Periodic solutions of differential algebraic equations with time delays: computation and stability analysis

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Abstract

This paper concerns the computation and local stability analysis of periodic solutions to semi-explicit *differential algebraic equations with time delays* (delay DAEs) of index 1 and index 2. By presenting different formulations of delay DAEs, we motivate our choice of a direct treatment of these equations. Periodic solutions are computed by solving a periodic two-point boundary value problem, which is an infinite-dimensional problem for delay DAEs. We investigate two collocation methods based on piecewise polynomials: collocation at Radau IIA and Gauss-Legendre nodes. Using the obtained collocation equations, we compute an approximation to the Floquet multipliers which determine the local asymptotic stability of a periodic solution. Based on numerical experiments, we present orders of convergence for the computed solutions and Floquet multipliers and compare our results with known theoretical convergence results for initial value problems for delay DAEs. We end with examples on bifurcation analysis of delay DAEs.

Keywords : delay differential algebraic equations, periodic solutions, collocation methods, stability.
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1 Introduction

We aim at the computation and local stability analysis of periodic solutions to index-1 and index-2 semi-explicit autonomous *delay differential algebraic equations* (delay DAEs, DDAEs) of retarded type. Since these equations have received relatively little attention in the literature so far, we first introduce the considered problems.

Index-1 semi-explicit delay DAEs of retarded type are given by

$$x_1'(t) = f(x_1(t), x_1(t-\tau), x_2(t), x_2(t-\tau)) \tag{1.1a}$$

$$0 = g(x_1(t), x_1(t-\tau), x_2(t)), \quad t \in [0, t_f], \tag{1.1b}$$

where $\frac{\partial g}{\partial x_2}(t)$ is nonsingular for each $t,\ t \in [0, t_f]$, $x_i(t) \in \mathbb{R}^{n_i},\ 0 < \tau < \infty$ is the time delay, functions f and g are sufficiently smooth, $f : \mathbb{R}^{2(n_1+n_2)} \to \mathbb{R}^{n_1},\ g : \mathbb{R}^{2n_1+n_2} \to \mathbb{R}^{n_2}$, and the initial functions $x_i(\theta) = \phi_i(\theta),\ i = 1, 2,\ -\tau \le \theta \le 0$, are smooth and satisfy the algebraic equation,

$$g(\phi_1(0), \phi_1(-\tau), \phi_2(0)) = 0.$$

In the literature, systems of the form (1.1) are also called hybrid systems of functional differential equations. The algebraic equation (1.1b) is also called algebraic constraint or functional equation. Note that we use the differentiation index of a semi-explicit delay DAE as it is usually defined for DAEs: the minimal number of differentiations of the constraints that are necessary to obtain differential equations for all variables.

Similarly to DAEs, two kinds of index-2 semi-explicit delay DAEs can be distinguished, *pure* index-2 DDAEs (or DDAEs in Hessenberg form) and *mixed* index-2 DDAEs (or general DDAEs). Pure index-2 DDAEs of retarded type can be written as

$$x_1'(t) = f(x_1(t), x_1(t-\tau), x_2(t)) \tag{1.2a}$$

$$0 = g(x_1(t)), \quad t \in [0, t_f], \tag{1.2b}$$

where $\frac{\partial g}{\partial x_1}(t)\frac{\partial f}{\partial x_2}(t)$ is nonsingular for each $t,\ t \in [0, t_f]$, $x_i(t) \in \mathbb{R}^{n_i}$, $n_1 > n_2$, functions f and g are sufficiently smooth, $f : \mathbb{R}^{2n_1+n_2} \to \mathbb{R}^{n_1},\ g : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$, and the initial function $x_1(\theta) = \phi_1(\theta),\ -\tau \le \theta \le 0$, satisfies the algebraic equation,

$$g(\phi_1(0)) = 0.$$

Mixed index-2 DDAEs involve index-1 and index-2 algebraic variables and, respectively, index-1 and index-2 constraints, e.g.,

$$x_1'(t) = f(x_1(t), x_1(t-\tau), x_2(t), x_2(t-\tau), x_3(t)) \tag{1.3a}$$

$$\begin{aligned} 0 &= g_1(x_1(t), x_1(t-\tau), x_2(t)) \\ 0 &= g_2(x_1(t)), \quad t \in [0, t_f]. \end{aligned} \tag{1.3b}$$

Here $\frac{\partial g_1}{\partial x_2}(t)$ and $\frac{\partial g_2}{\partial x_1}(t)\frac{\partial f}{\partial x_3}(t)$ are both nonsingular for each $t,\ t \in [0, t_f]$, $x_i(t) \in \mathbb{R}^{n_i}$, $n_1 > n_3$, functions $f,\ g_1$ and g_2 are sufficiently smooth, $f : \mathbb{R}^{2(n_1+n_2)+n_3} \to \mathbb{R}^{n_1},\ g_1 : \mathbb{R}^{2n_1+n_2} \to$

$\mathbb{R}^{n_2}$, $g_2 : \mathbb{R}^{n_1} \to \mathbb{R}^{n_3}$, and the initial functions $x_i(\theta) = \phi_i(\theta),\ i = 1, 2,\ -\tau \leq \theta \leq 0$, are smooth and satisfy the algebraic equations,

$$g_1(\phi_1(0), \phi_1(-\tau), \phi_2(0)) = 0, \quad g_2(\phi_1(0)) = 0.$$

In (1.3), x_2 and x_3 are index-1 and index-2 algebraic variables and constraints (1.3b) are, respectively, of index 1 and of index 2. Slightly different formulations of mixed index-2 DDAEs of retarded type are possible, cf. Sec. 5.1.

We introduced delays in Eqs. (1.1)-(1.3) in a way which preserves the retarded type of the problems. If other solution components appear with delays, the problems are of neutral type, which is substantially different and it is not considered in this paper. Clearly, the delayed variables in (1.1)-(1.3) can appear with any number of different delays.

Delay DAEs, as DAEs, are mostly used in engineering modeling, e.g. in the theory of electrical networks containing lossless transmission lines [Brayton, 1968], in models of transmitted light through ring cavities [Ikeda *et al.*, 1980; Berre *et al.*, 1986], optical bistable systems [Gibbs *et al.*, 1981], deep-hole vibratory drilling [Batzer *et al.*, 2001], in modeling power systems [Venkatasubramanian *et al.*, 1994; Hiskens, 2003], in control theory [Shampine & Gahinet, 2004].

The present work is motivated by the lack of computational tools for delay DAEs. There is a substantial literature on the analysis and solution of DAEs, cf. [Brenan *et al.*, 1996; Hairer & Wanner, 1996] and the references therein. Moreover, a number of software packages to handle initial and boundary value problems (IVPs, respectively BVPs) for nonlinear systems of DAEs is publicly available, e.g. DASSL [Brenan *et al.*, 1996], RADAU5 [Hairer & Wanner, 1996], MEXX [Lubich *et al.*, 1992], COLDAE [Ascher & Spiteri, 1994]. Recent works [Franke & Führer, 2001; Lamour *et al.*, 1998, 2003] are devoted to periodic solutions of DAEs of index 1 to 3. However, theoretical and numerical analysis of delay DAEs has received relatively little attention in the literature so far. Some results on IVPs for delay DAEs related to results of this paper are outlined in Sec. 4.3. In [Luzyanina & Roose, 2004], we showed how local stability of stationary solutions to semi-explicit delay DAEs can be investigated numerically. For this, we adapted the numerical methods developed for retarded delay differential equations (DDEs) [Engelborghs & Roose, 2002]. As far as we know, no prior work on numerical study of periodic solutions to delay DAEs exists.

In this paper we investigate, by means of numerical experiments, piecewise polynomial collocation schemes for the computation of periodic solutions to delay DAEs (1.1)-(1.3). Using the obtained collocation equations, we analyze local stability of periodic solutions. Dependence of a periodic solution on a system parameter is studied by computing a branch of periodic solutions as a function of the parameter using a continuation procedure.

The paper is structured as follows. In Sec. 2, by presenting different formulations of index-1 and index-2 delay DAEs, we motivate our choice of a direct treatment of these equations. In Sec. 3 we formulate BVPs for computing periodic solutions of delay DAEs (1.1)-(1.3) and discuss stability analysis of these solutions. Next, in Sec. 4 we explain the collocation schemes that we investigate and we outline related results on solving BVPs for DAEs and IVPs for delay DAEs. In Sec. 5 we present our numerical results. Section 6 contains conclusions.

2 Reformulations of Index-1 and Index-2 Delay DAEs

We are interested in periodic solutions of delay DAEs (1.1)-(1.3), i.e., solutions which satisfy $x^*(t+T) = x^*(t)$ for any $t \in \mathbb{R}$ and some $T \in (0, t_f]$. Here $x^*(t) = (x_1^*(t), x_2^*(t))^T$ and $x^*(t) = (x_1^*(t), x_2^*(t), x_3^*(t))^T$ for delay DAEs (1.1)-(1.2), respectively (1.3). The minimal T is called the period of the solution.

Periodic solutions of Eqs. (1.1)-(1.3) can be computed using DDE-formulations of these equations, *the essential underlying* DDE and a DDE *with an invariant.* Hence, the software package DDE-BIFTOOL [Engelborghs *et al.*, 2001, 2002], developed for stability and bifurcation analysis of retarded DDEs, can be used. Remind that periodic solutions of retarded DDEs with an arbitrarily smooth right-hand side are arbitrarily smooth since the solution operator to a retarded DDE smoothes the solution as time increases. The same property holds for periodic solutions of the considered DDAEs. Below we present the DDE-formulations to motivate our choice of a direct treatment of delay DAEs.

2.1 The essential underlying DDE

The *essential underlying* DDE (EUDDE) for a given delay DAE is a DDE in minimal coordinates which describes the dynamics of the delay DAE.

Index – 1 problems. Using the implicit function theorem, we can solve Eq. (1.1b) for x_2, $x_2(t) = G(x_1(t), x_1(t-\tau))$. Substituting this solution in (1.1a) yields a DDE in x_1,

$$x_1'(t) = f(x_1(t), x_1(t-\tau), G(x_1(t), x_1(t-\tau)), G(x_1(t-\tau), x_1(t-2\tau))). \tag{2.1}$$

Note that this DDE has the reduced dimension n_1 and that the largest delay is 2τ.

DDE (2.1), called *the essential underlying* DDE for index-1 delay DAEs (1.1), plays an important role in theoretical analysis of these DAEs. In particular, the following result is important in the context of local stability analysis of periodic solutions to index-1 DDAEs. Let x^* be a periodic solution to DDAE (1.1). The following two linearizations around x^*, the linearization of EUDDE (2.1),

$$y_1'(t) = A_0(t)y_1(t) + A_1(t)y_1(t-\tau) + A_2(t)y_1(t-2\tau), \tag{2.2}$$

and the direct linearization of DDAE (1.1),

$$\begin{aligned} y_1'(t) &= B_0(t)y_1(t) + B_1(t)y_1(t-\tau) + B_2(t)y_2(t) + B_3(t)y_2(t-\tau) \\ 0 &= C_0(t)y_1(t) + C_1(t)y_1(t-\tau) + C_2(t)y_2(t), \end{aligned} \tag{2.3}$$

are equivalent in the sense that each of them can be obtained from another one. The proof of this result is analogous to the one in [Zhu & Petzold, 1998] for the case of stationary solutions.

While this approach (the use of the implicit function theorem) is good as a theoretical tool, it is not always practical for numerical solution since Eq. (1.1b) can be non-uniquely solvable or non-solvable analytically for x_2.

Index – 2 problems. Using constraint (1.2b) at each $t,\ t \in [0, T]$, we can define a smaller set of $n_1 - n_2$ unknowns such that the DDE for these unknowns describes the dynamics while enforcing the constraint. This yields a DDE *on a manifold*, called *the essential underlying* DDE for pure index-2 DDAE (1.2). The EUDDE for mixed index-2 DDAE (1.3) is defined similarly and its size is $n_1 - n_3$. The main difficulty with this approach, when applying to nonlinear DAEs, arises in the presence of nonlinear terms. Due to this, it is not practical for numerical solution and is used rather as a theoretical tool, see [Ascher & Petzold, 1998] for details in case of DAEs. To give an idea about EUDDEs for index-2 problems, we present an example similar to one in [Ascher & Petzold, 1995] where details can be found. For a linear pure index-2 DDAE, which can be considered as the linearization of (1.2) around its T-periodic solution,

$$\begin{aligned} y_1'(t) &= A_0(t)y_1(t) + A_1(t)y_1(t-\tau) + A_2(t)y_2(t) \\ 0 &= B(t)y_1(t), \end{aligned} \tag{2.4}$$

the EUDDE can be written as

$$v'(t) = \Big(R'(t)S(t) + R(t)A_0(t)S(t)\Big)v(t) + R(t)A_1(t)S(t-\tau)v(t-\tau). \tag{2.5}$$

In (2.5), the new unknowns $v(t) \in \mathbb{R}^{n_1-n_2}$ are defined as

$$v(t) = R(t)y_1(t), \quad t \in [0,T], \tag{2.6}$$

with the inverse transformation given by

$$y_1(t) = \begin{pmatrix} R(t) \\ B(t) \end{pmatrix}^{-1} \begin{pmatrix} v(t) \\ 0 \end{pmatrix} \equiv S(t)v(t), \tag{2.7}$$

where $R(t) \in \mathbb{R}^{(n_1-n_2)\times n_1}$ and $S(t) \in \mathbb{R}^{n_1\times(n_1-n_2)}$ satisfy

$$R(t)A_2(t) = 0, \quad R(t)S(t) = I, \quad B(t)S(t) = 0, \quad t \in [0,T]. \tag{2.8}$$

In case of nonlinear DAEs, $R(t)$ can depend on the solution components. Hence the set of unknowns in which the EUDDE is expressed can change depending on the solution, leading to difficulties especially in the BVP case, see, e.g. [Ascher & Petzold, 1998] for DAEs problems.

2.2 DDE with an invariant

Delay DAEs (1.1)-(1.3) can be rewritten as DDEs for all variables through differentiations of the constraints. However, an isolated periodic solution of the original delay DAE is not an isolated solution of the obtained DDE, which is manifested, in particular, by extra Floquet multipliers at 1 for the DDE. Therefore we cannot obtain numerical approximations of isolated periodic solutions from the DDE alone. Information about the constraints, which form *an invariant set* for this DDE, is always necessary. To compute a unique T-periodic solution, it is necessary to solve the DDE together with algebraic conditions obtained by fixing the algebraic constraints at any time point $t_0 \in [0,T]$. For example, such a condition for index-1 DDAE (1.1) reads as

$$g(x_1(t_0), x_1(t_0-\tau), x_2(t_0)) = 0, \quad t_0 \in [0,T]. \tag{2.9}$$

Thus, a periodic solution of the DDE satisfies $g(x_1(t), x_1(t-\tau), x_2(t)) = 0$ for all $t \in [0, T]$, and hence it solves the DDAE and vice versa.

In case of index-2 problems, we can obtain a "smaller" DDE, a DDE for the differential variables only, if all algebraic variables can be eliminated after the first differentiation of the index-2 constraints. The resulting DDE in x_1 is of size n_1 and the index-2 constraints form its *invariant set*. Hence the DDE, coupled with the algebraic conditions obtained by fixing the index-2 constraints at any $t_0 \in [0, T]$, has the same periodic solutions as the index-2 delay DDAE.

In DDE-BIFTOOL, algebraic conditions can be defined (programmed) by the user in a code specifically designed for algebraic conditions. In practice, this approach can lead to certain computational difficulties (as a non-square matrix in a Newton iteration), especially if the number of constraints (and hence the number of algebraic conditions) is large. Moreover, the maximal delay in the obtained DDE can be much larger than the one in the DDAE, which may increase the computational cost significantly when computing stability of periodic solutions, cf. Sec. 4.

As we have seen, the DDE-formulations of delay DAEs (1.1)-(1.3) have certain drawbacks. Hence a direct solution of these equations is worth to be investigated.

3 Periodic Solutions of Index-1 and Index-2 Delay DAEs

In this section we formulate BVPs for computing periodic solutions of delay DAEs (1.1)-(1.3) and discuss stability analysis of these solutions.

3.1 Boundary value problems

Let $C_n := C([-\tau, 0], \mathbb{R}^n)$ be the vector space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$. For $s \in \mathbb{R}$, denote by $(x_i)_s \in C_n$ the segment of a solution x_i defined by $(x_i(\theta))_s = x_i(s+\theta),\ \theta \in [-\tau, 0]$. The space

$$\mathcal{C} = \{(\phi_1, \phi_2) \in C_{n_1} \times C_{n_2} : g(\phi_1(0), \phi_1(-\tau), \phi_2(0)) = 0\}$$

is the state space for index-1 DDAE (1.1), i.e., for any given s the solution segment $(x_1, x_2)_s \in \mathcal{C}$ uniquely determines, via (1.1), the values $(x_1(t), x_2(t))$ for all $t \geq s$. Consequently, any T-periodic solution to (1.1) can be found as the solution of the following *two-point boundary value problem*,

$$\begin{aligned} x_1'(t) &= f(x_1(t), x_1(t-\tau), x_2(t), x_2(t-\tau)) \\ 0 &= g(x_1(t), x_1(t-\tau), x_2(t)), \quad t \in [0, T], \\ (x_1, x_2)_0^T &= (x_1, x_2)_T^T, \\ p(x_1, T) &= 0. \end{aligned} \tag{3.1}$$

As in case of periodic BVPs for DDEs, the boundary condition $(x_1, x_2)_0^T = (x_1, x_2)_T^T$ is a condition in an *infinite-dimensional functional space*, here $\mathcal{C}$, which reflects that a solution of a delay equation is uniquely determined by a function segment on the delay interval. This is in contrast with the analogous problem for ODEs and DAEs, where the boundary condition applies in a *finite-dimensional space*. The last equation in (3.1) represents a suitable phase condition to remove the indeterminacy due to the fact that, for autonomous systems, a phase shift of a periodic solution is also a periodic solution.

The state spaces for pure and mixed index-2 DDAEs (1.2) and (1.3) are defined as

$$\mathcal{C} = \{(\phi_1, \phi_2) \in C_{n_1} \times \mathbb{R}^{n_2} : g(\phi_1(0)) = 0\},$$

respectively

$$\mathcal{C} = \{(\phi_1, \phi_2, \phi_3) \in C_{n_1} \times C_{n_2} \times \mathbb{R}^{n_3} : g_1(\phi_1(0), \phi_1(-\tau), \phi_2(0)) = 0,\ g_2(\phi_1(0)) = 0\}.$$

Periodic solutions of DDAEs (1.2) and (1.3) can be found as the solutions of the following boundary value problems,

$$\begin{aligned} x_1'(t) &= f(x_1(t), x_1(t-\tau), x_2(t)), \\ 0 &= g(x_1(t)), \quad t \in [0, T], \\ (x_1)_0 &= (x_1)_T, \\ p(x_1, T) &= 0, \end{aligned} \tag{3.2}$$

respectively,

$$\begin{aligned} x_1'(t) &= f(x_1(t), x_1(t-\tau), x_2(t), x_2(t-\tau), x_3(t)), \\ 0 &= g_1(x_1(t), x_1(t-\tau), x_2(t)), \\ 0 &= g_2(x_1(t)), \quad t \in [0, T], \\ (x_1, x_2)_0^T &= (x_1, x_2)_T^T, \\ p(x_1, T) &= 0. \end{aligned} \tag{3.3}$$

We do not impose boundary conditions for the index-2 algebraic variables x_2 in (3.2) and x_3 in (3.3) since their periodicity is determined by the differential variable, respectively by the differential and index-1 algebraic variables.

Note that the introduced state spaces reflect the fact that we treat delay DAEs directly. If one considers BVPs corresponding to the EUDDE formulation of delay DAEs, certain reduced state spaces must be introduced. For example, the reduced state space for the EUDDE formulation of an index-1 DDAE, cf. (2.1), is $C([-2\tau, 0], \mathbb{R}^{n_1})$. Accordingly, the periodic boundary condition is $x_1(\theta) = x_1(T + \theta),\ \theta \in [-2\tau, 0]$.

3.2 Stability of periodic solutions

A complete analysis of the local asymptotic stability of a periodic solution to a differential equation can be achieved by applying the Floquet theory. To our knowledge, the stability theory of periodic solutions to delay DAEs has not been developed in the mathematical literature yet. We can only refer to [Hale, 1996] where periodic solutions of an index-1 DDAE, arising due to Hopf bifurcations, are investigated aiming at their limiting behavior in case of a singular perturbation. In this work, standard tools from the DDEs' theory are used to determine the direction of the bifurcations and hence the local stability of the analyzed periodic solutions. We will consider the local stability of periodic solutions of delay DAEs (1.1)-(1.3) "going from DDEs to delay DAEs". Therefore we outline results on the local stability of periodic solutions to autonomous retarded DDEs which we use.

The local stability of a periodic solution to a retarded DDE is determined by the spectrum of the solution operator $S(T,0)$ which integrates the associated variational equation, e.g. (2.2), from time $t=0$ to $t=T$. This operator is called the *monodromy operator*. Any eigenvalue $\mu \neq 0$ of $S(T,0)$ is called a *Floquet multiplier*. Floquet multipliers are independent of the starting moment $t=0$. For autonomous systems there is always a *trivial* Floquet multiplier at 1, corresponding to a perturbation along the periodic solution. If $T \geq \tau$ then $S(T,0)$ is compact and the Floquet multipliers are eigenvalues of finite multiplicity with zero as their only cluster point. If $T < \tau$, $S(T,0)$ is non-compact. Then, there is a $k \in \mathbb{N}$ such that $\tau \leq kT$ and hence $S(kT,0) = S^k(T,0)$ is compact. In this case, the spectrum of $S(T,0)$ is at most countable, is a compact subset of the complex plane with the only possible accumulation point being zero and if $\mu \neq 0$ is in the spectrum of $S(T,0)$, then μ is in the point spectrum of $S(T,0)$. Details can be found in [Hale, 1977].

Let $x^* = (x_1^*, x_2^*)^T$ and $x^* = (x_1^*, x_2^*, x_3^*)^T$ denote a T-periodic solution of delay DAEs (1.1)-(1.2), respectively (1.3). Given a DDAE of index d with a periodic solution $x^*(t)$, let $S_d(T,0)$ denote the monodromy operator for the associated EUDDE, e.g.,

$$(S_1(T,0)y_1)(\theta) = y_1(T+\theta), \quad \theta \in [-2\tau, 0],$$

for the EUDDE (2.1). Let $\tilde{S}_d(T,0)$ denote the solution operator which integrates, over the period T, the direct linearization of the delay DAE around x^*, e.g.,

$$(\tilde{S}_1(T,0)(y_1,y_2)^T)(\theta) = (y_1(T+\theta), y_2(T+\theta))^T, \quad \theta \in [-\tau, 0], \tag{3.4}$$

for the direct linearization (2.3) in case of index-1 DDAEs. Based on the DDE-formulations of delay DAEs (1.1)-(1.3) presented in Sec. 2, we formulate the following result concerning local stability of periodic solutions to these delay DAEs.

Proposition 3.1 *Let $d = 1, 2$. For an index-d delay DAE with a T-periodic solution x^*, nonzero eigenvalues of the operators $S_d(T,0)$ and $\tilde{S}_d(T,0)$ coincide and determine the local asymptotic stability of x^*.*

For index-1 DDAEs, this result is due to the equivalence of the direct linearization of DDAE (1.1) and the linearization of the associated EUDDE at the periodic solution, cf. Sec. 2. For index-2 DDAEs, the result is based on the following consideration. Consider a pure index-2 DDAE. Differentiating the constraint and eliminating the algebraic variable x_2, we obtain an n_1-dimensional DDE with an invariant, cf. Sec. 2. Due to the differentiation, the monodromy operator for this DDE has n_2 eigenvalues at 1 and the remaining nonzero eigenvalues coincide with the ones of $\tilde{S}_2(T,0)$. Further, formally eliminating n_2 components of the differential variable x_1 (and hence n_2 equations), we obtain the EUDDE for the index-2 DDAE. Due to the elimination, the monodromy operator for this EUDDE, $S_2(T,0)$, does not have the additional n_2 eigenvalues at 1 and hence its nonzero eigenvalues coincide with the ones of $\tilde{S}_2(T,0)$. For mixed index-2 DDAEs, the consideration is similar.

For a given DDAE, the above results allow us to refer to the operator $S(T,0)$ and to its nonzero eigenvalues as the monodromy operator, respectively the Floquet multipliers for the DDAE. Since we treat delay DAEs directly, we deal, in our computations, with (an approximation to) the operator $\tilde{S}(T,0)$. Due to the proposition given above, its nonzero eigenvalues are the Floquet multipliers determining the local stability of a periodic solution to the DDAE.

Note that in case of periodic solutions of DAEs, two different approaches are presented in [Franke & Führer, 2001] and [Lamour *et al.*, 1998, 2003] to define monodromy operators and Floquet multipliers for DAEs. In [Franke & Führer, 2001], a monodromy operator for DAEs of index 1 to 3 is defined as the one for a corresponding unconstrained underlying ODE, i.e., for an ODE in minimal coordinates, and its eigenvalues are defined as the Floquet multipliers. In [Lamour *et al.*, 1998, 2003], the introduced monodromy operators for index-1 and index-2 DAEs have the full dimension of a given DAE and it is shown that the "extra" eigenvalues of these operators, w.r.t. the corresponding underlying ODE, are zeros.

4 Computation of Periodic Solutions and Stability Analysis

To compute solutions of the periodic BVPs introduced in Sec. 3.1, we use a collocation method based on piecewise polynomials. Collocation methods are popular methods for the numerical solution of BVPs for systems of ODEs, DAEs and DDEs, cf. the widely used software packages COLSYS/COLNEW [Ascher *et al.*, 1981; Bader & Ascher, 1987], AUTO [Doedel *et al.*, 1997], COLDAE [Ascher & Spiteri, 1994] and DDE-BIFTOOL. We investigate two modifications of the approach implemented in DDE-BIFTOOL to handle periodic BVPs for DDEs [Engelborghs *et al.*, 2000; Engelborghs & Doedel, 2002], i.e., collocation based on Radau IIA and Gauss-Legendre nodes. Using the collocation equations, we analyze the local stability of the computed periodic solutions. We end this section with related results on the solution of BVPs for DAEs and IVPs for delay DAEs.

4.1 Collocation approximation

We scale time by the factor T^{-1} and consider, instead of BVPs (3.1)-(3.3), transformed BVPs. BVP (3.1) now reads as

$$\begin{aligned} x_1'(t) &= Tf(x_1(t), x_1(t-\tilde{\tau}), x_2(t), x_2(t-\tilde{\tau})), \\ 0 &= g(x_1(t), x_1(t-\tilde{\tau}), x_2(t)), \quad t \in [0,1], \\ (x_1, x_2)_0^T &= (x_1, x_2)_1^T, \\ p(x_1, T) &= 0, \end{aligned} \tag{4.1}$$

where $\tilde{\tau} = \tau/T$, T is unknown and will be determined during computations. The transformed BVPs (3.2) and (3.3) read as

$$\begin{aligned} x_1'(t) &= Tf(x_1(t), x_1(t-\tilde{\tau}), x_2(t)), \\ 0 &= g(x_1(t)), \quad t \in [0,1], \\ (x_1)_0 &= (x_1)_1, \\ p(x_1, T) &= 0, \end{aligned} \tag{4.2}$$

respectively,

$$\begin{aligned} x_1'(t) &= Tf(x_1(t), x_1(t-\tilde{\tau}), x_2(t), x_2(t-\tilde{\tau}), x_3(t)), \\ 0 &= g_1(x_1(t), x_1(t-\tilde{\tau}), x_2(t)), \\ 0 &= g_2(x_1(t)), \quad t \in [0,1], \\ (x_1, x_2)_0^T &= (x_1, x_2)_1^T, \\ p(x_1, T) &= 0. \end{aligned} \tag{4.3}$$

First we introduce some notations. Let Π be a mesh, i.e., a collection of mesh points $0 = t_0 < t_1 < \ldots < t_L = 1$ on the interval $[0,1]$. Set $h_i := t_{i+1} - t_i$, $i = 0, \ldots, L-1$. Let $\{c_l\}$ with $0 \le c_1 < c_2 \cdots < c_m \le 1$ be a given, fixed set of collocation parameters and

$$X_c := \{c_{i,l} := t_i + c_l h_i,\ i = 0, 1, \ldots, L-1,\ l = 1, ..., m\} \tag{4.4}$$

be a set of collocation points in $[0,1]$ based on $\{c_l\}$. Let

$$X_m := \{t_{i+\frac{j}{m}},\ t_L,\ i = 0, 1, \ldots, L-1,\ j = 0, ..., m-1\} \tag{4.5}$$

be the set of the mesh points and so-called representation points, i.e. points between the mesh points,

$$t_{i+\frac{j}{m}} := t_i + \frac{j}{m} h_i, \quad j = 1, \ldots, m-1.$$

Denote by π_m the set of all (vector-valued) polynomials of degree not exceeding m.

Index – 1 problems. We approximate a solution $(x_1, x_2)^T$ to BVP (4.1) on the interval $[0,1]$ by an element u from the following space of piecewise polynomials:

$$S_m(\Pi) := \{u \in C : u|_{[t_i, t_{i+1}]} \in \pi_m,\ i = 0, \ldots, L-1\}.$$

Clearly, $\dim S_m(\Pi) = n \times (L \times m + 1) =: N$, where $n := n_1 + n_2$. Note that $u \in S_m(\Pi)$ is continuous but not necessarily continuously differentiable (at the mesh points).

We represent $u(t)$ on subinterval $[t_i, t_{i+1}]$, $i = 0, \ldots, L-1$, as

$$u(t) = \sum_{j=0}^{m} u(t_{i+\frac{j}{m}}) P_{i,j}(t), \quad P_{i,j}(t) = \prod_{k=0, k\neq j}^{m} \frac{t - t_{i+\frac{k}{m}}}{t_{i+\frac{j}{m}} - t_{i+\frac{k}{m}}}, \quad j = 0, 1 \ldots, m, \tag{4.6}$$

where $P_{i,j}(t)$ are Lagrange coefficients. Thus the function $u(t)$ is completely determined on $[0,1]$ by the vectors

$$u_{i+\frac{j}{m}} := u(t_{i+\frac{j}{m}}),\ i = 0, \ldots, L-1,\ j = 0, \ldots, m-1, \quad \text{and} \quad u_L = u(t_L). \tag{4.7}$$

Let $u \equiv (v, w)^T$ and the polynomials v and w approximate x_1, respectively x_2. For the differential equation in (4.1) we require that, for $i = 0, \dots, L-1,\ l = 1, \dots, m$,

$$v'(c_{i,l}) = Tf(v(c_{i,l}), v((c_{i,l} - \tilde{\tau}) \bmod 1), w(c_{i,l}), w((c_{i,l} - \tilde{\tau}) \bmod 1)). \tag{4.8}$$

Note that if the mesh points are included in the set of collocation points, the derivative of v used in (4.8) is that of v restricted to $[t_i, t_{i+1}]$. In (4.8), we used the (linear) periodic boundary condition to eliminate $u(t)$ for $t < 0$. We did not eliminate u_0 (because this value is used in $[0, t_1]$), so we need to require $u_0 = u_L$. Note that we allow, using the modulo operation, the period T to be less than the delay. Using $c = c_{i,l},\ \tilde{c} = (c - \tilde{\tau}) \bmod 1$ and k such that $t_k \le \tilde{c} < t_{k+1}$, the collocation equations (4.8) have the following form

$$\begin{aligned}\sum_{j=0}^{m} v_{i+\frac{j}{m}} P'_{i,j}(c) = Tf\Big(&\sum_{j=0}^{m} v_{i+\frac{j}{m}} P_{i,j}(c), \sum_{j=0}^{m} v_{k+\frac{j}{m}} P_{k,j}(\tilde{c}),\\ &\sum_{j=0}^{m} w_{i+\frac{j}{m}} P_{i,j}(c), \sum_{j=0}^{m} w_{k+\frac{j}{m}} P_{k,j}(\tilde{c})\Big),\end{aligned} \tag{4.9}$$

where $P'_{i,j}$ denotes the derivative of $P_{i,j}$.

The described approach is analogous to the one implemented in DDE-BIFTOOL for DDEs . For the algebraic equation in (4.1) we require that

$$0 = g(v(c), v(\tilde{c}), w(c)), \tag{4.10}$$

and hence

$$0 = g\Big(\sum_{j=0}^{m} v_{i+\frac{j}{m}} P_{i,j}(c), \sum_{j=0}^{m} v_{k+\frac{j}{m}} P_{k,j}(\tilde{c}), \sum_{j=0}^{m} w_{i+\frac{j}{m}} P_{i,j}(c)\Big). \tag{4.11}$$

The first collocation variant, which we will investigate, is collocation at Radau IIA points since Radau IIA methods are widely used for the solution of IVPs for DAEs and delay DAEs, cf. [Ascher & Petzold, 1998; Guglielmi & Hairer, 2001]. Note that when using Radau IIA collocation points, the algebraic equation is satisfied at the mesh points.

With respect to BVPs, it is known that a symmetric method is preferred. Collocation methods based on Gauss-Legendre points have been successfully used for the solution of BVPs for ODEs, DAEs and DDEs, cf. all software packages mentioned in the beginning of this section. However, collocation of the algebraic equation in (4.1) at Gauss-Legendre points leads to a singular linear system in the Newton iteration when solving the nonlinear collocation equations iteratively. This is due to the symmetry of Gauss-Legendre collocation points w.r.t. the mesh points leading to certain properties of Lagrange coefficients in (4.6). Such a singularity takes place when we collocate a delay DAE (its differential and algebraic parts) at symmetric points using continuous extension (4.6) with any representation points (not necessarily equidistant). Therefore the second collocation variant, which we will analyze, is the following. The differential equation in (4.1) is collocated at Gauss-Legendre points and the algebraic equation is collocated at the points of the set X_m except the point t_L,

$$0 = g\Big(v_{i+\frac{j}{m}}, \sum_{\nu=0}^{m} v_{s+\frac{\nu}{m}} P_{s,\nu}(\bar{c}), w_{i+\frac{j}{m}}\Big), \tag{4.12}$$

where $\bar{c} = (t_{i+\frac{j}{m}} - \tilde{\tau}) \bmod 1$ and s is such that $t_s \leq \bar{c} < t_{s+1}$, $i = 0, \ldots, L-1$, $j = 0, \ldots, m-1$. Note that in this case the algebraic equation is satisfied at the mesh and representation points.

So we obtained, for each variant, the $(N+1)$-dimensional system ($N-n$ collocation equations together with equations $u_0 = u_L$ and $p(v, T) = 0$) with N collocation unknowns and the unknown T. We solve this system iteratively by a Newton iteration.

Index − 2 problems. As for index-1 problems, our first collocation variant for BVPs (4.2) and (4.3) is collocation of the differential and algebraic equations at Radau IIA points. Collocation equations are constructed analogously to index-1 problems and we do not present them here.

In the second collocation variant, we also use Gauss-Legendre points and the points of the set X_m to collocate the differential, respectively algebraic parts of BVPs (4.2) and (4.3). However, the implementation of this variant for index-2 BVPs differs from the one for index-1 BVPs.

We first consider BVP (4.2). We approximate the differential solution component $x_1(t)$ by an element $v(t)$ from the $n_1 \times (L \times m + 1)$-dimensional space of piecewise polynomials

$$S_m(\Pi) := \{v \in C : v|_{[t_i, t_{i+1}]} \in \pi_m,\ i = 0, \ldots, L-1\}.$$

Analogously to index-1 BVPs, we use the representation (4.6) for $v(t)$. The algebraic solution component $x_2(t)$ is approximated by a function $z(t)$ computed at Gauss-Legendre collocation points. Thus, for the differential equation in (4.2) we require that

$$v'(c_{i,l}) = Tf(v(c_{i,l}), v((c_{i,l} - \tilde{\tau}) \bmod 1), z(c_{i,l})), \tag{4.13}$$

and, using $c = c_{i,l}$, $\tilde{c} = (c - \tilde{\tau}) \bmod 1$, $t_k \leq \tilde{c} < t_{k+1}$, and $z_{i,l} := z(c_{i,l})$, we obtain that

$$\sum_{j=0}^{m} v_{i+\frac{j}{m}} P'_{i,j}(c) = Tf\Big(\sum_{j=0}^{m} v_{i+\frac{j}{m}} P_{i,j}(c), \sum_{j=0}^{m} v_{k+\frac{j}{m}} P_{k,j}(\tilde{c}), z_{i,l}\Big). \tag{4.14}$$

Collocation equations for the algebraic part of (4.2) are obtained analogously to the ones for index-1 BVPs,

$$0 = g\Big(v_{i+\frac{j}{m}}\Big). \tag{4.15}$$

This approach has the following motivations: a continuous extension of an approximation to the index-2 algebraic variable x_2 is not necessary to solve BVP (4.2), and we remind that the use of the representation (4.6) as a continuous extension for this approximation leads to a singular linear system in a Newton iteration in case of symmetric collocation points. If a continuous extension is required for a dense output, it can be computed using Lagrange interpolation polynomials through the collocation points, e.g.,

$$\begin{aligned} z(t) &= \sum_{l=1}^{m} z_{i,l} \tilde{P}_{i,l}(t) + z_{i+1,1} \tilde{P}_{i,m+1}(t), \quad t \in [c_{i,1}, c_{i+1,1}], \quad i = 0, 1, \ldots, L-2, \\ z(t) &= \sum_{l=1}^{m} z_{L-1,l} \tilde{P}_{L-1,l}(t) + z_{0,1} \tilde{P}_{L-1,m+1}(t), \quad t \in [c_{L-1,1}, 1 + c_{0,1}], \end{aligned} \tag{4.16}$$

where we replaced the interval $[0, c_{0,1}]$ by $[1, 1+c_{0,1}]$ due to periodicity of the solution and

$$\begin{aligned}
&\tilde{P}_{i,l}(t) = \Big(\frac{t - c_{i+1,1}}{c_{i,l} - c_{i+1,1}}\Big) \prod_{k=1,k\neq l}^{m} \frac{t - c_{i,k}}{c_{i,l} - c_{i,k}}, \quad \tilde{P}_{i,m+1}(t) = \prod_{k=1}^{m} \frac{t - c_{i,k}}{c_{i+1,1} - c_{i,k}},\\
&\tilde{P}_{L-1,l}(t) = \Big(\frac{t - (1 + c_{0,1})}{c_{L-1,l} - (1 + c_{0,1})}\Big) \prod_{k=1,k\neq l}^{m} \frac{t - c_{L-1,k}}{c_{L-1,l} - c_{L-1,k}},\\
&\tilde{P}_{L-1,m+1}(t) = \prod_{k=1}^{m} \frac{t - c_{L-1,k}}{1 + c_{0,1} - c_{L-1,k}}.
\end{aligned} \tag{4.17}$$

Note that we use here, as in the first collocation variant, polynomials of degree m.

The obtained system of collocation equations together with the equation $v_0 = v_L$ and the phase condition is solved iteratively by a Newton iteration. Note that the computed v and z are represented at a different number of time points since they are computed at points from the sets X_m and X_c with a different dimension ($L \times m + 1$, respectively $L \times m$). To overcome this, we can introduce an additional component for the vector z and impose the periodic boundary condition $x_2(0) = x_2(1)$. This condition can be approximated, e.g., by

$$z_{L,0} = \sum_{l=1}^{m} z_{L-1,l} \tilde{P}_{L-1,l}(0) + z_{0,1} \tilde{P}_{L-1,m+1}(0), \tag{4.18}$$

where $z_{L,0} := x_2(1)$ and $x_2(0)$ is approximated by $z(0)$, cf. (4.16). Clearly, the boundary condition for the index-2 algebraic variables is redundant but consistent and both of these approaches (with and without this condition) give the same result.

To solve BVP (4.3), a similar approach is applied: $(x_1, x_2)^T$ is approximated by piecewise polynomials $(v, w)^T$ as $(x_1, x_2)^T$ for BVP (4.1) and the index-2 algebraic variable x_3 is approximated by z as x_2 for BVP (4.2). Collocation equations read as

$$\begin{aligned}
\sum_{j=0}^{m} v_{i+\frac{j}{m}} P'_{i,j}(c) = Tf\Big(&\sum_{j=0}^{m} v_{i+\frac{j}{m}} P_{i,j}(c), \sum_{j=0}^{m} v_{k+\frac{j}{m}} P_{k,j}(\tilde{c}), \sum_{j=0}^{m} w_{i+\frac{j}{m}} P_{i,j}(c),\\
&\sum_{j=0}^{m} w_{k+\frac{j}{m}} P_{k,j}(\tilde{c}), z_{i,l}\Big),\\
0 = g_1\Big(&v_{i+\frac{j}{m}}, \sum_{\nu=0}^{m} v_{s+\frac{\nu}{m}} P_{s,\nu}(\bar{c}), w_{i+\frac{j}{m}}\Big),\\
0 = g_2\Big(&v_{i+\frac{j}{m}}\Big),
\end{aligned} \tag{4.19}$$

where $c = c_{i,l}$, $\tilde{c} = (c - \tilde{\tau}) \bmod 1$, $t_k \leq \tilde{c} < t_{k+1}$, $\bar{c} = (t_{i+\frac{j}{m}} - \tilde{\tau}) \bmod 1$, $t_s \leq \bar{c} < t_{s+1}$, $z_{i,l} := z(c_{i,l})$.

4.2 Computing stability of periodic solutions

The approach we apply to compute an approximation to the Floquet multipliers determining the local stability of periodic solutions is similar to the one used in case of periodic

BVPs for DDEs in [Engelborghs *et al.*, 2002; Luzyanina & Engelborghs, 2002].

Index – 1 problems. We obtain an approximation to the Floquet multipliers by computing the eigenvalues of the discretized operator $\tilde{S}_1(1,0)$ (remind that we scaled time by T^{-1}), cf. Sec. 3.2. For this, we use the linearized collocation equations without the modulo operation for the delayed argument, i.e. equations on a mesh in $[-r/T, 1]$, where $r \geq \tau$. The discretization of $\tilde{S}_1(1,0)$, a matrix $\tilde{M}$, is obtained from the extended system using an elimination procedure. $\tilde{M}$ represents the linear map between the variables presenting the segment $[-r/T, 0]$ and those presenting the segment $[-r/T+1, 1]$, i.e. $\tilde{M}u_0 = u_1$, where $u = (v, w)^T$ and $u_0,\ u_1$ are variables at $[-r/T, 0]$, respectively $[-r/T+1, 1]$. When $T < r$, these segments overlap and part of the map $\tilde{M}$ is an identity matrix (presenting a shift of the variables). In this case, we compute eigenvalues of the matrix $\tilde{M}$ which is an approximation of the non-compact operator $\tilde{S}_1(1,0)$, cf. Sec. 3.2.

As the discretization is refined, and, correspondingly, the size of $\tilde{M}$ increases, more and more Floquet multipliers are well approximated by eigenvalues of $\tilde{M}$. Generally speaking, dominant multipliers are better approximated because their corresponding eigenfunctions are usually more smooth [Luzyanina & Engelborghs, 2002]. Note that the computational time depends significantly on the size of $\tilde{M}$ and, hence, on the value of the maximal delay.

Alternatively, using the linearized collocation equations, we can obtain a discretization of the operator $S(1,0)$, a matrix M, such that $Mv_0 = v_1$, where v_0 and v_1 present the segments $[-2r/T, 0]$ and $[-2r/T+1, 1]$. For this, we solve the linearization of (4.11) and (4.12) for w and substitute the solution in the linearization of (4.9) to obtain a system in v. Note that we use the "extended" boundary condition, $v(\theta) = v(1+\theta),\ \theta \in [-2\tau/T, 0]$, when computing "delayed" approximations w for (4.9). Clearly, the dominant eigenvalues of the matrices $\tilde{M}$ and M coincide (up to a roundoff error). Whether this approach requires less computational time than the previous one (i.e. M is smaller than $\tilde{M}$), depends on the size of the DDAE under study and on the value of the maximal delay. In our tests, presented in the next section, we used the first approach.

Index – 2 problems. For both collocation variants, we can construct, analogously to index-1 problems, a matrix $\tilde{M}$ which approximates the operator $\tilde{S}_2(1,0)$ so that $\tilde{M}(v,z)_0^T = (v,z)_1^T$ in case of pure index-2 DDAEs and $\tilde{M}(v,w,z)_0^T = (v,w,z)_1^T$ in case of mixed index-2 DDAEs.

The computational time can certainly be reduced by eliminating the approximations to the index-2 algebraic variables, z, from the linearized collocation equations. After this elimination, we construct a matrix representing a map for the variables v and (v, w) in case of pure, respectively mixed index-2 DDAEs. Dominant eigenvalues of this matrix coincide with the ones of $\tilde{M}$ up to a roundoff error. We used this approach in our computations. For mixed index-2 problems, the computational time may further be reduced by eliminating the approximations w to the index-1 algebraic variables as described above. We can also construct a matrix M, the discretization of the operator $S(1,0)$, such that $M\tilde{v}_0 = \tilde{v}_1$, where $\tilde{v}$ is a vector of $n_1 - n_2$ components of v ($n_1 - n_3$ components of v in case of mixed index-2 DDAEs). For this, we solve the linearized collocation equations corresponding to the index-2 constraints for n_2 (n_3 for mixed problems) components of the vector v. Substituting the solution into the linearization of collocation equations for the differential equation (with

z eliminated) we obtain a system of the size $(n_1 - n_2)(L-1)m$ (respectively, the size $(n_1 - n_3)(L-1)m$), i.e., a system in minimal coordinates. We did not use this approach in our computations due to its rather complicated implementation and the necessity to distinguish between pure and mixed index-2 DDAEs.

4.3 Related results

Here we outline (a) the approach implemented in COLDAE [Ascher & Spiteri, 1994] as related to the subject of this paper and (b) known theoretical convergence results for IVPs for delay DAEs which we compare, in the next section, with our numerical results. The used notations have the same meaning as above unless they are specified.

BVPs for DAEs. In COLDAE, a linearized BVP (for semi-explicit DAEs of index at most 2) is solved by piecewise polynomial collocation at Gauss-Legendre points. The differential and algebraic solution components of a DAE are approximated by elements from $S_m(\Pi)$, respectively $S_{m-1}(\Pi)$ so that approximations to the algebraic variables are generally discontinuous piecewise polynomials. A monomial representation is used on each mesh interval $[t_i, t_{i+1}]$, for the differential (v) and algebraic (w) collocation approximations,

$$v(t) = v_i + h_i \sum_{l=1}^{m} \psi_l\Big(\frac{t-t_i}{h_i}\Big) v'(c_{i,l}), \quad w(t) = \sum_{l=1}^{m} \psi_l\Big(\frac{t-t_i}{h_i}\Big) w(c_{i,l}), \tag{4.20}$$

where $\psi_l(0) = 0,\ \psi_l'(c_j) = \delta_{j,l},\ \psi_l(c_j) = \delta_{j,l},\ j,l = 1,\dots,m.$ The derivatives $v'(c_{i,l})$ and the algebraic variables $w(c_{i,l})$ are eliminated locally at each mesh interval. So, the resulting system to solve is a linear system for the mesh values of v. As emphasized in [Ascher & Spiteri, 1994], the resulting ODE discretization allows to obtain stability and simplifies error control and mesh selection. Note that the use of a similar approach for delay DAEs does not allow a local elimination of variables due to delayed terms.

The straightforward collocation retains properties of nonstiff ODE collocation only for index-1 problems. To retain the ODE convergence orders and, in particular, superconvergence at mesh points, a projected and selective projected collocation methods [Ascher & Petzold, 1991; Ascher & Petzold, 1992] are applied to pure, respectively mixed index-2 DAEs. The idea of projected collocation is that the computed approximations v_i to the differential solution components at the mesh points t_i are corrected as

$$v_i \longleftarrow v_i + A(t_i)\lambda_i, \quad i = 1,\dots,L,$$

where $A(t_i)$ is the derivative of the right-hand side of the differential equation w.r.t. the index-2 variable. λ_i is chosen such that the resulting v_i satisfies the linearized algebraic equation at t_i and it is eliminated locally at each mesh interval. In this way, the computed solution is projected on the constrained manifold at the mesh points. Note that the projections cause a discontinuity in the differential solution components at the mesh points. In the selective projected method, first the index-1 algebraic variables are eliminated to obtain a pure index-2 DAE and then the projection is applied. In [Franke & Führer, 2001], an extension of projected collocation is applied to solve periodic BVPs for an index-2 formulation of the original index-3 DAE. Note that superconvergence at mesh points is, in general, lost for DDEs [Engelborghs *et al.*, 2000] and, hence, we cannot expect such a convergence for delay DAEs.

IVPs for delay DAEs. In [Ascher & Petzold, 1995], stability and convergence of Runge-Kutta methods are analyzed for the solution of IVPs for index-1 and pure index-2 delay DAEs.

Index-1 delay DAEs (1.1) are solved by a standard s-stage implicit Runge-Kutta method combined with an interpolation approximation (of order k_i) of delayed arguments. The proven global (continuous) convergence order of the method equals $\min(k_s+1, k_d, p)$, where k_s and k_d are the stage and ODE orders of the method, $k_s \le k_i,\ k_d \ge k_s$, and p is the degree of continuity of x_1 on the interval of integration. This implies, e.g., the global order $s+1$ in case of the s-stage Radau IIA method if $p > s+1$.

To solve pure index-2 delay DAE (1.2), an extension of the projected implicit Runge-Kutta method is applied,

$$\begin{aligned} X_i' &= f(X_i, \phi x_h(t_i-\tau), Y_i), \quad i=1,\dots,s,\\ 0 &= g(X_i),\\ X_i &= x_{n+1} + h\sum_{j=1}^{s} a_{i,j}X_j', \quad x_n = x_{n-1} + h\sum_{i=1}^{s} b_i X_i' + \frac{\partial f}{\partial y}\lambda_n, \quad 0 = g(x_n), \end{aligned} \tag{4.21}$$

where $x := x_1,\ y := x_2,\ \phi x_h$ is an interpolant for the delayed argument, $a_{i,j}$ and $b_i,\ i,j = 1,\dots,s$, are the coefficients of the applied s-stage Runge-Kutta method and λ_n is chosen such that the resulting x_n satisfies the constraint at t_n. Note that no projection is needed (i.e. $\lambda_n = 0$) for Runge-Kutta methods which satisfy $0 = g(x_n)$ anyway, e.g. for collocation at Radau II A points. The proven convergence results for the differential variable are equivalent to the ones in case of index-1 DDAEs. For the algebraic variable, as the presented numerical experiments (with Gauss-Legendre nodes) suggest, the convergence is one order less.

In [Hauber, 1997], collocation methods of Runge-Kutta type are applied to index-1 and pure index-2 DAEs with state-dependent delays. Index-1 problems are converted to the associated EUDDEs using the implicit function theorem. For index-2 DDAEs, s-stage Runge-Kutta methods with Radau IIA nodes are applied. The global convergence orders $s+1$ and s of the differential, respectively algebraic components of the collocation solution were proven under certain conditions.

Some aspects of numerical solution of delay DAEs can be found in [Liu, 1999; Guglielmi & Hairer, 2001; Baker *et al.*, 2002]. We do not consider these works here since they are not related to our subject.

5 Numerical Results

In this section we first introduce the delay DAEs for which we computed periodic solutions and Floquet multipliers. Then we analyze the accuracy of the computed solutions and multipliers through observing their convergence orders. We end with examples on bifurcation analysis of delay DAEs. All our computations were done with an experimental code based on the package DDE-BIFTOOL.

5.1 Test problems

For our tests, we used five examples of index-1 and index-2 DDAEs. The periodic solution profiles, scaled to the interval $[0,1]$, are depicted in Fig. 1. Each of these solutions was obtained by a continuation of a branch of periodic solutions emanating from a Hopf bifurcation.

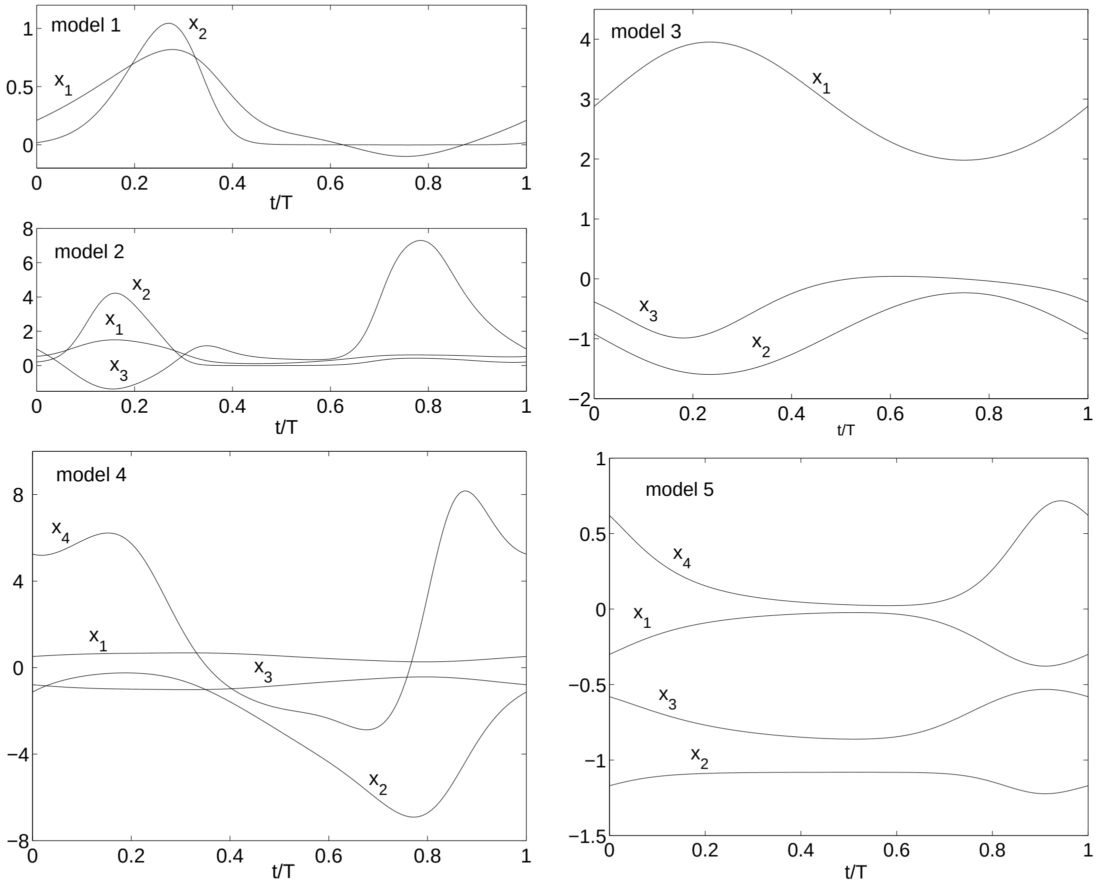

Figure 1: The periodic solutions under consideration.

Model 1 (index-1 DDAE)

$$\begin{aligned} x_1'(t) &= x_1^2(t) - 0.5x_1(t-\tau_1)x_2(t) + \alpha\sin(x_1(t-\tau_1)) - 8x_1(t)x_2(t-\tau_2) + 1, \\ 0 &= -2x_1^3(t) + x_2(t)(1 + x_1(t-\tau_1)). \end{aligned} \tag{5.1}$$

We consider a stable periodic solution with $T \approx 1.83$ computed for $\tau_1 = 2.5,\ \tau_2 = 0.3,\ \alpha = -2.2$. The corresponding branch of periodic solutions is shown in Fig. 7 (left).

Model 2 (index-1 DDAE)

$$\begin{aligned}x_1'(t) =& -5x_1^2(t) - 2x_1(t-\tau_1)x_2(t) + 1.5\sin(x_1(t-\tau_1)) - 5.15x_1(t)x_2(t-\tau_2) + \\ & x_3(t-\tau_1)(1+x_1(t)) + 1, \\ 0 =& -2x_1^3(t) + x_2(t)(1+x_1(t-\tau_1)), \\ 0 =& x_1(t-\tau_1)(x_3(t) + 2x_1(t)) - 1.\end{aligned} \tag{5.2}$$

Here we consider an unstable periodic solution with $T \approx 1.50$ computed for $\tau_1 = 0.5$ and $\tau_2 = 0.3$.

Model 3 (pure index-2 DDAE)

$$\begin{aligned}x_1'(t) =& x_1(t)(1+x_2(t-\tau)) - 0.8x_2(t)x_3(t), \\ x_2'(t) =& -2.3x_2^3(t) + 4x_1(t-\tau)x_3(t), \\ 0 =& x_1^2(t) + 2x_1(t)x_2(t) - 3.\end{aligned} \tag{5.3}$$

For $\tau = 1$, we computed a stable periodic solution with $T \approx 3.54$.

Model 4 (mixed index-2 DDAE)

$$\begin{aligned}x_1'(t) =& -0.5x_1^2(t) + 0.2x_1(t-\tau_1)x_2(t) + \alpha\tanh(x_3(t-\tau_2)) + 0.1, \\ x_2'(t) =& 0.6x_1(t)x_2(t-\tau_2) + 0.3x_4(t)(1-0.5x_3(t)), \\ 0 =& 1.7x_1(t) + x_3(t)(1+0.2x_1(t)), \\ 0 =& 0.4x_3^2(t-\tau_2) - 0.2x_1(t-\tau_1)x_2(t).\end{aligned} \tag{5.4}$$

Here we consider a stable periodic solution with $T \approx 15.34$ computed for $\tau_1 = 0.5,\ \tau_2 = 6,\ \alpha = -0.41$. The corresponding branch of periodic solutions is shown in Fig. 7 (right).

Model 5 (mixed index-2 DDAE)

$$\begin{aligned}x_1'(t) =& 2\tanh(x_1(t)) - 2x_3(t)x_4(t), \\ x_2'(t) =& 10x_1(t-\tau)(1+x_2(t)) - 0.9x_4(t), \\ 0 =& x_3(t)(1-2x_1(t)) + 0.13\sin(x_3(t)) + 1, \\ 0 =& x_1^2(t) + x_2(t) + 1.08.\end{aligned} \tag{5.5}$$

For $\tau = 1$, we computed an unstable periodic solution with $T \approx 10.15$.

5.2 Numerical order results

For all tests described in this section, we use equidistant meshes, i.e. $h_i = h = 1/L,\ i = 0, \dots, L-1$. Our computations proceed as follows. First we compute an "exact" periodic solution and its Floquet multipliers by the collocation variant with Gauss-Legendre points using $L = 800$ mesh subintervals for models 1-3, $L = 600$ for models 4,5 (the values of L here are due to a memory limit) and collocation polynomials of degree 4 ($m = 4$). For an "exact" periodic solution, let x_d and x_a denote its differential and algebraic components and μ^* denote its multipliers. Next, using both collocation variants, we compute periodic solutions and their multipliers using $L = 10, \dots, L_f$ (L_f is specified below) and $m = 2, 3, 4$.

Then, at a very fine mesh, we compute (approximations of) the *continuous error* defined by

$$Ed = \max_{t\in[0,1]} \|x_d(Tt) - u_d(t)\|, \quad Ea = \max_{t\in[0,1]} \|x_a(Tt) - u_a(t)\|,$$

and the error of the approximation of the two dominant multipliers,

$$E\mu_i = |\mu_i^* - \mu_i|, \quad i = 1, 2,$$

where u_d, u_a denote the differential and algebraic components of a collocation solution with multipliers μ and $\|\cdot\|$ is the Euclidean norm.

To distinguish the errors corresponding to the collocation methods with Radau IIA and Gauss-Legendre points, we use Ed_R, Ea_R, respectively Ed_G, Ea_G. Note that for all considered solutions, the trivial multiplier is either the dominant one (stable solutions) or the second one (unstable solutions). The accuracy of the computed trivial multiplier of the "exact" solutions is 10^{-14} for models 1-4 and 10^{-10} for model 5. We do not consider other multipliers because all computed multipliers, except the trivial one, have the same order of convergence and the only difference is in the error constant.

Below we present the observed convergence orders for the test models. As a phase condition for all BVPs, we fixed the value of one differential solution component at $t_0 = 0$.

Index − 1 problems. Figure 2 shows the evolution of the computed continuous errors for collocation solutions obtained using both collocation variants with $m = 3, 4$ (model 1) and $m = 4$ (model 2) as h goes to zero ($L_f = 200$). Note a slightly larger error in case of collocation at Radau IIA points, in particular for m small. We observe that as $h \to 0$, the slope of the upper bound on the errors tends to $m+1$ and hence an $\mathcal{O}(h^{m+1})$ convergence behavior is apparent. Similar results were obtained for both collocation variants with $m = 2, 3, 4$. These results are analogous to the ones for periodic BVPs for DDEs (when using collocation at Radau II A and Gauss-Legendre points) and to results on IVPs for index-1 delay DAEs, cf. Sec. 4.3.

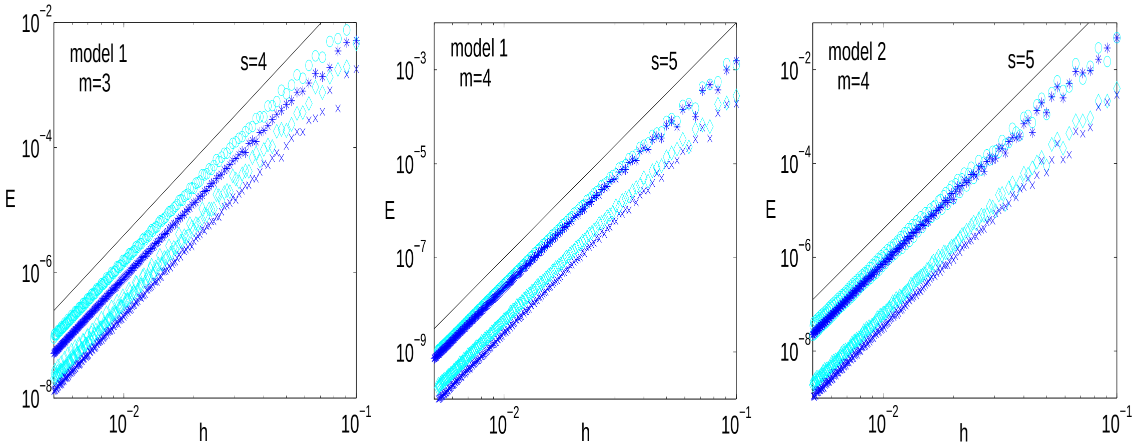

Figure 2: Evolution of $Ed_R(\diamond)$, $Ea_R(\circ)$, $Ed_G(\times)$ and $Ea_G(\star)$ for models 1 and 2. Straight lines indicate the slope s.

For both collocation variants, the convergence of the computed trivial multiplier is very fast, faster than $\mathcal{O}(h^{2m})$, cf. Fig. 3. Similar behavior is observed when solving periodic BVPs for DDEs by collocation at Gauss-Legendre and Radau IIA points on equidistant meshes. Some comments on this phenomenon can be found in [Luzyanina & Engelborghs, 2002]. For all other multipliers, analogously to periodic BVPs for DDEs [Luzyanina & Engelborghs, 2002], an $\mathcal{O}(h^{m+1})$ convergence behavior is apparent. The error constant for approximations of multipliers by both collocation variants is practically the same (cf. the left and middle figures in Fig. 3).

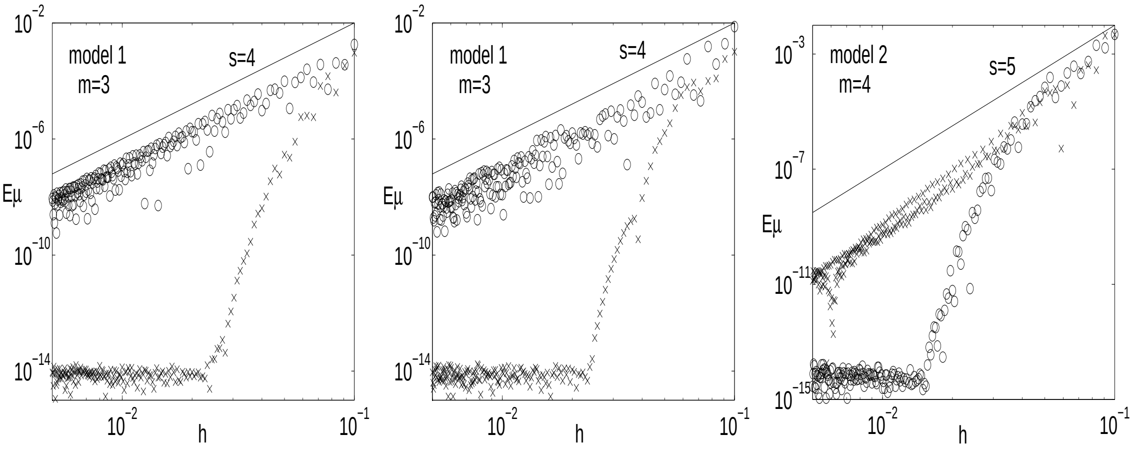

Figure 3: Evolution of $E\mu_1(\times)$ and $E\mu_2(\circ)$ computed using collocation at Gauss-Legendre points (left and right) and at Radau IIA points (middle). $\mu_1^* = 1$, $\mu_2^* \approx -0.937$ (model 1) and $\mu_1^* \approx 2.45$, $\mu_2^* = 1$ (model 2). Straight lines indicate the slope s.

To compare the accuracy of the computed periodic solutions and their multipliers with the one obtained by solving periodic BVPs for the associated EUDDEs, we performed similar convergence tests using DDE-BIFTOOL. The obtained errors differ non-significantly w.r.t. to the ones for delay DAEs.

Index − 2 problems. Figure 4 shows the evolution of the computed continuous errors for collocation solutions to models 3-5 obtained using both collocation variants with $m = 4$. For models 4 and 5, the error Ed is not depicted because it is practically independent of the collocation variant. Since these models involve index-1 and index-2 algebraic variables, we use the notations $Ea1$ and $Ea2$ for the corresponding errors. Note that, in case of collocation at Gauss-Legendre points, the continuous error for approximations to the index-2 algebraic variables was computed using the continuous extension (4.16).

For all index-2 models, we used $L_f = 400$ because, for some examples, the convergence order for index-2 algebraic variables changes as $h \to 0$, e.g. Fig. 4 (left). We observe that collocation at Gauss-Legendre points yields more accurate, in comparison with collocation at Radau IIA points, approximations to algebraic variables. The difference in the accuracy for the two variants is pronounced for solutions with rather steep gradients, cf. Fig. 5. For both variants, we observe an $\mathcal{O}(h^{m+1})$ and $\mathcal{O}(h^m)$ convergence behavior for the differential and index-1 algebraic variables, respectively index-2 algebraic variables. Similar results

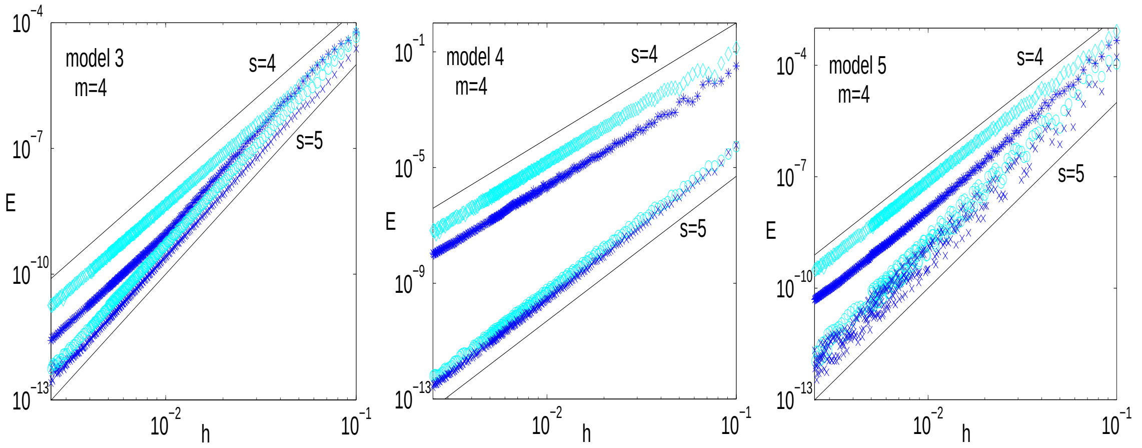

Figure 4: Left: Evolution of $Ed_R(\diamond)$, $Ea_R(\circ)$, $Ed_G(\times)$ and $Ea_G(\star)$ for model 3. Middle, right: Evolution of $Ea1_R(\diamond)$, $Ea2_R(\circ)$, $Ea1_G(\times)$ and $Ea2_G(\star)$ for models 4 and 5. Straight lines indicate the slope s.

were obtained for both collocation variants with $m = 2, 3, 4$. The obtained convergence results agree with the ones for IVPs for pure index-2 delay DAEs, cf. Sec. 4.3.

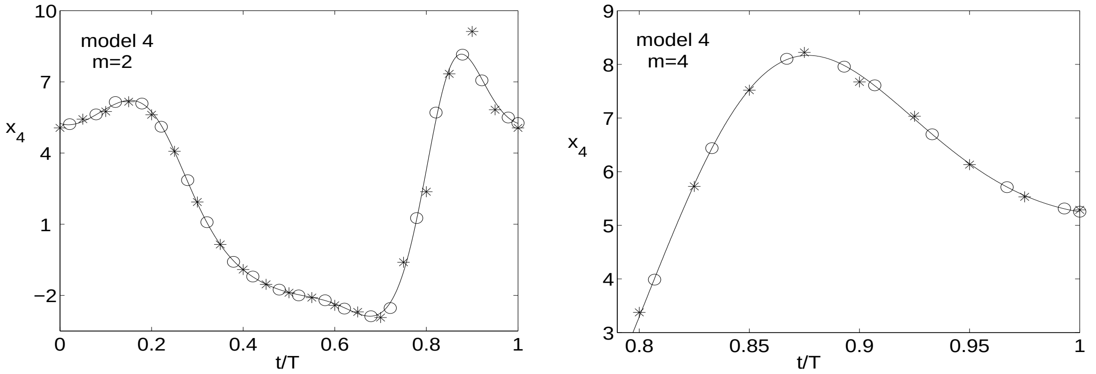

Figure 5: The "exact" profile for the index-2 algebraic solution component x_4 of model 4 (solid lines) and its approximations computed with $L = 10$ and $m = 2, 4$ by collocation at Radau IIA $(\star)$ and Gauss-Legendre $(\circ)$ points. Note that, for a better visibility, only a part of the solution is depicted in the right figure.

The convergence of the computed trivial multiplier in case of index-2 problems is also very fast for both collocation variants, cf. Fig. 6. For all other multipliers, an $\mathcal{O}(h^{m+1})$ convergence behavior is apparent. As for index-1 problems, the accuracy of the approximation of multipliers by both collocation variants is very similar and, therefore, the results for collocation at Radau IIA points are not depicted.

Performing similar convergence tests for periodic BVPs for DDEs with invariant sets, corresponding to the considered index-2 models, we obtained an analogous convergence behavior for approximations to the differential and index-1 solution components and to

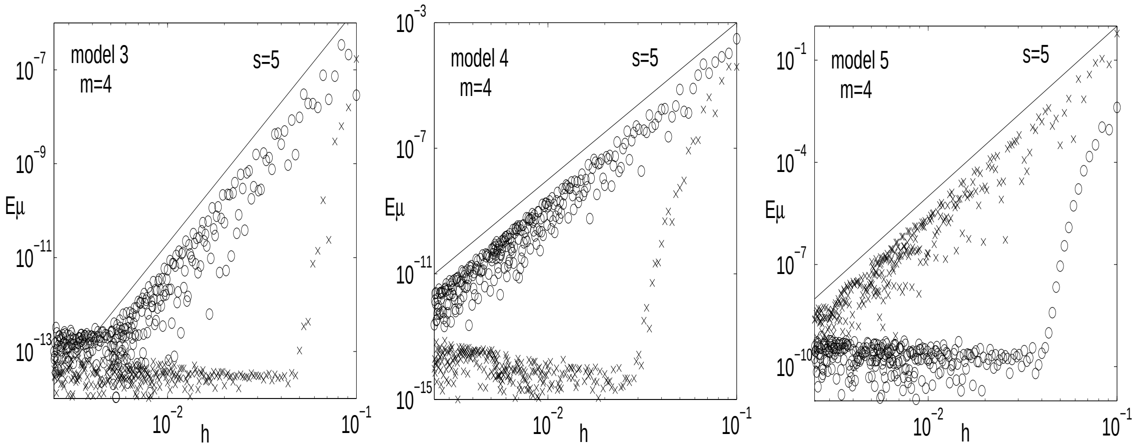

Figure 6: Evolution of $E\mu_1(\times)$ and $E\mu_2(\circ)$ computed using collocation at Gauss-Legendre points. $\mu_1^* = 1,\ \mu_2^* \approx 0.801$ (model 3), $\mu_1^* = 1,\ \mu_2^* \approx -0.107 + 0.052\mathrm{i}$ (model 4) and $\mu_1^* \approx\ 76.1,\ \mu_2^* = 1$ (model 5). Straight lines indicate the slope s.

the Floquet multipliers. Approximations to the index-2 algebraic variables converge as $\mathcal{O}(h^{m+1})$ since they are computed, afterwards, as functions of the differential and index-1 variables.

5.3 Continuation

Figure 7 shows branches of periodic solutions of model 1 (left) and model 4 (right) as functions of the parameter α. For model 1, the branch, emanating from a Hopf point, is initially unstable, gains stability in a turning point, looses it through a period doubling bifurcation and then undergoes a torus bifurcation. For model 4, the branch is initially stable and loses stability in a period doubling bifurcation. The computed dominant Floquet multipliers along the branches are depicted in Fig. 8.

Branches of periodic solutions were computed using the continuation strategy implemented in DDE-BIFTOOL. As a phase condition, we applied the integral phase condition, implemented in the package, to differential solution components. All our computations were done with equidistant meshes. An adaptive mesh selection is an open and nontrivial subject for delay DAEs in case when the algebraic solution components are not eliminated during computations (as in case of the collocation schemes described in Sec. 4.1) and, therefore, a mesh should also be adapted to these solution variables.

6 Conclusions

The results of this paper can be summarized as follows. We have investigated piecewise polynomial collocation schemes for the computation and local stability analysis of periodic solutions to semi-explicit index-1 and index-2 delay DAEs of retarded type. As numerical experiments have shown, the two presented schemes, collocation based on Radau IIA and

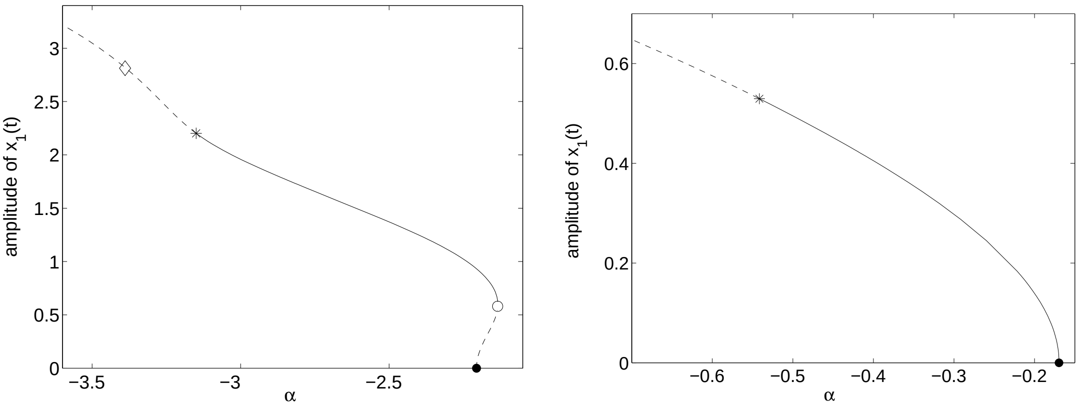

Figure 7: Branches of stable (solid lines) and unstable (dashed lines) periodic solutions of model 1 (left) and model 4 (right) versus the parameter α. Left: A Hopf bifurcation (.) at $\alpha \approx -2.206$, a turning point ($\circ$) at $\alpha \approx -2.135$, a period doubling bifurcation ($\star$) at $\alpha \approx -3.150$ and a torus bifurcation ($\diamond$) at $\alpha \approx -3.389$. Right: A Hopf bifurcation at $\alpha \approx -0.1670$ and a period doubling bifurcation at $\alpha \approx -0.5416$.

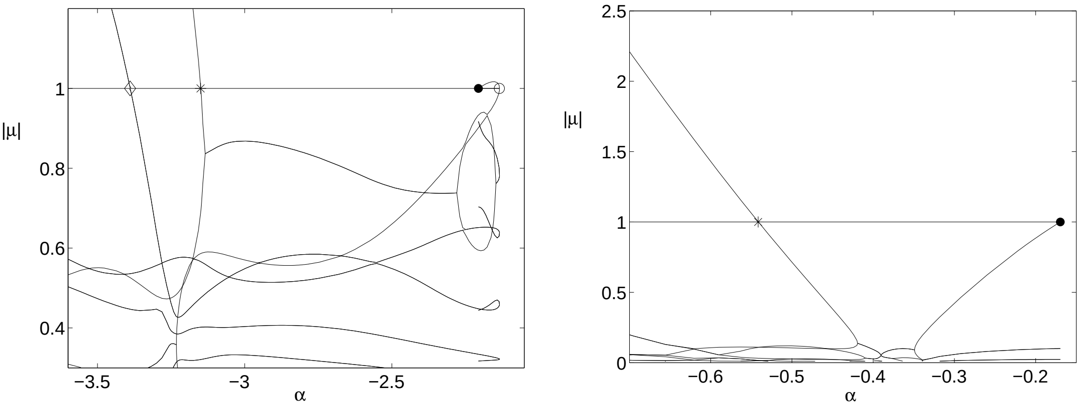

Figure 8: Modulus of the computed dominant Floquet multipliers along the branches of periodic solutions depicted in Fig. 7.

Gauss-Legendre nodes, provide reliable results. The second scheme yields a better accuracy of the algebraic components of solutions, especially in case of index-2 delay DAEs. Based on a set of numerical tests, we obtained approximations to the order of convergence of the computed solutions and Floquet multipliers. The convergence results for periodic solutions agree with known convergence results for IVPs for delay DAEs when using collocation at (m) Radau IIA and Gauss-Legendre nodes (although the collocation schemes are not identical): convergence order $m+1$ for the differential and index-1 algebraic variables and order m for the index-2 variables. The convergence results for the computed multipliers are analogous to those when solving periodic BVPs for DDEs by collocation at Radau IIA and Gauss-Legendre nodes: a very fast convergence of the computed trivial multiplier (using equidistant meshes) and convergence order $m+1$ for all other multipliers. We also showed that the presented collocation schemes, embedded into a parameter continuation procedure, allow to study bifurcation behavior of a delay DAE system.

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