\infty = \partial_{u_0} f(q_\infty, q_\infty), \qquad J_1^\infty = \partial_{u_1} f(q_\infty, q_\infty). \tag{2.4}$$

Nontrivial solutions of the equation $\mathcal{L}_\lambda^\infty u = 0$ in the form $u(x) = \mathrm{e}^{\nu x} u_0$ with $u_0 \in \mathbb{C}^n$ exist precisely when

$$\Delta(\lambda, \nu) := \det \left[D\nu^2 - c\nu + J_0^\infty - \lambda + \mathrm{e}^{-(\lambda+\nu c)\tau} J_1^\infty \right] \tag{2.5}$$

vanishes. We refer to the function $\Delta(\lambda, \nu)$ as the linear dispersion relation of $\mathcal{L}_\lambda^\infty$. We then have [20, Theorem 4.5]

$$\Sigma_{\text{ess}} = \{\lambda \in \mathbb{C};\ \Delta(\lambda, \mathrm{i}k) = 0 \text{ for some } k \in \mathbb{R}\}. \tag{2.6}$$

For later use, we state the following result whose proof we omit as it is a straightforward scaling argument.

Lemma 2.1 *There is a number $r \in \mathbb{R}$ such that the function $\Delta(\lambda, \nu)$ has, for each fixed $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda > r$, precisely n roots $\nu_1, \ldots, \nu_n$ with positive real part, while the remaining roots have negative real parts.*

With the essential spectrum given in (2.6), it suffices to determine the point spectrum. Referring again to [20, §4], a complex number λ is in Σ_{pt} if, and only if, there is a non-zero, exponentially decaying solution $u(x)$ of the eigenvalue problem

$$Du_{xx}(x) - cu_x(x) + [J_0(x) - \lambda]u(x) + \mathrm{e}^{-\lambda\tau} J_1(x) u(x - c\tau) = 0, \tag{2.7}$$

which we also write as the first-order system

$$U_x(x) = A_0(x; \lambda) U(x) + A_1(x; \lambda) U(x - c\tau)$$

where $U = (u, u_x) \in \mathbb{C}^{2n}$ and

$$A_0(x; \lambda) = \begin{pmatrix} 0 & 1 \\ D^{-1}[\lambda - J_0(x)] & cD^{-1} \end{pmatrix}, \qquad A_1(x; \lambda) = \begin{pmatrix} 0 & 0 \\ -D^{-1}\mathrm{e}^{-\lambda\tau} J_1(x) & 0 \end{pmatrix}.$$

2.3 The dynamical-systems formulation

It will turn out to be advantageous to describe the point spectrum using a dynamical-systems formulation of (2.7). Thus, we review here the results presented in [1, 2, 10, 14] where this formulation was investigated in detail. Define the spaces

$$Y_0 := L^2([-c\tau, 0], \mathbb{C}^{2n}) \times \mathbb{C}^{2n}, \qquad Y_1 := \{(\phi, a) \in Y_0;\ \phi \in H^1([-c\tau, 0], \mathbb{C}^{2n}) \text{ and } \phi(0) = a\},$$

where Y_1 is dense in Y_0, and the closed operator

$$\mathcal{A}(x; \lambda): \quad Y_0 \longrightarrow Y_0, \qquad \begin{pmatrix} \phi \\ a \end{pmatrix} \longmapsto \begin{pmatrix} \phi' \\ A_0(x; \lambda) a + A_1(x; \lambda) \phi(-c\tau) \end{pmatrix}$$

with domain Y_1, then solutions to (2.7) and the dynamical system

$$\frac{\mathrm{d}V}{\mathrm{d}x} = \mathcal{A}(x; \lambda) V \tag{2.8}$$

are in one-to-one correspondence via $V(x) = (u|_{[x-c\tau,x]}, u(x))$ by [1, Proposition 1] or [10, Lemma 2.2]. Similarly, the eigenvalue problem $\mathcal{L}_\lambda^\infty u = 0$ for the asymptotic constant-coefficient operator can be written in first-order form as

$$U_x(x) = A_0^\infty(\lambda)U(x) + A_1^\infty(\lambda)U(x - c\tau), \tag{2.9}$$

with $U = (u, u_x) \in \mathbb{C}^{2n}$, and in dynamical-systems form as

$$\frac{\mathrm{d}V}{\mathrm{d}x} = \mathcal{A}_\infty(\lambda)V \tag{2.10}$$

for $V \in Y_0$, where

$$\mathcal{A}_\infty(\lambda)\begin{pmatrix} \phi \\ a \end{pmatrix} = \begin{pmatrix} \phi' \\ A_0^\infty(\lambda)a + A_1^\infty(\lambda)\phi(-c\tau) \end{pmatrix}$$

with

$$A_0^\infty(\lambda) = \begin{pmatrix} 0 & 1 \\ D^{-1}[\lambda - J_0^\infty] & cD^{-1} \end{pmatrix}, \qquad A_1^\infty(\lambda) = \begin{pmatrix} 0 & 0 \\ -D^{-1}\mathrm{e}^{-\lambda\tau}J_1^\infty & 0 \end{pmatrix}.$$

The spectrum of the operator $\mathcal{A}_\infty(\lambda)$ on Y_0 is discrete: in fact, $\nu \in \mathbb{C}$ is in the spectrum of $\mathcal{A}_\infty(\lambda)$ if, and only if, $\det \Delta(\lambda, \nu) = 0$ in which case the corresponding eigenfunction is of the form

$$\begin{pmatrix} \phi(\eta) \\ a \end{pmatrix} = \begin{pmatrix} a\mathrm{e}^{\eta\nu} \\ a \end{pmatrix}.$$

We refer to the eigenvalues ν of $\mathcal{A}_\infty(\lambda)$ as spatial eigenvalues to distinguish them from the temporal eigenvalues λ.

We say that (2.8) and (2.10) are hyperbolic if the spectrum of $\mathcal{A}_\infty(\lambda)$ contains no points on the imaginary axis. Thus, these equations are hyperbolic precisely when λ is not in the essential spectrum Σ_{ess}. If (2.8) is hyperbolic, it has exponential dichotomies, see [1, 2, 10, 14], on $\mathbb{R}^\pm$ with projections $P^{\mathrm{u}}(x;\lambda)$ on $\mathbb{R}^-$ and $P^{\mathrm{s}}(x;\lambda)$ on $\mathbb{R}^+$. Furthermore, the range $\mathrm{R}(P^{\mathrm{u}}(x;\lambda))$ has finite dimension given by the number of spatial eigenvalues ν of $\mathcal{A}_\infty(\lambda)$ with $\mathrm{Re}\,\nu > 0$, counted with multiplicity. In line with this, (2.8) actually admits a strongly continuous semigroup $\mathcal{T}(x, y; \lambda)$, defined for $x \geq y$, of solution operators on Y_0.

Next, we shall consider the adjoint equation. We identify the dual space of Y_0 with the space

$$Y_0^* := L^2([0, c\tau], \mathbb{C}^{2n}) \times \mathbb{C}^{2n}$$

so that the dual product associated with Y_0 and Y_0^* becomes

$$\left\langle \begin{pmatrix} \phi \\ a \end{pmatrix}, \begin{pmatrix} \psi \\ b \end{pmatrix} \right\rangle = a \cdot \bar{b} + \int_{-c\tau}^0 \phi(\eta) \cdot \overline{\psi(\eta + c\tau)}\,\mathrm{d}\eta. \tag{2.11}$$

The adjoint equation associated with (2.8) is then given by

$$\frac{\mathrm{d}W}{\mathrm{d}x} = -\mathcal{A}(x;\lambda)^* W \tag{2.12}$$

where

$$\mathcal{A}(x;\lambda)^* : \quad Y_0^* \longrightarrow Y_0^*, \qquad (\psi, b) \longmapsto (-\psi', \psi(c\tau) + A_0(x;\lambda)^* b)$$

is defined on Y_0^* with dense domain Y_1^* given by

$$Y_1^* := \{(\psi, b) \in Y_0^*;\ \psi \in H^1([0, c\tau], \mathbb{C}^{2n}) \text{ and } \psi(0) = -A_1(x;\lambda)^* b\}.$$

Note that, while the domain of $\mathcal{A}(x;\lambda)^*$ depends on x, equation (2.12) admits the semigroup $\mathcal{T}^*(x,y;\lambda) = (\mathcal{T}(y,x;\lambda))^*$, computed with respect to (2.11), for $x \le y$ [1, 2]. A key feature of solutions $V(x)$ to (2.8) and $W(x)$ to (2.12) is that their dual product $\langle W(x), V(x)\rangle$ is independent of x. For $\lambda \notin \Sigma_{\rm ess}$, (2.8) is hyperbolic, and (2.12) then has exponential dichotomies on $\mathbb{R}^\pm$ with projections $Q^{\rm u}(x;\lambda)$ on $\mathbb{R}^-$ and $Q^{\rm s}(x;\lambda)$ on $\mathbb{R}^+$ with $\mathrm{R}(Q^{\rm s}_\lambda(x)) \perp \mathrm{R}(P^{\rm s}(x;\lambda))$ for $x \ge 0$. Furthermore, we have

$$\dim \mathrm{R}(P^{\rm u}(0;\lambda)) = \dim \mathrm{R}(Q^{\rm s}(0;\lambda)) = n_{\rm u},$$

where $n_{\rm u}$ is independent of λ in each connected component of $\mathbb{C} \setminus \Sigma_{\rm ess}$. Note that $n_{\rm u} = n$ for $\mathrm{Re}\,\lambda \gg 1$ by Lemma 2.1.

We are now in a position to state the definition of the Evans function constructed in this context in [1, 2]. For any $\lambda \notin \Sigma_{\rm ess}$, equations (2.8) and (2.12) are hyperbolic and therefore admit exponential dichotomies as outlined above. We choose analytic bases $\{V_i(0;\lambda)\}_{i=1,\dots,n_{\rm u}}$ and $\{W_i(0;\lambda)\}_{i=1,\dots,n_{\rm u}}$ of $\mathrm{R}(P^{\rm u}(0;\lambda))$ and $\mathrm{R}(Q^{\rm s}(0;\lambda))$, respectively. The Evans function $D(\lambda)$ is then defined as

$$D(\lambda) = \det\left(\langle V_i(0;\lambda), W_j(0;\lambda)\rangle\right)_{1\le i,j\le n_{\rm u}}. \tag{2.13}$$

It has been proved in [1] that $D(\lambda)$ is analytic in λ for $\lambda \notin \Sigma_{\rm ess}$, and $D(\lambda) = 0$ if, and only if, $\lambda \in \Sigma_{\rm pt}$.

Remark 2.2 *We emphasize that we shall use the Evans function (2.13), and other related Evans functions defined later on, only locally in λ. To ensure analyticity in this context, it suffices to construct, again locally in λ, analytic exponential dichotomies which can be done using the roughness theorem for exponential dichotomies [15]. Locally analytic bases can then be found by projecting a fixed basis at λ_* onto the relevant subspaces using the locally analytic projections associated with the exponential dichotomies.*

We assume that Evans function does not vanish identically:

Hypothesis 2.3 *The point spectrum $\Sigma_{\rm pt}$ is discrete.*

Proposition 2.4 *Assume that Hypothesis 2.3 is met, then the order of a complex number $\lambda \notin \Sigma_{\rm ess}$ as a root of the Evans function $D(\lambda)$ and the algebraic multiplicity of λ as an eigenvalue of $\mathcal{L}_\lambda$ coincide.*

Proof. The inequality $\mathrm{a.m.}(\lambda) \le \mathrm{ord}(\lambda, D)$ has been shown in [2, Proposition 3.16]. We shall strengthen this result by showing the opposite inequality.

Thus, we shall assume that $D(\lambda_*) = D'(\lambda_*) = 0$ and prove that the algebraic multiplicity is equal to or larger than two; the generalization to larger order is tedious but straightforward and involves combining the arguments given below with the proof given in [11] in finite dimensions.

Note that the algebraic multiplicity of λ_* is defined in terms of (2.8) by maximal Jordan chains of bounded solutions on $\mathbb{R}$ of

$$\frac{\mathrm{d}V_j}{\mathrm{d}x} = \mathcal{A}(x;\lambda)V_j + \mathcal{A}_\lambda(x;\lambda)V_{j-1}. \tag{2.14}$$

First, if the geometric multiplicity is equal to two, we are done. Hence, we can assume that λ_* has geometric multiplicity one. We need to prove that the algebraic multiplicity is at least two.

Exploiting that the geometric multiplicity is one, we may choose the bases $\{V_i(0;\lambda)\}_{i=1,\dots,n_\mathrm{u}}$ and $\{W_i(0;\lambda)\}_{i=1,\dots,n_\mathrm{u}}$ of $\mathrm{R}(P^\mathrm{u}(0;\lambda))$ and $\mathrm{R}(Q^\mathrm{s}(0;\lambda))$, respectively, with the following properties: $V_1(x;\lambda_*)$ corresponds to the eigenfunction belonging to λ_* (in particular, $V_1(x;\lambda_*)$ is a bounded solution on $\mathbb{R}$ of (2.8)) and

$$\begin{aligned} \langle V_1(0;\lambda_*), W_1(0;\lambda_*)\rangle &= 0 \\ \langle V_i(0;\lambda_*), W_i(0;\lambda_*)\rangle &= 1, \qquad i \neq 1 \\ \langle V_i(0;\lambda_*), W_j(0;\lambda_*)\rangle &= 0, \qquad i \neq j. \end{aligned} \tag{2.15}$$

In particular, $W_1(0;\lambda_*)$ is the unique element (up to scalar multiples) in Y_0^* which annihilates the subspace $\mathrm{R}(P^\mathrm{u}(0;\lambda_*)) + \mathrm{R}(P^\mathrm{s}(0;\lambda_*))$. Furthermore, it follows easily from the normalization (2.15) and the assumption that the derivative $D'(\lambda_*)$ of the Evans function vanishes that

$$\left\langle \frac{\mathrm{d}V_1}{\mathrm{d}\lambda}(0;\lambda_*), W_1(0;\lambda_*)\right\rangle + \left\langle V_1(0;\lambda_*), \frac{\mathrm{d}W_1}{\mathrm{d}\lambda}(0;\lambda_*)\right\rangle = 0. \tag{2.16}$$

Using the exponential dichotomies associated with (2.8), we see that the solutions $V_1^\pm(x;\lambda)$ of (2.8) defined by

$$V_1^+(x;\lambda) := \mathcal{T}(x,0;\lambda)P^\mathrm{s}(0;\lambda)V_1(0;\lambda_*), \qquad V_1^-(x;\lambda) := V_1(x;\lambda) \tag{2.17}$$

are bounded for $x \in \mathbb{R}^\pm$, respectively, and smooth in λ. Furthermore, taking derivatives in (2.8) and using again smoothness of exponential dichotomies in exponentially weighted norms, we see that the derivatives $\frac{\mathrm{d}V_1^\pm}{\mathrm{d}\lambda}(x;\lambda_*)$ are bounded solutions of

$$\frac{\mathrm{d}V}{\mathrm{d}x} = \mathcal{A}(x;\lambda_*)V + \mathcal{A}_\lambda(x;\lambda_*)V_1(x;\lambda_*) \tag{2.18}$$

on $\mathbb{R}^\pm$, respectively. Thus, the general bounded solutions of (2.18) on $\mathbb{R}^\pm$ are given by

$$V^+(x) := \mathcal{T}(x,0;\lambda_*)V_0^+ + \frac{\mathrm{d}V_1^+}{\mathrm{d}\lambda}(x;\lambda_*), \qquad V^-(x) := \mathcal{T}(x,0;\lambda_*)V_0^- + \frac{\mathrm{d}V_1^-}{\mathrm{d}\lambda}(x;\lambda_*),$$

respectively, where

$$V_0^+ \in \mathrm{R}(P^\mathrm{s}(0;\lambda_*)), \qquad V_0^- \in \mathrm{R}(P^\mathrm{u}(0;\lambda_*)) \tag{2.19}$$

are arbitrary.

The next task is to find $V_0^\pm$ subject to (2.19) so that $V^+(0) = V^-(0)$, that is,

$$V_0^+ + \frac{\mathrm{d}V_1^+}{\mathrm{d}\lambda}(0;\lambda_*) = V_0^- + \frac{\mathrm{d}V_1^-}{\mathrm{d}\lambda}(0;\lambda_*). \tag{2.20}$$

Indeed, we would then obtain a solution $V(x)$ of (2.18) which is continuous at $x = 0$ and bounded for $x \in \mathbb{R}$, and therefore corresponds to the desired generalized eigenfunction of the operator pencil $\mathcal{L}_{\lambda_*}$. Elements $V_0^\pm$ that satisfy (2.19) and (2.20) can be found if, and only if,

$$\left\langle \frac{\mathrm{d}V_1^-}{\mathrm{d}\lambda}(0;\lambda_*) - \frac{\mathrm{d}V_1^+}{\mathrm{d}\lambda}(0;\lambda_*), W_1(0;\lambda_*)\right\rangle = 0, \tag{2.21}$$

since the comment right after (2.15) shows that (2.21) is equivalent to

$$\frac{\mathrm{d}V_1^-}{\mathrm{d}\lambda}(0;\lambda_*) - \frac{\mathrm{d}V_1^+}{\mathrm{d}\lambda}(0;\lambda_*) \in \mathrm{R}(P^\mathrm{u}(0;\lambda_*)) + \mathrm{R}(P^\mathrm{s}(0;\lambda_*)).$$

It remains to prove that our starting assumption (2.16) implies (2.21) whose left-hand side we can write as

$$\left\langle \frac{\mathrm{d}V_1}{\mathrm{d}\lambda}(0;\lambda_*) - \frac{\mathrm{d}V_1^+}{\mathrm{d}\lambda}(0;\lambda_*), W_1(0;\lambda_*) \right\rangle \tag{2.22}$$

on account of (2.17). Comparing (2.22) with (2.16), we see that we need to prove that

$$-\left\langle \frac{\mathrm{d}V_1^+}{\mathrm{d}\lambda}(0;\lambda_*), W_1(0;\lambda_*) \right\rangle = \left\langle V_1(0;\lambda_*), \frac{\mathrm{d}W_1}{\mathrm{d}\lambda}(0;\lambda_*) \right\rangle.$$

Since $V_1^+(x;\lambda)$ is a decaying solution on $\mathbb{R}^+$ of (2.8), we have by construction that

$$\langle V_1^+(0;\lambda), W_1(0;\lambda)\rangle = 0$$

for all λ. Taking the derivative with respect to λ evaluated at λ_*, we get

$$\begin{aligned} 0 &= \left\langle \frac{\mathrm{d}V_1^+}{\mathrm{d}\lambda}(0;\lambda_*), W_1(0;\lambda_*) \right\rangle + \left\langle V_1^+(0;\lambda_*), \frac{\mathrm{d}W_1}{\mathrm{d}\lambda}(0;\lambda_*) \right\rangle \\ &\overset{(2.17)}{=} \left\langle \frac{\mathrm{d}V_1^+}{\mathrm{d}\lambda}(0;\lambda_*), W_1(0;\lambda_*) \right\rangle + \left\langle V_1(0;\lambda_*), \frac{\mathrm{d}W_1}{\mathrm{d}\lambda}(0;\lambda_*) \right\rangle \end{aligned}$$

as needed. This completes the proof of the proposition. ∎

3 Spectra of pulses on large bounded intervals

We consider the same setting as in the previous section and assume, in particular, that the assumptions in §2.1 are met.

3.1 Periodic boundary conditions

We begin with the dynamical system (2.8)

$$\frac{\mathrm{d}V}{\mathrm{d}x} = \mathcal{A}(x;\lambda)V \tag{3.1}$$

on Y_0. For each fixed $\ell \gg 1$, we define $\Sigma_\ell^{\mathrm{per}}$ as the set of all $\lambda \in \mathbb{C}$ for which (3.1) has a nonzero solution $V(x)$ on $[-\ell, \ell]$ with

$$V(\ell) = V(-\ell). \tag{3.2}$$

Before we state our result on the behaviour of $\Sigma_\ell^{\mathrm{per}}$ in the limit $\ell \to \infty$, we introduce a technical hypothesis. We define the subset $\mathcal{S}_{\mathrm{per}}$ of Σ_{ess} to be the set of all $\lambda \in \Sigma_{\mathrm{ess}}$ for which

$$\mathrm{spec}(\mathcal{A}_\infty(\lambda)) \cap \mathrm{i}\mathbb{R} = \{\nu(\lambda_*)\}$$

is a simple spatial eigenvalue $\nu(\lambda_*) \in \mathrm{i}\mathbb{R}$ with the additional property that $\mathrm{d}\,\mathrm{Im}\,\nu/\mathrm{d}\lambda|_{\lambda_*} \neq 0$ where $\nu(\lambda)$ is the eigenvalue of $\mathcal{A}_\infty(\lambda)$ that is close to $\nu(\lambda_*)$ for λ close to λ_*.

Hypothesis 3.1 *The set $\mathcal{S}_{\mathrm{per}}$ is dense in Σ_{ess}.*

We can now state the announced approximation result.

Theorem 1 *Assume that the assumptions in §2.1 as well as Hypotheses 2.3 and 3.1 are met.*

(i) Fix a root λ_* of $D(\lambda)$ of order ρ in $\Sigma_{\rm pt}$. As $\ell \to \infty$, there are precisely ρ elements in $\Sigma_\ell^{\rm per}$, counted with multiplicity, close to λ_*, and these elements converge to λ_* as $\ell \to \infty$.

(ii) Fix $\lambda_* \in \Sigma_{\rm ess}$, then λ_* is approached by infinitely many elements in $\Sigma_\ell^{\rm per}$ as $\ell \to \infty$.

(iii) Fix a bounded open set $\Omega \subset \mathbb{C}$. For any $\delta > 0$, there is an ℓ_* such that $(\Sigma_\ell^{\rm per} \cap \Omega) \subset \mathcal{U}_\delta(\Sigma)$ for all $\ell > \ell_*$, where $\mathcal{U}_\delta(\Sigma)$ denotes a δ-neighborhood of Σ.

In particular, $\Sigma_\ell^{\rm per}$ converges, in the symmetric Hausdorff distance and in each bounded region of the complex plane, to Σ as $\ell \to \infty$.

Proof. Assertion (i) can be proved most easily using Lyapunov–Schmidt reduction near each root λ of $D(\lambda)$ (see [18] for details). The statement about the multiplicity follows from Proposition 2.4.

The convergence of $\Sigma_\ell^{\rm per}$ to $\Sigma_{\rm ess}$ in bounded regions of $\mathbb{C}$ follows as in [19, §5.2] where Lyapunov–Schmidt reduction was utilized.

Lastly, using exponential dichotomies on $\mathbb{R}^\pm$, it is straightforward to show that 2ℓ-periodic solutions of (3.1) do not exist for all $\ell \gg 1$ whenever $\lambda \notin \Sigma$ (see, for instance, [18, §3.3] and specifically [18, Lemma 3.3] with $D = G = \gamma = 0$). ∎

Next, consider the operator

$$\begin{aligned} \mathcal{L}_\lambda^{\rm per} \quad &: \quad L^2(S_\ell^1) \longrightarrow L^2(S_\ell^1) \\ [\mathcal{L}_\lambda^{\rm per} u](x) \quad &:= \quad D u_{xx}(x) - c u_x(x) + [J_0(x) - \lambda] u(x) + \mathrm{e}^{-\lambda\tau} J_1(x) u(x - c\tau) \end{aligned} \tag{3.3}$$

with domain $H^2(S_\ell^1)$ where $S_\ell^1 := [-\ell, \ell]/\sim$ denotes the circle with circumference 2ℓ.

Lemma 3.2 *Assume that the assumptions in §2.1 as well as Hypotheses 2.3 and 3.1 are met, then $\Sigma_\ell^{\rm per}$ coincides with the set of all $\lambda \in \mathbb{C}$ for which $\mathcal{L}_\lambda^{\rm per}$ has a non-trivial null space, and the algebraic multiplicities of elements in the two sets are equal to each other.*

Proof. The first statement follows from the correspondence of solutions to (3.1)-(3.2) and $\mathcal{L}_\lambda^{\rm per} u = 0$, while the assertion regarding the algebraic multiplicity can be proved by comparing (2.14) and $\mathcal{L}_\lambda^{\rm per} u_j = (\partial_\lambda \mathcal{L}_\lambda^{\rm per}) u_{j-1}$. We refer to [1, 2, 10] for similar results. ∎

Lastly, we compare (3.3) with the linear DPDE

$$u_t(x,t) = D u_{xx}(x,t) - c u_x(x,t) + J_0(x) u(x,t) + J_1(x) u(x - c\tau, t - \tau)$$

on $S_\ell^1 = [-\ell, \ell]/\sim$. On account of the results stated in [22, §2.1], this equation generates a semiflow $\mathcal{T}_\ell(t)$ on $X_\ell^{\rm per} := \mathcal{C}^0(S_\ell^1 \times [-\tau, 0])$, and the spectrum of $\mathcal{T}_\ell(t)$ is given by the set $\exp(\Sigma_\ell^{\rm per} t)$ plus possibly $\{0\}$.

3.2 Separated boundary conditions

Next, we discuss the approximation of spectra upon using separated boundary conditions. We shall use again the dynamical system (2.8)

$$\frac{\mathrm{d}V}{\mathrm{d}x} = \mathcal{A}(x; \lambda) V \tag{3.4}$$

on Y_0 and add the boundary conditions

$$V(-\ell) \in E^-_{\rm bc}, \qquad V(\ell) \in E^+_{\rm bc} \tag{3.5}$$

where the subspaces $E^{\pm}_{\rm bc}$ of Y_0 satisfy the following assumption.

Hypothesis 3.3 *We assume that $E^-_{\rm bc}$ and $E^+_{\rm bc}$ are closed subspaces of Y_0 such that $\dim E^-_{\rm bc} = n$ and $\operatorname{codim} E^+_{\rm bc} = n$.*

We remark that Lemma 2.1 is the reason why the dimension n is singled out and refer to the discussion below for more details.

We define $\Sigma^{\rm sep}_\ell$ as the set of all $\lambda \in \mathbb{C}$ for which (3.4)-(3.5) has a non-zero solution. Multiplicity is again defined using (2.14). We wish to determine whether, and in what sense, the limit of $\Sigma^{\rm sep}_\ell$ as $\ell \to \infty$ exists. Before we can state the result, we need to introduce additional definitions.

Recall from Lemma 2.1 that $\Delta(\lambda, \nu)$ has precisely n roots ν with positive real part for each fixed $\lambda \in \mathbb{C}$ with sufficiently large real part. We also know, for instance from [4, Theorem I.4.4(ii)], that we can label the roots $\nu_j(\lambda)$ of $\Delta(\lambda, \nu)$ with respect to their real part so that

$$\ldots \le \operatorname{Re}\nu_{j+1}(\lambda) \le \operatorname{Re}\nu_j(\lambda) \le \operatorname{Re}\nu_{j-1}(\lambda) \le \ldots \le \operatorname{Re}\nu_1(\lambda), \qquad j \in \mathbb{N}.$$

Lemma 2.1 can then also be stated as $\operatorname{Re}\nu_{n+1}(\lambda) < 0 < \operatorname{Re}\nu_n(\lambda)$ for all $\lambda \in \mathbb{C}$ with large real part. The absolute spectrum $\Sigma_{\rm abs}$ is now defined via

$$\Sigma_{\rm abs} = \{\lambda \in \mathbb{C};\ \operatorname{Re}\nu_{n+1}(\lambda) = \operatorname{Re}\nu_n(\lambda)\}.$$

Lastly, we define the subset $\mathcal{S}_{\rm sep}$ of $\Sigma_{\rm abs}$ as follows: $\lambda_* \in \mathcal{S}_{\rm sep}$ precisely if

$$\begin{gathered}\operatorname{Re}\nu_{n+2}(\lambda_*) < \operatorname{Re}\nu_{n+1}(\lambda_*) = \operatorname{Re}\nu_n(\lambda_*) < \operatorname{Re}\nu_{n-1}(\lambda_*),\\ \operatorname{Im}(\nu_{n+1}(\lambda_*) - \nu_n(\lambda_*)) \neq 0, \qquad \frac{\mathrm{d}\operatorname{Im}(\nu_{n+1} - \nu_n)}{\mathrm{d}\lambda}\Big|_{\lambda=\lambda_*} \neq 0.\end{gathered}$$

In the main result described at the end of this section, we shall need the following non-degeneracy assumption.

Hypothesis 3.4 *The subset $\mathcal{S}_{\rm sep}$ of the absolute spectrum $\Sigma_{\rm abs}$ is dense in $\Sigma_{\rm abs}$.*

We focus now on $\lambda \notin \Sigma_{\rm abs}$. For each such λ, there is an $\eta = \eta(\lambda) \in \mathbb{R}$ such that

$$\operatorname{Re}\nu_{n+1}(\lambda) < \eta < \operatorname{Re}\nu_n(\lambda).$$

Using the results in §2.3, we see that the equation

$$\frac{\mathrm{d}V}{\mathrm{d}x}(x) = [\mathcal{A}(x;\lambda) - \eta \mathbb{1}]\, V(x) \tag{3.6}$$

is hyperbolic and that the asymptotic matrix $\mathcal{A}_\infty(\lambda) - \eta\mathbb{1}$ has precisely n unstable eigenvalues. Note that solutions of (3.4) and (3.6) are related via $V(x) = U(x)\mathrm{e}^{-\eta x}$. We denote the projections of the exponential dichotomies of (3.6) on $\mathbb{R}^-$ by $\check{P}^{\rm u}(x;\lambda)$ and those of the associated adjoint equation

$$\frac{\mathrm{d}W}{\mathrm{d}x}(x) = -\left[\mathcal{A}(x;\lambda)^* - \eta\mathbb{1}\right] W(x) \tag{3.7}$$

on $\mathbb{R}^+$ by $\check{Q}^{\rm s}(x;\lambda)$. By construction, we have

$$\dim \mathrm{R}(\check{P}^{\rm u}(0;\lambda)) = \dim \mathrm{R}(\check{Q}^{\rm s}(0;\lambda)) = n.$$

Next, we choose analytic bases $\{\check{V}_i(\lambda)\}_{i=1,\dots,n}$ and $\{\check{W}_i(\lambda)\}_{i=1,\dots,n}$ of $\mathrm{R}(\check{P}^{\rm u}(0;\lambda))$ and $\mathrm{R}(\check{Q}^{\rm s}(0;\lambda))$, respectively. Similarly, let $\{V_i^\infty(\lambda)\}_{i=1,\dots,n}$ and $\{W_i^\infty(\lambda)\}_{i=1,\dots,n}$ be analytic bases of the generalized unstable and stable eigenspaces of the asymptotic systems associated with (3.6) and (3.7), respectively. Lastly, due to Hypothesis 3.3, we can choose analytic bases $\{V_i^-\}_{i=1,\dots,n}$ of $E_{\rm bc}^-$ and $\{W_i^+\}_{i=1,\dots,n}$ of the annihilator of $E_{\rm bc}^+$ in Y_0^*, computed with respect to the dual product (2.11).

We then define the Evans functions

$$\begin{aligned}
\check{D}(\lambda) &= \det\left(\langle \check{V}_i(\lambda), \check{W}_j(\lambda)\rangle\right)_{1\le i,j\le n}\\
D_{\rm bc}^-(\lambda) &= \det\left(\langle V_i^-, W_j^\infty(\lambda)\rangle\right)_{1\le i,j\le n}\\
D_{\rm bc}^+(\lambda) &= \det\left(\left\langle V_i^\infty(\lambda), W_j^+\right\rangle\right)_{1\le i,j\le n}.
\end{aligned}$$

It follows from [1] that these functions are well defined and analytic for $\lambda \notin \Sigma_{\rm abs}$ (see again Remark 2.2). The next hypothesis states that none of the Evans functions defined above vanishes identically on open sets.

Hypothesis 3.5 *There is a discrete (possibly empty) set $\mathcal{C} \subset \mathbb{C}$ with no accumulation points in $\mathbb{C}$ so that $\check{D}(\lambda)$ and $D_{\rm bc}^\pm(\lambda)$ are non-zero for all $\lambda \notin \Sigma_{\rm abs} \cup \mathcal{C}$.*

We can now define the extended point spectrum.

Definition 3.6 *We define the extended point spectrum $\Sigma_{\rm pt}^{\rm sep}$ as*

$$\Sigma_{\rm pt}^{\rm sep} := \{\lambda \notin \Sigma_{\rm abs};\ \check{\rho} + \rho_- + \rho_+ > 0\}$$

where $\check{\rho}$ and $\rho_\pm$ denote the order of λ as a root of the functions $\check{D}$ and $D_{\rm bc}^\pm$ defined above. We call $\rho = \check{\rho} + \rho_- + \rho_+$ the multiplicity of λ for $\lambda \in \Sigma_{\rm pt}^{\rm sep}$.

The next theorem states that the set $\Sigma_\ell^{\rm sep}$ does not approximate the spectrum $\Sigma = \Sigma_{\rm pt} \cup \Sigma_{\rm ess}$ of the travelling wave on $\mathbb{R}$ but the set $\Sigma_{\rm pt}^{\rm sep} \cup \Sigma_{\rm abs}$.

Theorem 2 *Assume that the assumptions in §2.1 as well as the Hypotheses 2.3, 3.3, 3.4 and 3.5 are met.*

(i) Fix an element λ_* with multiplicity ρ in $\Sigma_{\rm pt}^{\rm sep}$. As $\ell \to \infty$, there are precisely ρ elements in $\Sigma_\ell^{\rm sep}$, counted with multiplicity, close to λ_*, and these elements converge to λ_* as $\ell \to \infty$.

(ii) Fix $\lambda_* \in \Sigma_{\rm abs}$, then λ_* is approached by infinitely many elements of $\Sigma_\ell^{\rm sep}$ as $\ell \to \infty$.

(iii) Fix a bounded open domain $\Omega \subset \mathbb{C}$. For any $\delta > 0$, there is an ℓ_* such that $(\Sigma_\ell^{\rm sep} \cap \Omega) \subset \mathcal{U}_\delta(\Sigma_{\rm pt}^{\rm sep} \cup \Sigma_{\rm abs})$ for all $\ell > \ell_*$.

Thus, $\Sigma_\ell^{\rm sep}$ converges, in the symmetric Hausdorff distance, to the union $\Sigma_{\rm pt}^{\rm sep} \cup \Sigma_{\rm abs}$ as $\ell \to \infty$.

Proof. Assertions (i) and (iii) follow upon applying the arguments in [19, §4.3] to the Evans function

$$D_{\rm sep}(\lambda) := \det\left(\left\langle \mathcal{T}(0,-\ell;\lambda)\check{V}_i^-(\lambda), \mathcal{T}^*(0,\ell;\lambda)\check{W}_j^+(\lambda)\right\rangle\right)_{1\le i,j\le n}$$

where $\mathcal{T}(x,y;\lambda)$ and $\mathcal{T}^*(x,y;\lambda)$ are the semigroups associated with (3.4) and its adjoint.

The convergence of $\Sigma_\ell^{\rm sep}$ to $\Sigma_{\rm ess}$ in bounded regions of $\mathbb{C}$ is a consequence of the results in [19, §5.3] which were proved using Lyapunov–Schmidt reduction and therefore apply also to the current situation. ∎

We focus now on Dirichlet boundary conditions that we shall use in the numerical computations presented in §5 below. Thus, we consider the operator

$$\begin{aligned} \mathcal{L}_\lambda^{\rm dir} \quad &: \quad L^2([-\ell,\ell]) \longrightarrow L^2([-\ell,\ell]) \\ [\mathcal{L}_\lambda^{\rm dir}u](x) \quad &:= \quad Du_{xx}(x) - cu_x(x) + [J_0(x)-\lambda]u(x) + \mathrm{e}^{-\lambda\tau}J_1(x)u(x-c\tau) \end{aligned} \tag{3.8}$$

with dense domain $H_0^1([-\ell,\ell])$ where we extend functions by zero to $[-\ell-c\tau,-\ell]$. Upon setting

$$E_{\rm bc}^- = \{U=(u,v,a,b)\in Y_0;\ u=v=a=0\}, \qquad E_{\rm bc}^+ = \{U=(u,v,a,b)\in Y_0;\ a=0\},$$

we see that the eigenvalue problem associated with $\mathcal{L}_\lambda^{\rm dir}$ fits into the framework of this section. As in the case of periodic boundary conditions, we can use the results in [22, §2.1] to conclude that the spectrum of $\mathcal{T}_\ell(t)$ is given by the set $\exp(\Sigma_\ell^{\rm sep}t)$ plus possibly $\{0\}$, where $\mathcal{T}_\ell(t)$ is the semiflow associated with the linear DPDE

$$\begin{aligned} u_t(x,t) &= Du_{xx}(x,t) - cu_x(x,t) + J_0(x)u(x,t) + J_1(x)u(x-c\tau,t-\tau) \\ u(x,t) &= 0 \qquad \text{for } x\in[-\ell-c\tau,-\ell] \text{ and at } x=\ell \end{aligned}$$

on $\mathcal{C}^0(L^2([-\ell,\ell])\times[-\tau,0])$, where functions are again extended by zero to $[-\ell-c\tau,-\ell]$.

4 Numerical continuation of pulses

We assume that $q(x)$ is a solution of (1.2)

$$Dq''(x) - cq'(x) + f(q(x), q(x-c\tau)) = 0 \tag{4.1}$$

for $c>0$ with $q(x)\to 0$ as $x\to\pm\infty$. Written in first-order form with $U=(q,q_x)$, we obtain an equation of the form

$$U'(x) = F(U(x), U(x-\check{\tau});\mu), \tag{4.2}$$

where we fix $\check{\tau}=c\tau>0$ and include a continuation parameter $\mu\in\mathbb{R}$. We shall assume that the nonlinearity f is analytic (this will guarantee backward uniqueness of solutions).

Note that the spatial delay $\check{\tau}=c\tau$ in (4.1) is fixed. We may use the wave speed parameter c in front of the term $cq_x(x)$ as the free parameter μ which is then determined by the condition that (4.1) has a homoclinic orbit. Afterwards, from the knowledge of $c\tau$ and c, we obtain the value of τ for which the computed homoclinic orbit is a travelling pulse. If we wish to find a pulse for a given delay τ, we would need to use continuation in $c\tau$ and c with the hope of realizing a specified value of τ.

4.1 The boundary-value problem formulation

Associated with (4.2) is the dynamical system

$$V_x = \mathcal{F}(V;\mu) \tag{4.3}$$

on the phase space Y_0. The mapping

$$\mathcal{F}(V,\mu): \quad Y_0 \longrightarrow Y_0, \qquad (\phi, a) \longmapsto (\phi', F(a, \phi(-\check{\tau}); \mu))$$

is well defined provided f satisfies certain growth conditions: since we are interested in the dynamics near a given pulse of (4.1), we can modify f outside a bounded region in $\mathbb{R}^n \times \mathbb{R}^n$.

Pulses $q(x)$ of (4.1) correspond to homoclinic orbits $Q(x) := (q(x+\cdot), q(x))$ of (4.3). The idea is therefore to follow the numerical algorithm studied in [3] for ODEs which consists of approximating the stable and unstable manifolds of the asymptotic equilibria $V = 0$ by their tangent spaces.

In addition to the standing hypotheses stated in §2.1, we assume that the equilibrium $V = 0$ of (4.3) is hyperbolic for $\mu = 0$, which is equivalent to the assumption that $0 \notin \Sigma_{\text{ess}}$. Next, we assume that $Q(x)$ is a homoclinic orbit, converging to $V = 0$, of (4.3) at $\mu = 0$. Moreover, we assume that $Q_x(x)$ is the only nontrivial bounded solution of the linearization of (4.3) about $Q(x)$ which is given by (2.8) with $\lambda = 0$. Lastly, we assume that the Melnikov integral

$$\int_{-\infty}^{\infty} \langle \Psi(x), \mathcal{F}_\mu(Q(x);0) \rangle \, \mathrm{d}x \neq 0$$

is not zero, where $\Psi(x)$ denotes the unique nontrivial bounded solution on $\mathbb{R}$ of the adjoint equation (2.12) at $\lambda = 0$: in the notation of §2.3, the solution $\Psi(x)$ satisfies $\Psi(0) \perp \mathrm{R}(P^{\mathrm{u}}(0;0))$, and we further have $\Psi(x) = W_1(x;0)$ in (2.15).

We denote by E_0^{s} and E_0^{u} the stable and unstable generalized eigenspaces of the operator $\mathcal{A}_\infty(0)$ defined in (2.10). Furthermore, we choose positive constants ν^{u} and ν^{s} so that either $\operatorname{Re}\nu < -\nu^{\mathrm{s}} < 0$ or $\operatorname{Re}\nu > \nu^{\mathrm{u}} > 0$ for each root ν of the linear dispersion relation $\Delta(0;\nu)$ defined in (2.5).

We may then seek approximations $V(x)$ of $Q(x)$ as solutions to the boundary-value problem

$$\begin{aligned} V_x(x) &= \mathcal{F}(V(x);\mu), \qquad x \in (-\ell, \ell) && (4.4)\\ V(-\ell) &\in E_0^{\mathrm{u}} && (4.5)\\ V(\ell) &\in E_0^{\mathrm{s}} && (4.6)\\ 0 &= \int_{-\ell}^{\ell} \langle Q_x(x), V(x) - Q(x) \rangle_{Y_0} \, \mathrm{d}x && (4.7) \end{aligned}$$

on the interval $[-\ell, \ell]$ with $\ell \gg 1$. Note that (4.5) involves a finite number of conditions as does (4.6) since it can be written as

$$\langle V(\ell), W_j^\infty \rangle = 0, \qquad j = 1, \ldots, n_{\mathrm{u}}$$

where $\{W_j^\infty\}_{j=1,\ldots,n_{\mathrm{u}}}$ is a basis of the finite-dimensional generalized stable eigenspace of $\mathcal{A}_\infty^*(0)$; see §3.2.

In this setup, we have the following result regarding the solvability of (4.4)–(4.7).

Theorem 3 *Under the assumptions stated in this section, the boundary-value problem (4.4)–(4.7) has a unique solution (V_ℓ, μ_ℓ) near $(V,\mu) = (Q,0)$ for all sufficiently large $\ell \gg 1$. Moreover, there is a constant $C > 0$ such that the solution (V_ℓ, μ_ℓ) with $V_\ell =: (\phi_\ell, q_\ell)$ satisfies the estimates*

$$\begin{aligned} \sup_{x\in[-\ell,\ell]} |q_\ell(x) - q(x)| &\leq Ce^{-2\ell \min\{\nu^{\mathrm{u}},\nu^{\mathrm{u}}\}} \\ |\mu_\ell| &\leq Ce^{-\ell \min\{2\nu^{\mathrm{u}}+\nu^{\mathrm{s}}, 2\nu^{\mathrm{s}}+\nu^{\mathrm{u}}\}} \end{aligned} \tag{4.8}$$

uniformly in $\ell \gg 1$.

Proof. Using the exponential dichotomies from [1, 10, 12, 14], we can write (4.4) in a weak integral form. From this point on, the proof given in [13] with $Q_\rho = \mathbb{1}$ for elliptic PDEs applies verbatim, thus yielding the desired result for DDEs. ∎

4.2 Practical implementation

Firstly, we argue that the boundary-value problem (4.4)–(4.7) can be written in the following simpler form. For the sake of simplicity, we assume that the unstable spatial eigenvalues ν_j^{u} of the linearization about the equilibrium $V = 0$ are semisimple, so that the associated eigenvectors V_j^∞ and V_j^∞ of the linearization and its adjoint are of the form

$$V_j^\infty = \begin{pmatrix} U_j^\infty e^{\nu_j^{\mathrm{u}} x} \\ U_j^\infty \end{pmatrix}, \qquad W_j^\infty = \begin{pmatrix} -F_{u_1}(0,0;0)^* \hat{U}_j^\infty e^{-\nu_j^{\mathrm{u}} x} \\ \hat{U}_j^\infty \end{pmatrix}.$$

Consider the DDE

$$U'(x) = F(U(x), U(x-\check{\tau}); \mu), \qquad x \in [-\ell, \ell] \tag{4.9}$$

together with the boundary conditions

$$U(x) = \sum_{j=1}^{n^{\mathrm{u}}} \alpha_j U_j^\infty e^{\nu_j^{\mathrm{u}} x}, \qquad x \in [-\check{\tau}, 0] \tag{4.10}$$

$$0 = U(\ell)\cdot \hat{U}_j^\infty - \int_{-\check{\tau}}^0 e^{-\nu_j^{\mathrm{u}}(y+\check{\tau})} \left[F_{u_1}(0,0;0) U(\ell+y)\right] \cdot \hat{U}_j^\infty \, dy = 0, \quad j = 1,\ldots,n^{\mathrm{u}} \tag{4.11}$$

and the, slightly modified, phase condition

$$\int_{-\ell}^{\ell} Q_2'(x) \cdot (a(x) - Q_2(x))\, dx = 0 \tag{4.12}$$

where $V = (\phi, a)$ and $Q = (Q_1, Q_2)$. We remark that, in practice, one would replace the exact, but unknown, solution Q that appears in (4.12) by an approximate solution, for instance, by the solution obtained at a previous continuation step.

The system (4.9)–(4.12) is a boundary-value problem which we solve using a collocation method. The algorithm has been implemented in DDE-BIFTOOL, a MATLAB package for the bifurcation analysis of DDEs [5, 6].

As a test case, we consider the system

$$\begin{aligned} \frac{du_1}{dx}(x) &= u_2(x) \\ \frac{du_2}{dx}(x) &= u_1(x) - u_1(x)u_1(x-\tau) + \mu u_2(x) + \frac{1}{2}u_1(x)u_2(x), \end{aligned} \tag{4.13}$$

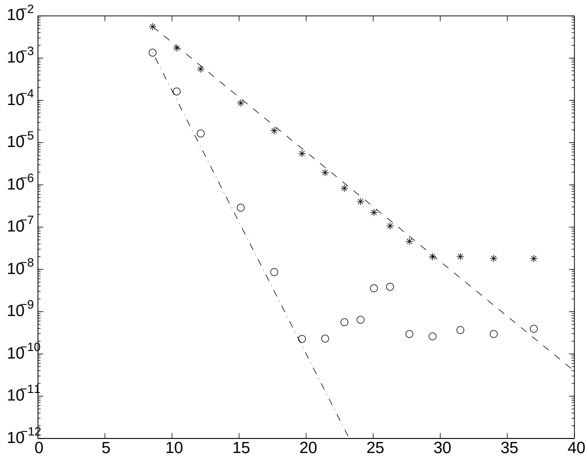

Figure 4.1: *Plotted are the truncation errors* $\|q_\ell - q_*\|$ *(*)* *and* $|\mu_\ell - \mu_*|$ *(*$\circ$*) from (4.8) for the homoclinic orbit of (4.13) as functions of* 2ℓ*. For comparison, we also plotted two lines with slope* $-\nu^{\mathrm{u}}$ *and* $-(\nu^{\mathrm{u}} + \nu^{\mathrm{s}}/2)$*.*

which has two steady states given by $(0,0)$ and $(1,0)$. Equation (4.13) with $\tau = 0$ was used in [3] to examine the effect of truncation on the accuracy of numerically computed homoclinic orbits in ODEs. For convenience, we used the same model with an artificially introduced delay to illustrate the assertions made in Theorem 3. For $\tau = 0.8255$, equation (4.13) has a homoclinic orbit q_* to the saddle (0,0) for $\mu = \mu_* \approx -1.0796$. The linearization about the steady state $(0,0)$ has one unstable spatial eigenvalue given by $\nu^{\mathrm{u}} = 0.59695$, while the rightmost stable spatial eigenvalue is given by $-\nu^{\mathrm{s}} = -1.6752$.

The results of our numerical computations are shown in Figure 4.1, where we compare the solution to (4.9)–(4.11) and the phase condition $u_2(0) = 0$ for a range of values of ℓ with the numerically obtained solution for $2\ell = 591.44$.

5 Numerical computation of spectra

In this section, we shall discuss our strategy for computing the spectra of travelling pulses numerically on the interval $(-\ell, \ell)$ with periodic or Dirichlet boundary conditions. Ideally, one would like to use the eigenvalue problems (3.3) and (3.8) to compute these spectra since the relevant operators act only on functions in the x-variable. However, the eigenvalue parameter λ enters these operators in a transcendental fashion, and it is therefore not clear how to proceed.

Instead, we consider the original linear equation (2.2)

$$u_t(x,t) = Du_{xx}(x,t) - cu_x(x,t) + J_0(x)u(x,t) + J_1(x)u(x - c\tau, t - \tau) \tag{5.1}$$

on the interval $(-\ell, \ell)$, supplemented by periodic

$$u(x,t) := u(x + 2\ell, t) \qquad \text{for } x \in [-\ell - c\tau, -\ell] \tag{5.2}$$

or Dirichlet boundary conditions

$$u(x,t) = 0 \qquad \text{for } x \in [-\ell - c\tau, -\ell] \text{ and at } x = \ell. \tag{5.3}$$

In the previous section, we gave characterizations of the spectra of the semigroups associated with (5.1)-(5.2) and (5.1)-(5.3) in terms of the sets $\Sigma_\ell^{\rm per}$ and $\Sigma_\ell^{\rm dir}$ in the limit $\ell \to \infty$.

The numerical strategy pursued here is to calculate the spectra of these semigroups. To this end, we discretize (5.1) and the boundary equations in the spatial variable x, and compute the eigenvalues of the semiflow as growth rates of solutions of the resulting large system of ODEs with time t being the evolution variable.

5.1 Discretization and time integration

We discretize equation (5.1) in the space variable x using centered finite differences of second order on an equidistant mesh on $(-\ell, \ell)$ with mesh points $\{x_i\}_{i=1,\dots,N}$. This yields a system of N coupled delay differential equations (DDEs) for the functions $u_i(t) := u(x_i, t)$ where $i = 1, \dots, N$.

Note that some care has to be taken when evaluating the function $J_1(x)$ defined in (2.3) as it depends on the delay through the pulse profile $q(x - c\tau)$. We assume that we have computed a numerical approximation of the pulse profile $q(x)$ on $[-\ell - c\tau, \ell]$, for instance using collocation. We then use interpolation to compute the values $q(x_i - c\tau)$ on the mesh points $\{x_i\}_{i=1,\dots,N}$.

We also use linear interpolation between $u_{m_i}(t)$ and $u_{m_i+1}(t)$ to approximate the intermediate term $u(x_i - c\tau, t)$ in (5.1), where the indices m_i are determined so that $x_i - c\tau \in [x_{m_i}, x_{m_i+1})$. Of course, this procedure for computing $u(x_i - c\tau, t)$ works only for indices $i \geq N_0$ where N_0 is the smallest index for which $x_{N_0} - c\tau \geq 0$. We use the boundary conditions to compute the remaining terms for $j < N_0$: for periodic boundary conditions, we take $u_{j-N_0}(t) = u_{N-N_0+j}(t)$ and $u_{N+1}(t) = u_1(t)$, while we choose $u_{j-N_0}(t) = 0$ and $u_{N+1}(t) = 0$ when using Dirichlet boundary conditions.

In summary, we obtain the system

$$\begin{aligned}\frac{\mathrm{d}u_i}{\mathrm{d}t}(t) &= \left(-\frac{c}{2\Delta x} + \frac{1}{\Delta x^2}\right) u_{i+1}(t) + \left(J_0(x_i) - \frac{2}{\Delta x^2}\right) u_i(t) + \left(\frac{c}{2\Delta x} + \frac{1}{\Delta x^2}\right) u_{i-1}(t) \\ &\quad + J_1(x_i)\left[(1-\xi)u_{m_i}(t-\tau) + \xi u_{m_i+1}(t-\tau)\right]\end{aligned} \tag{5.4}$$

of DDEs for $i = 1, \dots, N$, where

$$\xi = \frac{x_i - c\tau - x_{m_i}}{x_{m_i+1} - x_{m_i}}$$

and Δx is the spatial step size.

The discretized equation (5.4) is a large linear system of autonomous DDEs which we write in short as

$$\frac{\mathrm{d}U}{\mathrm{d}t}(t) = B_0 U(t) + B_1 U(t-\tau), \qquad U \in \mathbb{R}^N. \tag{5.5}$$

The growth rates λ of solutions of (5.5) are the roots of the transcendental characteristic equation

$$\det(\lambda \mathbb{1} - B_0 - B_1 \mathrm{e}^{-\lambda\tau}) = 0. \tag{5.6}$$

We solve (5.6) using Newton iterations in λ. Appropriate initial guesses for λ are obtained through time integration of (5.5) [7, 8], which guarantees that the rightmost (stability-determining) roots will be found.

In the rest of this section, we discuss the time integration of (5.5). Using the notation

$$U_t(\theta) := U(t+\theta) \qquad \theta \in [-\tau, 0],$$

we can write the solution operator of (5.5) as

$$\begin{aligned} \mathcal{T}_{\bar{t}} \quad &: \quad \mathcal{C}^0([-\tau,0],\mathbb{R}^N) \longrightarrow \mathcal{C}^0([-\tau,0],\mathbb{R}^N), \qquad U_t \longmapsto \mathcal{T}_{\bar{t}} U_t \\ [\mathcal{T}_{\bar{t}} U_t](\theta) \quad &:= \quad U_{t+\bar{t}}(\theta). \end{aligned} \tag{5.7}$$

Note that the eigenvalues μ of $\mathcal{T}_{\bar{t}}$ and the solutions λ of (5.6) are related via $\mu = \mathrm{e}^{\lambda \bar{t}}$ (possibly supplemented by $\mu = 0$) [9, §7.4].

We consider an equidistant mesh $\Pi := \{-\tau = t_0 < t_1 < \ldots < t_{M-1} = 0\}$ with $t_j - t_{j-1} = h$ for $j = 0, \ldots, M-1$. The time discretized state $\bar{U}_t \in \mathbb{R}^{MN}$ is given by the function segment U_t discretized on Π so that $\bar{U}_t = (U_t(t_0), U_t(t_1), \ldots, U_t(t_{M-1}))$. A single time-step with a linear multi-step (LMS) method with step length h is given by

$$\bar{U}_{t+h} = \check{\mathcal{T}}_h \bar{U}_t,$$

where the first row of the matrix $\check{\mathcal{T}}_h$ is determined by the LMS method, while the rest of the matrix is a shifted identity matrix. The eigenvalues of the matrix $\check{\mathcal{T}}_h$ will provide good approximations of the eigenvalues of the solution operator $\mathcal{T}_h$ provided h is chosen appropriately. It has been proved in [8] that one needs $h\lambda \in \mathbb{D}$ for all λ with

$$\lambda \in \bigcup_{|z|<1} \mathrm{spec}(B_0 + zB_1)$$

to preserve the stability of the original operator, where $\mathbb{D}$ denotes the stability region of the LMS method. Based on this result, a heuristic step length h_{LMS} has been given in [8, Heuristic 4.1] such that for all $h < h_{\mathrm{LMS}}$ the discretized operator $\check{\mathcal{T}}_h$ should have the same stability as the original operator $\mathcal{T}_h$. Choosing h smaller results in a better approximation of roots that are more to the left. Note however, that smaller step sizes will require much more memory. For stiff problems, a less pessimistic heuristic step length has been derived in [21]. In our computations, we chose h to be the smallest number compatible with the memory available on the machine we used for the computations.

5.2 FitzHugh–Nagumo equations with delay

To test the prediction made in §3, we consider the FitzHugh–Nagumo system

$$\begin{aligned} u_t(x,t) \quad &= \quad u_{xx}(x,t) + u(x,t-\tau) - \frac{1}{3}u^3(x,t) - v(x,t) \\ v_t(x,t) \quad &= \quad \epsilon v_{xx}(x,t) + d[u(x,t-\tau) + a - bv(x,t)] \end{aligned} \tag{5.8}$$

with an artificial delay τ. We set $\epsilon = 0.1$, $a = 0.7$, $b = 0.8$ and $d = 0.08$. We use the approach given in [16] as outlined in §4 to compute pulses of (5.8) numerically as homoclinic orbits of the delay differential equation (4.1).

Periodic boundary conditions

Fixing $c\tau = 0.5$, we computed the pulse shown in Figure 5.1 with wave speed $c = 1.0931$ and associated delay $\tau = 0.4574$. We then compute the spectrum of the pulse on a periodic domain with length $2\ell = 59.5$ using a spatial discretization of $N = 120$ and a step length $h = 0.01$ for the LMS method.

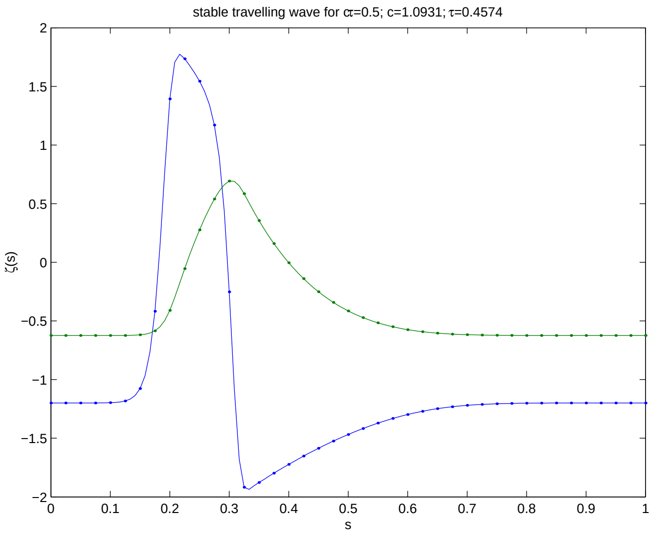

Figure 5.1: *Pulse profile of the FitzHugh–Nagumo equation for* $\tau = 0.4574$.

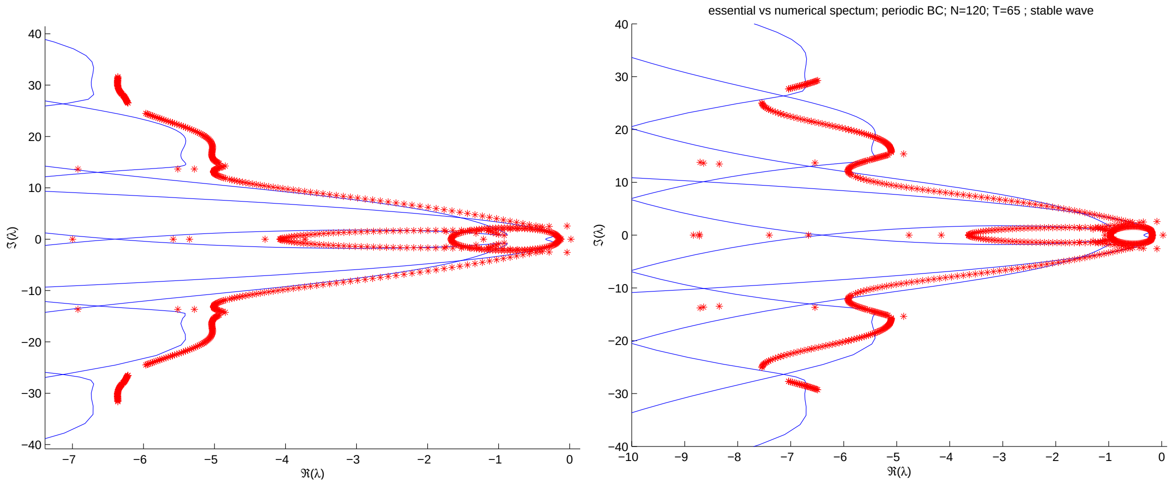

Figure 5.2: *Essential spectrum (solid line) of the asymptotic rest state* (u_∞, v_∞) *and the spectrum (stars) of the pulse with periodic boundary conditions are shown for* $\tau = 0.4574$.

For comparison, we also computed the essential spectrum of the background state (u_∞, v_∞) using continuation in λ applied to the system (2.9) supplemented by the condition that it has a purely imaginary spatial eigenvalue. This procedure was implemented in DDE-BIFTOOL.

In Figure 5.2, we plotted the rightmost eigenvalues of the numerically computed spectrum and the essential spectrum of the background state (u_∞, v_∞) on the unbounded domain. In general, there is a reasonable correspondence between the numerically computed and the theoretically predicted spectra, and we clearly also capture point spectrum. Note that the spectra begin to look quite different the further we go to the left: this disagreement is due to the space-time discretization of the DPDE.

Separated boundary conditions

We fixed $c\tau = 0.1$ and computed a pulse with speed $c = 0.5545$ and delay $\tau = 0.1800$. The spectrum of the pulse was computed using Dirichlet boundary conditions on the interval of length $2\ell = 59.5$ using a spatial discretization of $N = 200$ and a step length $h = 0.01$ for the LMS method.

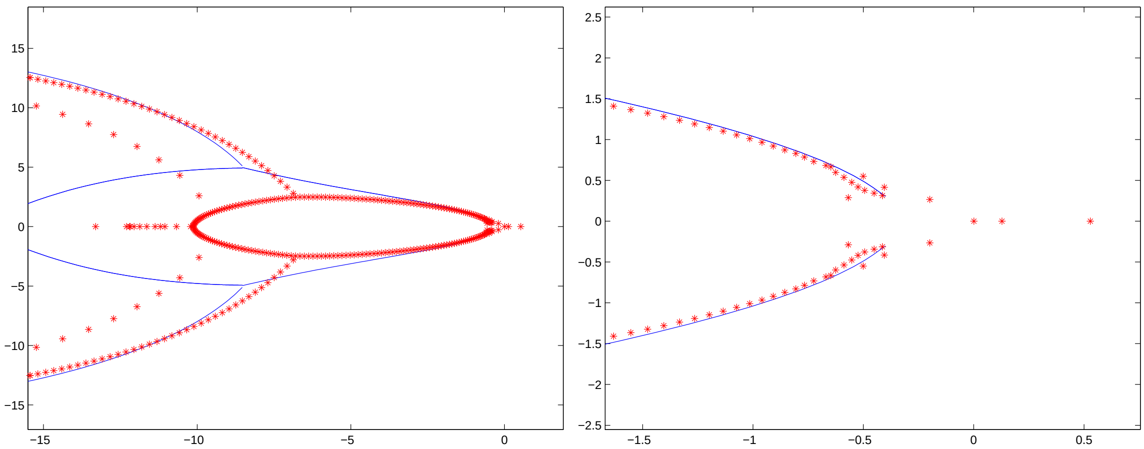

Figure 5.3: *Absolute spectrum (solid line) of the asymptotic rest state* (u_∞, v_∞) *and the spectrum (stars) of the pulse with Dirichlet conditions are shown for* $\tau = 0.18$.

We also computed the absolute spectrum of the asymptotic rest state since we expect from Theorem 2(ii) that the spectrum, computed with Dirichlet conditions, approaches this set. We again use DDE-BIFTOOL for continuation in λ applied to the system (2.9) supplemented by the condition that the second and third rightmost spatial eigenvalues ν_2 and ν_3 have the same real part.

Both the absolute spectrum of (u_∞, v_∞) and the spectrum of the pulse under Dirichlet conditions are shown in Figure 5.3. Again, we capture isolated eigenvalues, which are part of the extended point spectrum, in addition to the absolute spectrum.

6 Conclusions

We compared spectra of pulses on the unbounded real line with spectra computed on the interval $(-\ell, \ell)$ for $\ell \gg 1$ under separated and periodic boundary conditions. Through the collection of appropriate reformulations, we showed that known results for ODEs and PDEs apply also to partial differential equations with temporal delays. We confirmed these findings by numerical computations using the continuation package DDE-BIFTOOL.

Even though we focused entirely on pulses in this paper, the results apply also to the computation of spectra of fronts under separated boundary conditions; note that periodic boundary conditions are not appropriate in this case. We refer to [17, 19] for statements of the relevant results for ODEs.

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References

[1] S. Benzoni-Gavage. Stability of semi-discrete shock profiles by means of an Evans function in infinite dimensions. *J. Dynam. Diff. Eqns.* **14** (2002) 613–674.

[2] S. Benzoni-Gavage, P. Huot and F. Rousset. Nonlinear stability of semidiscrete shock waves. *SIAM J. Math. Anal.* **35** (2003) 639–707.

[3] W. J. Beyn. The numerical computation of connecting orbits in dynamical systems. *IMA J. Numer. Anal.* **9** (1990) 379–405.

[4] O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel and H.-O. Walther. *Delay Equations: Functional, Complex, and Nonlinear Analysis.* Springer-Verlag, New York, 1995.

[5] K. Engelborghs, T. Luzyanina and D. Roose. Numerical bifurcation analysis of delay differential equations using DDE-BIFTOOL. *Trans. Math. Software* **28** (2002) 1–21.

[6] K. Engelborghs, T. Luzyanina and G. Samaey. DDE-BIFTOOL v2.00: A MATLAB package for bifurcation analysis of delay differential equations. Technical Report TW-330, Dept. Comp. Sci., KULeuven, 2001.

[7] K. Engelborghs and D. Roose. Numerical computation of stability and detection of Hopf bifurcations of steady state solutions of delay differential equations. *Adv. Comp. Math.* **10** (1999) 271–289.

[8] K. Engelborghs and D. Roose. On stability of LMS-methods and characteristic roots of delay differential equations. *SIAM J. Numer. Anal.* **40** (2002) 629–650.

[9] J. K. Hale. *Theory of Functional Differential Equations.* Springer-Verlag, 1977.

[10] J. Härterich, B. Sandstede and A. Scheel. Exponential dichotomies for linear non-autonomous functional differential equations of mixed type. *Indiana Univ. Math. J.* **51** (2002) 1081–1109.

[11] T. Kapitula, N. Kutz and B. Sandstede. The Evans function for nonlocal equations. *Indiana Univ. Math. J.* (in press).

[12] X.-B. Lin. Exponential dichotomies and homoclinic orbits in functional-differential equations. *J. Diff. Eqns.* **63** (1986) 227–254.

[13] G. L. Lord, D. Peterhof, B. Sandstede and A. Scheel. Numerical computation of solitary waves on infinite cylinders. *SIAM J. Numer. Anal.* **37** (2000) 1420–1454.

[14] J. Mallet-Paret and S. M. Verduyn Lunel. Exponential dichotomies and Wiener–Hopf factorizations for mixed-type functional differential equations. *J. Diff. Eqns.* (to appear).

[15] D. Peterhof, B. Sandstede and A. Scheel. Exponential dichotomies for solitary-wave solutions of semilinear elliptic equations on infinite cylinders. *J. Diff. Eqns.* **140** (1997) 266–308.

[16] G. Samaey, K. Engelborghs and D. Roose. Numerical computation of connecting orbits in delay differential equations. *Numer. Algor.* **30** (2002) 335–352.

[17] B. Sandstede. Stability of travelling waves. In: *Handbook of Dynamical Systems II: Towards Applications* (B. Fiedler, ed.). Elsevier, (2002) 983–1055.

[18] B. Sandstede and A. Scheel. On the stability of periodic travelling waves with large spatial period. *J. Diff. Eqns.* **172** (2001) 134–188.

[19] B. Sandstede and A. Scheel. Absolute and convective instabilities of waves on unbounded and large bounded domains. *Physica D* **145** (2000) 233–277.

[20] K. W. Schaaf. Asymptotic behaviour and traveling wave solutions for parabolic functional differential equations. *Trans. Amer. Math. Soc.* **302** (1987) 587–615.

[21] K. Verheyden, T. Luzyanina and D. Roose. Efficient computation of characteristic roots of delay differential equations using LMS methods. Technical Report TW-383, Dept.Comp. Sci., KULeuven, 2003.

[22] J. Wu. *Theory and Applications of Partial Functional Differential Equations.* Springer-Verlag, 1996.