# Location and numerical preservation of characteristic roots of delay differential equations by LMS methods

*Koen Verheyden*　　*Tatyana Luzyanina*　　*Dirk Roose*

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Katholieke Universiteit Leuven
Department of Computer Science
Celestijnenlaan 200A – B-3001 Heverlee (Belgium)

# Location and numerical preservation of characteristic roots of delay differential equations by LMS methods [*]

*Koen Verheyden*  *Tatyana Luzyanina*[†]  *Dirk Roose*

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## Abstract

The local stability of steady state solutions of differential equations with time delays is determined by the roots of a nonlinear characteristic equation. These characteristic roots can be computed by e.g. the discretization of the solution operator using linear multistep (LMS) methods. Ideally, this numerical procedure ensures that *all* characteristic roots with real part larger than a given constant are computed accurately. This requires some knowledge on the location of these roots. The reliability of the numerical results depends on the preservation of these roots by the discrete approximation. Here we present theoretical results for both issues. The theoretical foundation obtained in this paper allows effective improvements to the numerical procedure.

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[†] On leave from the Institute of Mathematical Problems in Biology, RAS, Pushchino, Moscow region, 142290, Russia.

# 1 Introduction

We consider a system of linear *delay differential equations* (*DDEs*) of the form

$$y'(t) = A_0 y(t) + \textstyle\sum_{j=1}^{m} A_j y(t-\tau_j), \quad \text{where } y(t) \in \mathbb{R}^{n\times 1}, \tag{1.1}$$

with $A_0$, $A_j \in \mathbb{R}^{n\times n}$ and *delays* $\tau_j > 0$, for $j = 1, \ldots, m$. The stability (of the zero steady state solution) of (1.1) is determined by the *characteristic roots* $\lambda$ of the *characteristic equation*

$$\det(\lambda I - A_0 - \textstyle\sum_{j=1}^{m} A_j \mathrm{e}^{-\lambda\tau_j}) = 0. \tag{1.2}$$

System (1.1) is asymptotically stable if all characteristic roots $\lambda$ of (1.2) lie in the open left half-plane, i.e., $\Re(\lambda) < 0$, see e.g. [5]. Note that (1.2) has an infinite number of roots $\lambda$. However, the number of roots in any right half-plane, i.e., with $\Re(\lambda) \geqslant r \in \mathbb{R}$ is finite. Hence the stability of (1.1) is always determined by a finite number of roots.

Remark that system (1.1) can be considered as the linearization of the nonlinear DDE system

$$x'(t) = f\left(x(t), x(t-\tau_1), \ldots, x(t-\tau_m)\right) \tag{1.3}$$

about a steady state solution $x(t) \equiv x^\star$, where $f(\cdot)$ is continuously differentiable. The *local stability* of the steady state $x^\star$ of (1.3) is determined by the stability of (1.1).

In this paper, we present a theoretical foundation for a numerical procedure that aims at computing *all* characteristic roots $\lambda$ with $\Re(\lambda) \geqslant r$ for a given $r \in \mathbb{R}$. Specifically, we deal with the location of these roots in the complex plane and the preservation of the roots by a discrete approximation. The theoretical foundation obtained in this paper is used effectively in [8]. There, a numerical procedure to compute accurately all roots $\lambda$ with $\Re(\lambda) \geqslant r$ is made significantly more efficient. The computational approach used is the same as in [2]. That is, one considers a discretization of the linear *solution operator* to (1.1) over a certain time interval. Its eigenvalues before discretization, $\mu$, correspond (by an exponential transform) to the characteristic roots $\lambda$ of (1.2). The dominant eigenvalues after discretization, $\tilde{\mu}$, correspond to approximations $\tilde{\lambda}$ of the rightmost roots $\lambda$ of (1.2). The discrete approximation we consider uses a *linear multistep* (*LMS*) method combined with polynomial interpolation to evaluate the delayed terms.

This paper is structured as follows. Section 2 introduces notations and results needed later on. Section 3 treats the location of the characteristic roots. The preservation of roots by the discrete approximation is considered in Section 4. Section 5 briefly outlines the practical use of the obtained theoretical foundation.

# 2 Preliminaries

Sections 2.1 and 2.2, respectively, consider the (delay-independent) stability of the linear DDE system (1.1) and its discrete approximation using LMS methods.

## 2.1 Stability of the linear DDE system

Let $\mathbb{R}_0^+ := \{r \in \mathbb{R} \, : \, r > 0\}$ and $\mathbb{R}^+ := \{r \in \mathbb{R} \, : \, r \geqslant 0\}$. Similarly

$$\mathbb{C}_0^+ := \{\lambda \in \mathbb{C} \, : \, \Re(\lambda) > 0\} \quad \text{and} \quad \mathbb{C}^+ := \{\lambda \in \mathbb{C} \, : \, \Re(\lambda) \geqslant 0\},$$

denote the open and closed right half-plane, respectively. The definitions for the open and closed left half-plane, $\mathbb{C}_0^-$ and $\mathbb{C}^-$, respectively, are analogous.

Let $\sigma(\cdot)$ denote the spectrum of a matrix. We define the set-valued function

$$\Sigma_\tau(\lambda) := \sigma(A_0 + \textstyle\sum_{j=1}^m A_j \mathrm{e}^{-\lambda\tau_j}), \quad \text{where } \lambda \in \mathbb{C}. \tag{2.1}$$

Hence the characteristic equation (1.2) can be written as

$$\lambda \in \Sigma_\tau(\lambda), \tag{2.2}$$

which resembles a *fixed-point condition*. Clearly, the roots $\lambda$ that lie in $\mathsf{D} \subseteq \mathbb{C}$ are included in $\Sigma_\tau(\mathsf{D}) := \bigcup_{\lambda \in \mathsf{D}} \Sigma_\tau(\lambda)$, e.g. an "unstable root" $\lambda \in \mathbb{C}^+$ is included in $\Sigma_\tau(\mathbb{C}^+)$.

Let $\Omega(\cdot)$ be the set-valued function that maps $\vec{\chi} := (\chi_1, \ldots, \chi_m) \in \mathbb{R}^m$ unto

$$\Omega(\vec{\chi}) := \bigcup_{\vec{\omega} \in [0, 2\pi[^m} \sigma(A_0 + \textstyle\sum_{j=1}^m A_j \mathrm{e}^{-(\chi_j + \mathrm{i}\omega_j)}). \tag{2.3}$$

Remark that $\Omega(\cdot)$, on the contrary to $\Sigma_\tau(\cdot)$, does not depend on $\vec{\tau} := (\tau_1, \ldots, \tau_m)$. Denote by $\vec{\tau} > \vec{0}$ that $\tau_j > 0$, for $j = 1, \ldots, m$ and let $\Omega((\mathbb{R}^+)^m) := \bigcup_{\vec{\chi} \in (\mathbb{R}^+)^m} \Omega(\vec{\chi})$. Obviously, $\Sigma_\tau(\mathbb{C}^+) \subseteq \Omega((\mathbb{R}^+)^m)$, where the equality holds for $m = 1$. Moreover, if $\vec{\tau} > \vec{0}$, then $\mathrm{cl}\, \Sigma_\tau(\mathbb{C}^+) = \mathrm{cl}\, \Omega((\mathbb{R}^+)^m)$, where "cl" denotes the closure, as we will show in Section 3.2.

Using (2.3), we reformulate a result obtained in [2] concerning the *delay-independent stability* and *instability* of (1.1).

**Theorem 2.1.** *(parts* (i) *and* (ii) *of Theorem 3.4 in [2])*

(i) *If* $\Omega((\mathbb{R}^+)^m) \subseteq \mathbb{C}_0^-$, *then* (1.1) *is stable for all* $\vec{\tau} > \vec{0}$.

(ii) *If* $\Omega((\mathbb{R}^+)^m) \subseteq \mathbb{C}_0^+$, *then* (1.1) *is unstable for all* $\vec{\tau} > \vec{0}$.

Note that in our reformulation zero delays are not considered. It was also shown that, possibly except for a degenerate case, these sufficient conditions for delay-independent (in-)stability are also necessary conditions.

Theorem 2.1 allows to obtain information on the location of the characteristic roots in $\mathbb{C}^+$ by investigating the location of $\Omega((\mathbb{R}^+)^m)$. One of the aims of the paper is to make Theorem 2.1 more precise and usable as a building block of a numerical procedure. To achieve the latter, a less cumbersome expression than $\Omega((\mathbb{R}^+)^m)$ must be used. Indeed, locating $\Omega((\mathbb{R}^+)^m)$ in the complex plane, by using its definition, could be computationally inefficient. This problem will be solved naturally, in deriving an extension for Theorem 2.1. Moreover, the extension presented in Sections 3 and 4 allows to locate more precisely the characteristic roots in any right half-plane $\mathbb{C}^+ + r$, where $r \in \mathbb{R}$.

## 2.2 Stability of a discrete approximation

The rightmost, stability-determining roots of the characteristic equation (1.2) can be approximated by the roots associated with a discretization of the solution operator to (1.1). In [2], (1.1) is discretized using a $k$-step LMS method coupled with interpolation to evaluate delayed terms, i.e.,

$$\sum_{i=0}^{k} \alpha_i y_i = h \sum_{i=0}^{k} \beta_i \Big( A_0 y_i + \sum_{j=1}^{m} A_j \sum_{\ell=-s_-}^{s_+} \psi_\ell(\epsilon_j) y_{i+\ell-L_j} \Big), \tag{2.4}$$

where $h$ is the steplength and $y_i$ approximates $y(t_0 + ih)$ for some $t_0$. Here $L_j := \lceil \tau_j / h \rceil$ and $\epsilon_j := L_j - \tau_j/h \in [0, 1[$. A delayed term in (1.1), $y(t_i - \tau_j)$, is approximated in (2.4) by interpolation using the Lagrange polynomials of degree $s_- + s_+$, i.e.,

$$y(t_i - \tau_j) \approx \sum_{\ell=-s_-}^{s_+} \psi_\ell(\epsilon_j) y_{i+\ell-L_j}, \quad \text{with} \quad \psi_\ell(\epsilon_j) := \prod_{o=-s_-,\, o\neq\ell}^{s_+} \frac{\epsilon_j - o}{\ell - o}. \tag{2.5}$$

To avoid the use of future mesh points $y_{k+1}, \ldots$ in (2.4), we require for each $j = 1, \ldots, m$ that $s_+ - L_j \leqslant 0$. Moreover, we impose that $L_j = s_+$ only if $\tau_j = L_j h$ (i.e., $\epsilon_j = 0$), which is necessary for the results in Section 4.2. These conditions are satisfied if the steplength $h$ is bounded above by

$$h_{\max} := \tau_{\min} / s_+, \tag{2.6}$$

where $\tau_{\min} := \min_j \tau_j$ is the minimal delay.

The characteristic equation for the discrete scheme (2.4) is a polynomial in $\tilde{\mu}$ that reads as

$$\det\left( \Big( \sum_{i=0}^{k} \alpha_i \tilde{\mu}^i \Big) I - h \Big( \sum_{i=0}^{k} \beta_i \tilde{\mu}^i \Big) \Big( A_0 + \sum_{j=1}^{m} A_j \sum_{\ell=-s_-}^{s_+} \psi_\ell(\epsilon_j) \tilde{\mu}^{\ell - L_j} \Big) \right) = 0. \tag{2.7}$$

The finite number of (complex conjugate) solutions $\tilde{\mu}$ can be transformed to values $\tilde{\lambda}$ using the relation $\tilde{\mu} = \mathrm{e}^z$ with $|\Im(z)| \leqslant \pi$ and $z = \tilde{\lambda} h$. Moreover, by using the definition

$$\mathrm{LMS}(z) := \frac{\alpha(\mathrm{e}^z)}{\beta(\mathrm{e}^z)}, \quad \text{with} \quad \alpha(\tilde{\mu}) := \sum_{i=0}^{k} \alpha_i \tilde{\mu}^i \quad \text{and} \quad \beta(\tilde{\mu}) := \sum_{i=0}^{k} \beta_i \tilde{\mu}^i, \tag{2.8}$$

one obtains that (2.7) is equivalent to

$$\det\left( \frac{1}{h} \mathrm{LMS}(\tilde{\lambda} h) I - A_0 - \sum_{j=1}^{m} A_j \mathrm{e}^{-\tilde{\lambda}\tau_j} \sum_{\ell=-s_-}^{s_+} \psi_\ell(\epsilon_j) \mathrm{e}^{\tilde{\lambda}(\ell - \epsilon_j) h} \right) = 0. \tag{2.9}$$

The polynomials $\alpha(\cdot)$ and $\beta(\cdot)$ in (2.8) are required to be *irreducible*, such that (2.9) has no roots $\tilde{\lambda}$ that are only caused by the LMS scheme. Thus, the stability of the discrete scheme (2.4) is determined by the real parts of the (finite number of) roots $\tilde{\lambda}$ satisfying (2.9) and $|\Im(\tilde{\lambda})| \leqslant \pi/h$. These roots $\tilde{\lambda}$ approximate roots $\lambda$ of (1.2).

We now define the set-valued function $\Sigma_{\tau,h}(\cdot)$ as

$$\Sigma_{\tau,h}(\tilde{\lambda}) := \sigma\biggl(A_0 + \sum_{j=1}^{m} A_j \mathrm{e}^{-\tilde{\lambda}\tau_j} \sum_{\ell=-s_-}^{s_+} \psi_\ell(\epsilon_j)\mathrm{e}^{\tilde{\lambda}(\ell-\epsilon_j)h}\biggr), \tag{2.10}$$

which allows, analogously to the equivalence between (1.2) and (2.2), to rewrite (2.9) in the form

$$\frac{1}{h}\mathrm{LMS}(\tilde{\lambda}h) \in \Sigma_{\tau,h}(\tilde{\lambda}). \tag{2.11}$$

The delay-independent stability of (2.4) depends on $h$ as given by the following theorem.

**Theorem 2.2.** *(parts* (i) *and* (ii) *of Theorem 4.3 in [2]) Assume that* $\mathrm{LMS}(\mathbb{C}^+) \cap \mathrm{LMS}(\mathbb{C}_0^-) = \emptyset$ *and* $s_- \leqslant s_+ \leqslant s_- + 2$.

(i) *If* $\Omega((\mathbb{R}^+)^m) \subseteq \frac{1}{h}\mathrm{LMS}(\mathbb{C}_0^-)$, *then* (2.4) *is stable for all* $\vec{\tau} > \vec{0}$.

(ii) *If* $\Omega((\mathbb{R}^+)^m) \subseteq \frac{1}{h}\mathrm{LMS}(\mathbb{C}_0^+)$, *then* (2.4) *is unstable for all* $\vec{\tau} > \vec{0}$.

Analogously to Theorem 2.1, Theorem 2.2 can be extended, cf. Section 4.

In [2], Theorems 2.1 and 2.2 are combined to obtain requirements on the LMS method and the steplength $h$ used in the discrete approximation. These requirements ensure that the numerical procedure (approximately) preserves the delay-independent stability of (1.1). In [8], the requirements on the LMS methods used are reconsidered in the light of the extension of Theorems 2.1 and 2.2, presented in Section 4. This extension concerning the preservation of roots allows to make the numerical procedure significantly more efficient, cf. [8].

## 3 The location of characteristic roots

The characteristic roots $\lambda$ that belong to $\mathsf{D} \subseteq \mathbb{C}$ are located in $\Sigma_\tau(\mathsf{D})$, by (2.2). In this section we identify regions in the complex plane that correspond to "parts" of $\Sigma_\tau(\mathsf{D})$ which can partially or totally overlap. Section 3.1 describes the decomposition of $\Sigma_\tau(\mathsf{D})$ for a general $\mathsf{D} \subseteq \mathbb{C}$. Section 3.2 treats $\Sigma_\tau(\mathbb{C}^+ + r)$ where $r \in \mathbb{R}$. A related decomposition of $\Omega(\cdot)$ into "$\Omega_i$-regions" is considered. Furthermore, properties of these regions are discussed as well as their practical use, which is important for Section 4.

### 3.1 Decomposition of $\Sigma_\tau(\mathsf{D})$

We now decompose $\Sigma_\tau(\mathsf{D})$ into $n$ sets which can partially or totally overlap. Remark that a decomposition

$$\Sigma_\tau(\lambda) \equiv \{\sigma_1(\lambda), \ldots, \sigma_n(\lambda)\}, \quad \text{where } \lambda \in \mathsf{D} \subseteq \mathbb{C}, \tag{3.1}$$

into $n$ "*globally continuous*" functions is in general not possible, see e.g. [6]. If $\sigma_i(\lambda)$ is a simple eigenvalue for *all* $\lambda \in \mathsf{D}$, it is a globally continuous function w.r.t. $\lambda \in \mathsf{D}$. However, in general, the global continuity of an individual $\sigma_i(\lambda)$ w.r.t. $\lambda$ cannot be guaranteed. As an example, consider the roots $w_{1,2}(z)$ of $w^2 - z = 0$ [1, Section 5.2]. If $z = 1$, then "its square root" $w_1(1)$ equals 1. Now, let $z$ cycle around the origin of the complex plane. Clearly, $w_1(z)$ changes continuously with $z$. However, $w_1(z)$ "lags behind" : its argument is half of that of $z$. After one cycle, $z$ is back in its original position, $z = 1$, while $w_1(z)$ has changed from $w_1(1) = 1$ to $w_1(1) = -1$. (Likewise, $w_2(z)$ has changed from $-1$ to 1.) Although this example is very simple, it illustrates that a continuous decomposition (3.1) with a fixed numbering is not always possible.

Because we want to keep global continuity in the decomposition, certain eigenvalues in (3.1) "cannot be separated". We now outline the decomposition of $\Sigma_\tau(\lambda)$ into a set of continuous (and even analytic) functions on an appropriate *Riemann surface*. For more details, see e.g. [7, 3]. Let

$$g(\varphi, \lambda) := \det(\varphi I - A_0 - \textstyle\sum_{j=1}^{m} A_j \mathrm{e}^{-\lambda\tau_j}), \tag{3.2}$$

be a polynomial of degree $n$ in $\varphi$. Hence $\Sigma_\tau(\lambda)$ is the set of roots $\varphi$ of $g(\varphi, \lambda) = 0$. First, one defines the set of *branch points* $\Pi_i$ as the set of the $\lambda \in \mathsf{D}$ for which $\sigma_{\hat{\imath}}(\lambda) = \sigma_i(\lambda)$ for some $\hat{\imath} \neq i$, i.e., $\sigma_i(\lambda)$ is a root with multiplicity $\nu > 1$. The sets $\Pi_i$ are countable, by analyticity. Next, certain pairs of branch points $b, c \in \Pi_i$, where $b \neq c$, are connected by curves called *cuts*. All points on these cuts, except the end-points, are collected in the set $\Gamma_i^\circ$. It holds that $\sigma_i(\lambda)$ is continuous on $\mathsf{D}\backslash\Gamma_i^\circ$.

The construction of the appropriate Riemann surface proceeds as follows. First, $\mathsf{D}$ is copied $n$ times to obtain $n$ disconnected *sheets* $\mathcal{R}_{\mathsf{D},i}$. Thus $\lambda \in \mathsf{D}$ corresponds to $n$ *places* $\zeta \in \mathcal{R}_{\mathsf{D},i}$. Then the $i^{\text{th}}$ sheet is cut along $\Gamma_i^\circ$, for $i = 1, \ldots, n$. Next, sheets are connected along the cuts as follows. Let $b$ be a branch point that belongs to $\nu$ sets $\Pi_i$. The corresponding $\nu$ sheets are merged at the places *above* $b$ and the adjacent cuts are rejoined in a circular manner. Thus those $\nu$ sheets are visited sequentially by a place $\zeta$ that circles $\nu$ times around $b$. The connected components of the resulting Riemann surface $\mathcal{R}_{\mathsf{D}} := \bigcup_{i=1}^{n} \mathcal{R}_{\mathsf{D},i}$ correspond to the irreducible factors of $g(\varphi, \lambda)$, seen as a polynomial in $\varphi$ with coefficients that are analytic functions in $\lambda$ on $\mathsf{D}$. Accordingly, the set $\{1, \ldots, n\}$ is partitioned into as many index sets, say $\ell$. Let $\mathfrak{I}(i)$ denote the index set to which $i$ belongs and let $\mathcal{R}_{\mathsf{D},\mathfrak{I}(i)} := \bigcup_{\hat{\imath}\in\mathfrak{I}(i)} \mathcal{R}_{\mathsf{D},\hat{\imath}}$ denote a connected component.

Finally, on each $\mathcal{R}_{\mathsf{D},\mathfrak{I}(i)}$ the $\sigma_{\hat{\imath}}(\lambda)$ for $\hat{\imath} \in \mathfrak{I}(i)$ are "glued together" to obtain (the restriction to $\mathcal{R}_{\mathsf{D},\mathfrak{I}(i)}$ of) the function $\varphi(\zeta)$. Thus, $\Sigma_\tau(\lambda)$ can be decomposed into $\ell$ analytic functions, where each function corresponds to a connected component of $\mathcal{R}_{\mathsf{D},\mathfrak{I}(i)}$. This decomposition is unique up to a permutation of the indices. Accordingly, $\Sigma_\tau(\mathsf{D})$ can be decomposed into $\ell$ sets, $\varphi(\mathcal{R}_{\mathsf{D},\mathfrak{I}(i)})$. Note that $\varphi(\mathcal{R}_{\mathsf{D},\mathfrak{I}(i)})$ and $\varphi(\mathcal{R}_{\mathsf{D},\mathfrak{I}(\hat{\imath})})$ with $\mathfrak{I}(i) \neq \mathfrak{I}(\hat{\imath})$ can partially or totally overlap.

**Remark 3.1.** As outlined above, $\Sigma_\tau(\lambda)$ is decomposed by first constructing an equivalent single-valued function $\varphi(\zeta)$ on a particular Riemann surface. Analogously, $\Sigma_{\tau,h}(\lambda)$ can be decomposed via an equivalent function $\varphi_h(\zeta)$ on an appropriate Riemann surface.

### 3.2 The $\Omega_i$-regions

Let us now turn to the decomposition of $\Sigma_\tau(\mathbb{C}^+ + r)$, where $r \in \mathbb{R}$. First, we define the "$\Omega_i$-regions". Then, the relationship between the decomposition of $\Sigma_\tau(\mathbb{C}^+ + r)$ and these $\Omega_i$-regions is clarified. Some properties of these regions are also discussed. Finally, we briefly consider an example and explain the practical use, which is important for Section 4.2.

Let $\Sigma_\tau(\mathbb{C}^+ + r)$ be decomposed into $\varphi(\mathcal{R}_{\mathbb{C}^+ + r, \mathfrak{I}(i)})$, for $i = 1, \ldots, n$, as described in Section 3.1. Reconsider the resemblance between (2.1) and (2.3). This motivates the following decomposition : $\Omega(\cdot\vec{\tau}) \equiv \bigcup_{i=1}^{n} \Omega_i(\cdot\vec{\tau})$. Let $\varphi^{[\vec{\omega}]}(\cdot)$ on $\mathcal{R}^{[\vec{\omega}]}_{\mathbb{C}^+ + r}$ be as $\varphi(\cdot)$ on $\mathcal{R}_{\mathbb{C}^+ + r}$, but with $A_j$ replaced by $A_j \mathrm{e}^{-\mathrm{i}\omega_j}$, for $j = 1, \ldots, m$. Denote by $\mathsf{Z}^{[\vec{\omega}]}_{r,i}(\mathsf{D})$ the set of places $\zeta \in \mathcal{R}^{[\vec{\omega}]}_{\mathbb{C}^+ + r, \mathfrak{I}^{[\vec{\omega}]}(i)}$ above elements of $\mathsf{D} \subseteq \mathbb{C}$. Define

$$\Omega_i(r\vec{\tau}) := \bigcup_{\vec{\omega} \in [0, 2\pi[^m} \varphi^{[\vec{\omega}]}(\mathsf{Z}^{[\vec{\omega}]}_{r,i}(\{r\})), \quad \text{for } i = 1, \ldots, n, \tag{3.3}$$

where $r \in \mathbb{R}$. Clearly, the union of these functions equals $\Omega(r\vec{\tau})$. Note that if $r$ decreases, the number of branch points corresponding to $\mathsf{Z}^{[\vec{\omega}]}_{r,i}(\{r\})$ increases. Hence the number of distinct $\Omega_i(r\vec{\tau})$-regions decreases with $r$.

Denote by $\mathsf{S}_i$ the subset of $\sigma(A_0)$ such that $\Omega_i(\xi\vec{\tau})$ converges to $\mathsf{S}_i$ in the *Hausdorff metric*, if $\xi \to +\infty$. As mentioned before, $\Sigma_\tau(\mathbb{C}^+ + r)$ is decomposed into $\varphi(\mathcal{R}_{\mathbb{C}^+ + r, \mathfrak{I}(i)})$, for $i = 1, \ldots, n$. The following lemma clarifies the relationship between the elements of the latter decomposition and

$$\Omega_i((\mathbb{R}^+ + r)\vec{\tau}) := \bigcup_{\xi \in \mathbb{R}^+ + r} \Omega_i(\xi\vec{\tau}), \quad \text{for } i = 1, \ldots, n.$$

We will call the latter "$\Omega_i((\mathbb{R}^+ + r)\vec{\tau})$-regions".

**Lemma 3.1.** *Let* $\vec{\tau} > \vec{0}$ *and* $r \in \mathbb{R}$. *Then* $\mathrm{cl}\,\varphi(\mathcal{R}_{\mathbb{C}^+ + r, \mathfrak{I}(i)}) = \mathrm{cl}\,\Omega_i((\mathbb{R}^+ + r)\vec{\tau}) = \Omega_i((\mathbb{R}^+ + r)\vec{\tau}) \cup \mathsf{S}_i$.

*Proof.* We approximate the $m$-vector $\vec{\tau} > \vec{0}$ arbitrarily close (componentwisely) by a $\vec{t} \in (\mathbb{R}_0^+)^m$ such that $t_1, \ldots, t_m$ and 1 are incommensurate. Then, according to Kronecker's theorem, $\left\{ (\theta\vec{t}) \bmod 2\pi \ : \ \theta \in 2\pi\mathbb{Z}_0^+ \right\}$ is dense in $[0, 2\pi[^m$. Therefore $\mathrm{e}^{-\mathrm{i}\omega_j}$ can be approximated arbitrarily close by $\mathrm{e}^{-\mathrm{i}\theta\tau_j}$, where $\theta \in 2\pi\mathbb{Z}_0^+$, for $j = 1, \ldots, m$. Hence by continuity, $\Omega_i(\vec{0}) = \bigcup_{\vec{\omega} \in [0, 2\pi[^m} \varphi^{[\vec{\omega}]}(\mathsf{Z}^{[\vec{\omega}]}_{0,i}(\{0\})) \subseteq \mathrm{cl}\,\varphi(\mathsf{Z}^{[\vec{0}]}_{0,i}(\mathrm{i}\mathbb{R}))$. Moreover, $\varphi(\mathsf{Z}^{[\vec{0}]}_{0,i}(\mathrm{i}\mathbb{R})) \subseteq \Omega_i(\vec{0})$ by (3.3). Because $\Omega_i(\vec{0})$ is compact, $\mathrm{cl}\,\varphi(\mathsf{Z}^{[\vec{0}]}_{0,i}(\mathrm{i}\mathbb{R})) = \Omega_i(\vec{0})$.

The above results also hold if $A_j$ is replaced by $A_j \mathrm{e}^{-\xi\tau_j}$, for $j = 1, \ldots, m$, which proves that $\mathrm{cl}\,\varphi(\mathsf{Z}^{[\vec{0}]}_{\xi,i}(\mathrm{i}\mathbb{R} + \xi)) = \Omega_i(\xi\vec{\tau})$ holds for arbitrary $\xi \in \mathbb{R}$. Thus $\mathrm{cl}\,\varphi(\mathsf{Z}^{[\vec{0}]}_{r,i}(\mathbb{C}^+ + r)) = \mathrm{cl}\,\Omega_i((\mathbb{R}^+ + r)\vec{\tau})$. Clearly, $\mathsf{Z}^{[\vec{0}]}_{r,i}(\mathbb{C}^+ + r) = \mathcal{R}_{\mathbb{C}^+ + r, \mathfrak{I}(i)}$. □

It follows from Lemma 3.1 that $\mathrm{cl}\,\Sigma_\tau(\mathbb{C}^+ + r) = \mathrm{cl}\,\Omega((\mathbb{R}^+ + r)\vec{\tau}) = \Omega((\mathbb{R}^+ + r)\vec{\tau}) \cup \sigma(A_0)$.

We would like to obtain a convenient expression for the boundary of the $\Omega_i((\mathbb{R}^+ + r)\vec{\tau})$-regions. This would simplify the practical investigation of the location of these regions. We now discuss the most practical result we could obtain. First, another concept is introduced.

We say that a (set-valued) function has the open mapping property over a domain $\mathsf{D} \subseteq \mathbb{C}$ if it maps open subsets of $\mathsf{D}$ onto open sets. Let $r \in \mathbb{R}$ and $\vec{\tau} > \vec{0}$. Assume that $\varphi(\cdot)$ restricted to $\mathcal{R}_{\mathbb{C}^+ + r, \mathfrak{I}(i)}$ is nonconstant. This means that no eigenvector of $A_0$ belongs to the null spaces of *all* $A_j$, for $j = 1, \ldots, m$. Then,

$\varphi(\cdot)$ and $\Omega_i(\cdot\vec{\tau})$ have the open mapping property over the domain $\mathcal{R}_{\mathbb{C}^+ + r}$ and $\mathbb{R}^+ + r$, respectively. The former is implied by the open mapping theorem for analytic functions on Riemann surfaces [3, Corollary 2.4]. The latter result follows from (3.3) : it suffices to replace $r$ by $\xi$ and $\{r\}$ by e.g. $\mathrm{i}]0,1[+\xi$. As a consequence, if $\Omega_i(\cdot\vec{\tau})$ is nonconstant and $\hat{r} > r$, $\Omega_i((\mathbb{R}^+ + \hat{r})\vec{\tau}) \cup \mathsf{S}_i$ lies strictly inside $\Omega_i((\mathbb{R}^+ + r)\vec{\tau}) \cup \mathsf{S}_i$. These results allow to obtain the following lemma concerning the boundary of the closure of an $\Omega_i((\mathbb{R}^+ + r)\vec{\tau})$-region.

**Lemma 3.2.** *Let* $\vec{\tau} > \vec{0}$, $r \in \mathbb{R}$ *and* $i \in \{1, \ldots, n\}$. *Then* $\partial\big(\mathrm{cl}\,\Omega_i((\mathbb{R}^+ + r)\vec{\tau})\big) \subseteq \Omega_i(r\vec{\tau})$.

*Proof.* If $\Omega_i(\cdot\vec{\tau})$ is constant, the result trivially holds. Else, by the open mapping property, the interior of $\mathrm{cl}\,\Omega_i((\mathbb{R}^+ + r)\vec{\tau})$ contains $\Omega_i((\mathbb{R}_0^+ + r)\vec{\tau})$. Hence $\partial\big(\mathrm{cl}\,\Omega_i((\mathbb{R}^+ + r)\vec{\tau})\big) \subseteq \Omega_i(r\vec{\tau}) \cup \mathsf{S}_i$. The result follows from the fact that $\mathsf{S}_i$ lies strictly inside $\mathrm{cl}\,\Omega_i((\mathbb{R}^+ + r)\vec{\tau})$. □

We also proved the following result, already mentioned in Section 2.1.

**Lemma 3.3.** *If* $\vec{\tau} > \vec{0}$, *then* $\mathrm{cl}\,\Sigma_\tau(\mathbb{C}^+) = \mathrm{cl}\,\Omega((\mathbb{R}^+)^m) = \Omega((\mathbb{R}^+)^m) \cup \sigma(A_0)$.

*Proof.* Lemma 3.1 implies that $\mathrm{cl}\,\Sigma_\tau(\mathbb{C}^+) = \mathrm{cl}\,\Omega(\mathbb{R}^+\vec{\tau}) = \Omega(\mathbb{R}^+\vec{\tau}) \cup \sigma(A_0)$. Since the latter is a subset of $\Omega((\mathbb{R}^+)^m) \cup \sigma(A_0)$, it suffices to prove that $\mathrm{cl}\,\Omega(\mathbb{R}^+\vec{\tau}) = \mathrm{cl}\,\Omega((\mathbb{R}^+)^m)$. Consider $\Omega(\mathbb{R}^+\vec{\tau})$ as a continuous set-valued function w.r.t. $\vec{\tau}$. Clearly, $\mathrm{cl}\bigcup_{\vec{\tau}>\vec{0}}\Omega(\mathbb{R}^+\vec{\tau}) = \mathrm{cl}\,\Omega\,((\mathbb{R}^+)^m)$. Lemmas 3.1 and 3.2 imply that $\partial\Omega(\mathbb{R}^+\vec{\tau}) \subseteq \Omega(\vec{0}) \cup \sigma(A_0)$. Hence by continuity, $\mathrm{cl}\,\Omega(\mathbb{R}^+\vec{\tau}) = \mathrm{cl}\,\Omega((\mathbb{R}^+)^m)$ for arbitrary $\vec{\tau}$. □

Remark that Lemma 3.1 is a refinement of Lemma 3.3 and uses a less cumbersome expression than $\Omega((\mathbb{R}^+)^m)$. Lemma 3.3 clarifies the occurrence of $\Omega((\mathbb{R}^+)^m)$ in Theorem 2.1. Our refinement, Lemma 3.1, will be used in extending Theorem 2.1, cf. Section 4.2.

As an example, consider a system of four DDEs and one delay, $\tau = 1$, with

$$A_0 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -10 & -4 \\ 0 & 0 & 4 & -10 \end{bmatrix} \quad \text{and} \quad A_1 = \begin{bmatrix} 3 & 3 & 3 & 3 \\ 0 & -1.5 & 0 & 0 \\ 0 & 0 & 3 & -5 \\ 0 & 5 & 5 & 5 \end{bmatrix}. \tag{3.4}$$

The $\Omega_i((\mathbb{R}^+ + r)\vec{\tau})$-regions, for $i = 1, \ldots, 4$, in case of $r = 0$ and $r = -1$ are shown in Fig. 1. These graphs illustrate that all roots $\lambda \in \mathbb{C}^+ + r$ are included in $\Omega((\mathbb{R}^+ + r)\vec{\tau})$. In case $r = 0$ (Fig. 1, top), the graph shows that there are not more than three distinct $\Omega_i(\mathbb{R}^+\vec{\tau})$-regions. The largest region is obtained two times, cf. (3.3). Consider $\lambda I - A_0 - A_1\mathrm{e}^{-\lambda\tau}$ with $A_0$ and $A_1$ from (3.4). It readily follows that the two small $\Omega_i((\mathbb{R}^+ + r)\vec{\tau})$-regions correspond to factors of (3.2) that are linear in $\varphi$. In case $r = -1$ (Fig. 1, bottom), the graph could suggest that there are four distinct $\Omega_i(\mathbb{R}^+\vec{\tau})$. However, according to our definition, cf. (3.3), the two largest regions are counted only once.

Fig. 1 also illustrates Lemma 3.2. In the case of commensurate delays or a single delay (cf. Fig. 1), the sets $\Omega_i(r\vec{\tau})$ are curves. However in general, $\Omega_i(r\vec{\tau})$ is

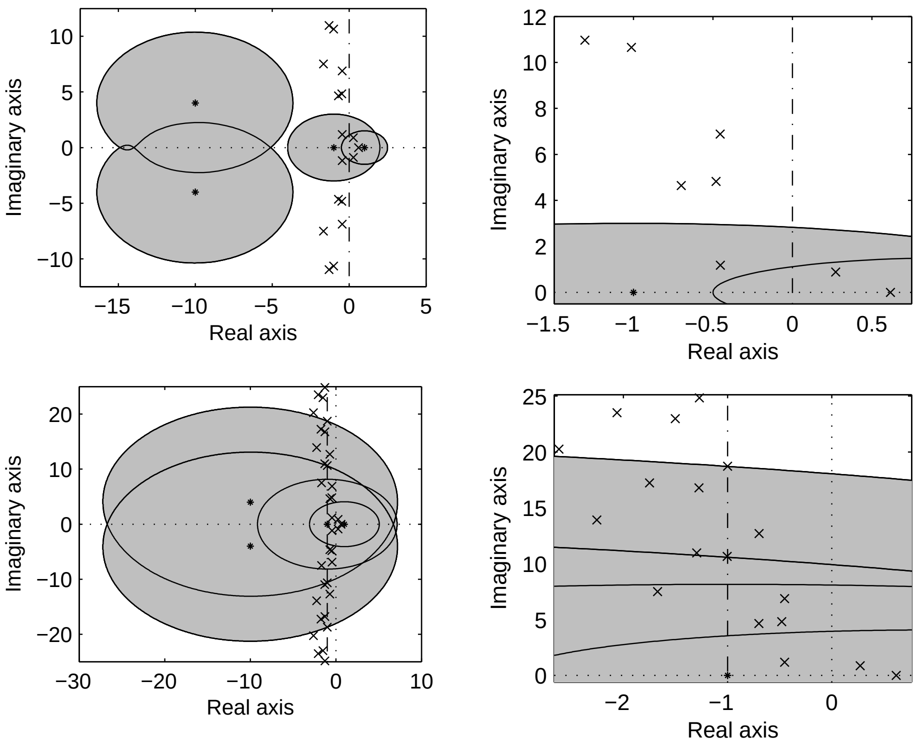

Figure 1: For the DDE system (3.4) with $n = 4$ and $m = 1$ : The characteristic roots $\lambda$ ($\times$), eigenvalues of $A_0$ ($*$) and the vertical line $\mathrm{i}\mathbb{R}+r$ (dashed line). The graphs at the right are blow-ups of the respective graphs at the left. Top ($r = 0$) : $\mathrm{cl}\,\Omega((\mathbb{R}^+)^m) = \mathrm{cl}\,\Omega(\mathbb{R}^+\vec{\tau})$ (colored in gray) and $\Omega(\vec{0})$ (solid line). Bottom ($r = -1$) : $\mathrm{cl}\,\Omega((\mathbb{R}^+ + r)\vec{\tau})$ (colored in gray) and $\Omega(r\vec{\tau})$ (solid line).

a two-dimensional subset of the complex plane. This justifies the name "$\Omega_i(r\vec{\tau})$-*region*". It follows from Lemma 3.1 that

$$\mathrm{cl}\,\Sigma_\tau(\mathbb{C}^+ + r) = \bigcup_{i=1}^{n} \mathrm{cl}\,\varphi(\mathcal{R}_{\mathbb{C}^+ + r, \mathfrak{I}(i)}) = \bigcup_{i=1}^{n} \mathrm{cl}\,\Omega_i((\mathbb{R}^+ + r)\vec{\tau}) = \mathrm{cl}\,\Omega((\mathbb{R}^+ + r)\vec{\tau}). \tag{3.5}$$

Together with Lemma 3.2, this implies that, when investigating the location of $\Sigma_\tau(\mathbb{C}^+ + r)$ w.r.t. $\mathbb{C}^+ + r$, it suffices to check the location of the $\Omega_i(r\vec{\tau})$-regions w.r.t. $\mathbb{C}^+ + r$. Indeed, if an $\Omega_i(r\vec{\tau})$-region belongs to $\mathbb{C}^+ + r$, then $\varphi(\mathcal{R}_{\mathbb{C}^+ + r, \mathfrak{I}(i)}) \subseteq \Omega_i((\mathbb{R}^+ + r)\vec{\tau}) \subseteq \mathbb{C}^+ + r$, for $\hat{\imath} \in \mathfrak{I}(i)$. This result is used in Section 4.2.

Finally remark that characteristic roots in $\mathbb{C}^+ + r$ belong to the set $\bigcup_{\xi \in \mathbb{R}^+ + r} \big(\Omega(\xi\vec{\tau}) \cap (\mathrm{i}\mathbb{R} + \xi)\big)$. In the case of commensurate delays, this set consist of at most $n$ curves.

# 4 Numerical preservation of characteristic roots

Section 4.1 considers the preservation of the characteristic roots in a general subset $\mathsf{D} \subseteq \mathbb{C}$. Section 4.2 treats the case $\mathsf{D} = \mathbb{C}^+ + r$, where $r \in \mathbb{R}$. This section also presents the extension of Theorems 2.1 and 2.2, announced earlier.

## 4.1 Preservation of roots in $\mathsf{D} \subseteq \mathbb{C}$

We first obtain the following general result.

**Theorem 4.1.** *Let* $\mathsf{D}$, $\tilde{\mathsf{D}} \subseteq \mathbb{C}$ *and* $h \in \mathbb{R}_0^+$. *Assume that* (3.2) *is an irreducible polynomial in* $\varphi$ *over* $\mathsf{D}$ *(i.e.,* $\ell = 1$ *in Section 3.1). Then*

(i)
- *If* $\Sigma_\tau(\mathsf{D}) \cap \mathsf{D} = \emptyset$, *then* (1.2) *has no roots* $\lambda$ *in* $\mathsf{D}$.
- *If* $\Sigma_{\tau,h}(\tilde{\mathsf{D}}) \cap \mathsf{D} = \emptyset$ *and* (2.9) *has a root* $\tilde\lambda \in \tilde{\mathsf{D}}$, *then* $\frac{1}{h}\mathrm{LMS}(h\tilde\lambda) \notin \mathsf{D}$.

(ii)
- *Let* $\mathsf{D}$ *be multiple connected. If* $\Sigma_\tau(\mathsf{D})$ *lies strictly inside* $\mathsf{D}$, *then* (1.2) *has exactly* $n$ *roots* $\lambda$ *(counting multiplicities) in the interior of* $\Sigma_\tau(\mathsf{D})$.
- *Let* $\tilde{\mathsf{D}}$ *be multiple connected. If* $\Sigma_{\tau,h}(\tilde{\mathsf{D}})$ *lies strictly inside* $\frac{1}{h}\mathrm{LMS}(h\tilde{\mathsf{D}})$, *then* (2.9) *has at least* $n$ *roots* $\tilde\lambda$ *(counting multiplicities) such that* $\frac{1}{h}\mathrm{LMS}(h\tilde\lambda)$ *lies in the interior of* $\Sigma_{\tau,h}(\tilde{\mathsf{D}})$. *Moreover, if* $\frac{1}{h}\mathrm{LMS}(h\cdot)$ *restricted to* $\tilde{\mathsf{D}}$ *is invertible, then the number of roots is exactly* $n$.

*Proof.* Part (i) of Theorem 4.1 is readily obtained by considering (2.2) and (2.11), respectively.

We first prove the first result of part (ii). Consider a compact, simply connected set $\mathsf{E} \subseteq \mathsf{D}$ such that its interior covers $\Sigma_\tau(\mathsf{D})$. Such a set $\mathsf{E}$ exists since $\mathsf{D}$ is multiple connected, $\Sigma_\tau(\mathsf{D})$ is connected and there is an $\varepsilon$-environment of $\Sigma_\tau(\mathsf{D})$ that belongs to $\mathsf{D}$. Hence $\partial\mathsf{E}$ is free of roots $\lambda$. Denote by $\varphi(\zeta)$ the function defined on $\mathcal{R}_{\mathsf{E}} \subseteq \mathcal{R}_{\mathsf{D}}$, from the decomposition in Section 3.1. We now apply the argument principle to $\varphi(\zeta) - \zeta$ on $\mathcal{R}_{\mathsf{E}}$. Since $\varphi(\cdot)$ "maps inside" $\mathsf{E}$ and there are $n$ sheets, $\varphi(\zeta) - \zeta$ has exactly $n$ zeros (counting multiplicities) in the interior of $\mathcal{R}_{\mathsf{E}}$, which proves the theorem. Remark that $\mathsf{E}$ does not have to be convex for the above to hold.

The second result of part (ii) is obtained analogously to the above by considering $\varphi_h(\zeta) - \frac{1}{h}\mathrm{LMS}(h\zeta)$, where $\varphi_h(\zeta)$ corresponds to $\Sigma_{\tau,h}(\cdot)$, cf. Remark 3.1. If $\frac{1}{h}\mathrm{LMS}(h\cdot)$ restricted to $\tilde{\mathsf{D}}$ is invertible, then it holds that $\partial\mathrm{LMS}(h\tilde{\mathsf{D}}) = \mathrm{LMS}(h\partial\tilde{\mathsf{D}})$. Hence the roots $\tilde\lambda$ correspond to the $n$ zeros of $\frac{1}{h}\mathrm{LMS}^{-1}(h\varphi_h(\zeta)) - \zeta$ on $\mathcal{R}_{\frac{1}{h}\mathrm{LMS}^{-1}(h\tilde{\mathsf{D}})}$. Else, part of $\mathrm{LMS}(h\partial\tilde{\mathsf{D}})$ can "curl inside" $\mathrm{LMS}(h\tilde{\mathsf{D}})$ and the number of roots can be higher. □

Remark that the extension to $\ell \geqslant 2$ is straightforward.

## 4.2 Preservation of roots in $\mathbb{C}^+ + r$

We now presents results for the preservation of characteristic roots in the half-plane $\mathbb{C}^+ + r$ for a given $r \in \mathbb{R}$. The case $r = 0$, which corresponds to the preservation of stability and instability, receives a more detailed treatment.

Let $s_-$, $s_+$, $\psi_\ell(\cdot)$ and $\epsilon_j$ be as defined in Section 2.2. We define the set $\Psi$ as

$$\Psi := \left\{ \tilde\lambda \in \mathbb{C} \ :\ r \leqslant \Re(\tilde\lambda) - \tfrac{1}{\tau_j} \log \left| \sum_{\ell=-s_-}^{s_+} \psi_\ell(\epsilon_j) \mathrm{e}^{\tilde\lambda(\ell-\epsilon_j)h} \right|, \ \text{for } j = 1, \ldots, m \right\}, \tag{4.1}$$

which is a function of $r$, $\vec{\tau}$, $h$, $s_-$ and $s_+$. The location of the set $\Psi$ is related to the quality of the polynomial interpolation (2.5). Let $\rho$ be the minimal value of $\varrho$ for which $\mathbb{C}^+ + \varrho \subseteq \Psi$. We use that the set $\Psi$ approximates $\mathbb{C}^+ + r$ in the following sense.

**Lemma 4.1.** *Let* $\vec{\tau} > \vec{0}$, $h \in ]0, h_{\max}]$ *(cf.* (2.6)*) and* $s_- \leqslant s_+ \leqslant s_- + 2$. *Then*

- $\Sigma_{\tau,h}(\Psi) \subseteq \Omega((\mathbb{R}^+ + r)\vec{\tau})$, *where* $r \in \mathbb{R}$ *and*
- *if* $r \leqslant 0$ *then* $r \leqslant \rho \leqslant 0$.

This result can easily be proven along the lines of Lemma 4.2 in [2], using the maximum modulus theorem. As a consequence, $\mathbb{C}^+ \subset \Psi$, if $r = 0$. Furthermore, due to the quality of the polynomial interpolation, $r$ is approximated well by $\rho$, hence $\mathbb{C}^+ + r$ is approximated well by $\Psi$.

We have obtained the following theorem on the preservation of roots.

**Theorem 4.2.** *Let* $r \in \mathbb{R}$, $\vec{\tau} > 0$, $h \in ]0, h_{\max}]$ *and* $s_- \leqslant s_+ \leqslant s_- + 2$.

(i) *Let* $\Omega(r\vec{\tau}) \cap (\mathbb{C}^+ + r) = \emptyset$. *Then*

  - (1.2) *has no roots* $\lambda$ *in* $\mathbb{C}^+ + r$ *and*
  - (2.9) *has no roots* $\tilde{\lambda}$ *in* $\Psi$ *that satisfy* $\frac{1}{h}\mathrm{LMS}(h\tilde{\lambda}) \in \mathbb{C}^+ + r$.

(ii) *Let* $\Psi_\circ \subseteq \Psi$ *be multiple connected. Assume there are* $n_r$ $\Omega_i(r\vec{\tau})$*-regions that lie in* $\mathbb{C}_0^+ + r$. *If those* $\Omega_i(r\vec{\tau})$*-regions also lie in the interior of* $\frac{1}{h}\mathrm{LMS}(h\Psi_\circ)$, *then*

  - (1.2) *has at least* $n_r$ *roots* $\lambda$ *(counting multiplicities) that lie in the interior of* $(\mathbb{C}_0^+ + r) \cap \frac{1}{h}\mathrm{LMS}(h\Psi_\circ)$ *and*
  - *those roots are approximated by* $n_r$ *roots* $\tilde{\lambda}$ *(counting multiplicities) of* (2.9) *that lie in the interior of* $\Psi_\circ$.

*Proof.* Let $\mathsf{D} = \mathbb{C}^+ + r$ and $\tilde{\mathsf{D}} \subseteq \Psi$ when using Theorem 4.1. It suffices to check the location of the $\Omega_i(r\vec{\tau})$-regions w.r.t. $\mathsf{D}$, by Section 3.2. By Lemma 4.1, it also suffices to check these regions in case of $\Sigma_{\tau,h}(\tilde{\mathsf{D}})$.

(i) This follows from part (i) of Theorem 4.1 with $\tilde{\mathsf{D}} = \Psi$.

(ii) Now let $\tilde{\mathsf{D}} = \Psi_\circ$. Since the $\Omega_i(r\vec{\tau})$-regions are closed while $\mathbb{C}_0^+ + r$ is open, the "strictly inside" condition in Theorem 4.1 is fulfilled. The first result is obtained by considering each irreducible factor of (3.2) for which the corresponding $\varphi(\mathcal{R}_{\mathsf{D},\mathfrak{I}(i)})$ lies in $\mathbb{C}_0^+ + r$. Analogously, the second result is obtained for $\varphi_h(\cdot)$. The total order of the irreducible factors, hence the lower bound for the number of roots, $n_r$, is the same. This follows from Lemma 4.1 and the convergence of both $\Sigma_\tau(\xi + \mathrm{i}\theta)$ and $\Sigma_{\tau,h}(\xi + \mathrm{i}\theta)$ to $\sigma(A_0)$, if $\xi \to +\infty$ and $h \leqslant h_{\max}$.

$\square$

Note that in part (ii) of Theorem 4.2 an $\Omega_i(r\vec{\tau})$-region is counted as many times as it appears in (3.3).

Consider the case $r = 0$. We first define the *stability preserving region* of an LMS method as

$$\mathcal{S}_{\mathrm{pr}}^+ := \left\{ z \in \mathbb{C}^+ \;:\; \mathrm{LMS}(z) \in \mathbb{C}^+ \right\}. \tag{4.2}$$

This region (4.2) is the domain where $\mathrm{LMS}(\cdot)$ "maps unstable points on unstable points". The stability preserving region differs from the stability region of an LMS method, defined as the subset of the complex plane where no $\lambda \in \mathbb{C}_0^+$ are mapped upon by $\mathrm{LMS}(\cdot)$. For a more precise definition of the latter, see e.g. [4].

The following corollary is implied by Theorem 4.2 and Lemma 4.1.

**Corollary 4.1.** *Under the assumptions of Theorem 4.2, the following holds.*

(i) *If $\Omega(\vec{0}) \cap \mathbb{C}^+ = \emptyset$, then* (1.2) *has no roots $\lambda \in \mathbb{C}^+$ and* (2.9) *has no roots in $\frac{1}{h}\mathcal{S}_{\mathrm{pr}}^+ \subset \mathbb{C}^+$ for all $\vec{\tau} > \vec{0}$.*

(ii) *Assume that $n_r$ $\Omega_i(\vec{0})$-regions lie in the interior of $\frac{1}{h}\mathrm{LMS}(\mathcal{S}_{\mathrm{pr}}^+) \subset \mathbb{C}_0^+$. Then* (1.2) *has at least $n_r$ roots $\lambda$ (counting multiplicities) in the interior of $\frac{1}{h}\mathrm{LMS}(\mathcal{S}_{\mathrm{pr}}^+)$ and those roots are approximated by $n_r$ roots $\tilde{\lambda}$ (counting multiplicities) of* (2.9) *that lie in the interior of $\frac{1}{h}\mathcal{S}_{\mathrm{pr}}^+ \subset \mathbb{C}^+$ for all $\vec{\tau} > \vec{0}$.*

The meaning of the stability preserving region (4.2) is explained by part (i) of Corollary 4.1. Indeed, under the appropriate conditions, the stability properties are maintained inside the scaled stability preserving region $\frac{1}{h}\mathcal{S}_{\mathrm{pr}}$.

The results of Corollary 4.1 are refinements of Theorems 2.1 and 2.2. Both results concern the delay-independent (in-)stability. In the former case, the delay-independence of the set $\Omega(r\vec{\tau})$ is obtained by the choice $r = 0$.

## 5 Practical use

The theoretical foundation obtained in this paper is used effectively in [8]. There, a numerical procedure presented in [2] to compute accurately all roots $\lambda$ with $\Re(\lambda) \geqslant r$ is made significantly more efficient. This procedure considers a discretization of the linear solution operator to (1.1) over a certain time interval using an LMS method, cf. Section 2. The dominant eigenvalues of the resulting matrix, $\tilde{\mu}$, correspond to approximations $\tilde{\lambda}$ of the rightmost roots $\lambda$ of (1.2). We now summarize how the theoretical results of Sections 3 and 4 are used in [8] to reduce the computational cost of this procedure while maintaining the reliability of the numerical results.

First, consider the results of Section 3 on the location of the characteristic roots in $\mathbb{C}^+ + r$ for a given $r \in \mathbb{R}$. After the computation of $\Omega(r\vec{\tau}) \cap (\mathbb{C}^+ + r)$, one knows which part of the complex plane has to be considered in the remainder of the numerical procedure. This knowledge is important for the following reason. By using the above discretization, the characteristic roots in the neighborhood of the origin are approximated best. The size of this neighborhood grows as the steplength $h$ used in the discretization decreases. However, a small steplength results in a large linear eigenvalue problem, hence a high computational cost. This cost can be reduced by using a sharp bound on the spectrum in $\mathbb{C}^+ + r$ when determining the steplength. The reliability of the numerical results is ensured since all characteristic roots in $\mathbb{C}^+ + r$ are "captured" by the $\Omega_i$-regions in a region enclosed by $\Omega(r\vec{\tau})$. In particular, Hopf bifurcation points caused by roots with large imaginary part cannot be overlooked.

Notice that the computation of $\Omega(r\vec{\tau})$ is cheap in practice. Indeed, points on $\Omega(r\vec{\tau})$ can be obtained by solving linear eigenvalue problems of the size of the DDE system (1.1). These points form a good estimate of $\Omega(r\vec{\tau})$.

Second, the discretization is improved based on the result on the preservation of roots, cf. Section 4. Specifically, Theorem 4.2 motivates (partially) which requirements on LMS methods are important for our objective. Notice that our objective of approximating the rightmost roots imposes other requirements on the LMS method used. Using these requirements, special-purpose LMS methods are constructed. These requirements also guarantee that the neighborhood of the origin, where the approximation is best, mentioned above, is also significantly larger. This is the second reason why a larger steplength can be used.

Both issues outlined above are worked out in detail in [8]. Their combination results in a procedure that computes efficiently and accurately *all* characteristic roots $\lambda$ with $\Re(\lambda) \geqslant r$.

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