More about five- and six-dimensional lattice rules generated by structured matrices

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Abstract

We describe the results of a computer-based search for 5 and 6-dimensional lattice rules of specified trigonometric degree. In this search only lattice rules that can be generated by a circulant or skew-circulant generator matrix are considered, which makes this approach significantly faster than earlier approaches. The drawback is that we do not necessarily obtain optimal lattice rules. We also present some new families of lattice rules parametrized by the trigonometric degree.

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1 Introduction

We consider s-dimensional integration rules or cubature formulas of the form

$$Q[f] := \sum_{j=1}^{N} w_j f(\mathbf{x}_j) \tag{1}$$

which approximate in some way the integral

$$I[f] := \int_{[0,1)^s} f(\mathbf{x}) d\mathbf{x}. \tag{2}$$

We are particularly interested in cubature formulas that are exact for a space of trigonometric polynomials.

An s-dimensional cubature rule is of trigonometric degree d if it integrates exactly all s-dimensional trigonometric monomials of degree $\le d$. These are functions of the form $\exp(2\pi i \mathbf{h} \cdot \mathbf{x})$, with $\mathbf{h} \in \mathbb{Z}^s$ and $\|\mathbf{h}\|_1 := \sum_{k=1}^{s} |h_k| \le d$.

1.1 Introduction to lattice rules

The construction of cubature formulas of a specified algebraic or trigonometric degree turns out to be nontrivial for dimensions > 2. Almost all known optimal cubature formulas of trigonometric degree are *lattice rules*. A lattice rule is an equal-weight cubature formula whose abscissa lie on the intersection of an *integration lattice* Λ and $[0,1)^s$. An integration lattice Λ is defined as a subset of $\mathbb{R}^s$ which is discrete and closed under addition and subtraction, and which contains $\mathbb{Z}^s$ as a subset. It is conventional to refer to $\mathbb{Z}^s$ as the s-dimensional unit lattice denoted by Λ_0^s.

A lattice can always be described in terms of s linearly independent vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_s\}$. These vectors are known as a set of generators of Λ. Associated with the generators is a $s \times s$ *generator matrix* A whose rows are $\mathbf{a}_1, \ldots, \mathbf{a}_s$. All elements $\mathbf{h}$ of Λ are of the form $\mathbf{h} = \lambda A$ for some $\lambda \in \Lambda_0^s$. One must keep in mind that a generator matrix of a lattice is not unique.

Two lattices are called *geometric equivalent* or *symmetric copies* if the first is obtained from the second by a linear transformation that takes the unit lattice into itself. Let P be a $s \times s$ matrix with in each row and each column exactly one element equal to $+1$ or -1, and all other elements equal to zero. Post-multiplying a generator matrix by P results in a generator matrix of a geometric equivalent lattice (that might be identical to the original one).

Associated with every lattice Λ is its *dual lattice* $\Lambda^\perp$. This is the set $\{\mathbf{h} \in \mathbb{R}^s : \mathbf{h} \cdot \mathbf{x} \in \mathbb{Z} \text{ for all } \mathbf{x} \in \Lambda\}$. In terms of matrices, the dual lattice $\Lambda^\perp$ may be defined as having generator matrix $B = (A^{-1})^T$, with A a generator matrix of Λ. In case of integration lattices: $\Lambda \supseteq \Lambda_0^s$, as a consequence $\Lambda_0^s \supseteq \Lambda^\perp$ and the generator matrix B of $\Lambda^\perp$ is integer-valued. It can be shown that the number of points N, required by the lattice rule, coincides with $|\det A|^{-1} = |\det B|$. The dual of an integration lattice plays an important role in the theory of lattice rules because it can be used to specify an error expansion of the lattice rule $Q[f]$ in terms of the Fourier coefficients $\hat{f}(\mathbf{h})$ of the integrand function as follows:

$$Q[f] - I[f] = \sum_{\mathbf{h} \in \Lambda^\perp \setminus \{0\}} \hat{f}(\mathbf{h}). \tag{3}$$

For more details on the topics introduced above, we refer to [8].

1.2 Lattice rules of trigonometric degree

Our aim is to construct lattice rules with the number of points N as low as possible for a given dimension s and trigonometric degree d. These are called *optimal lattice rules.* In this section we provide some geometric insight in this problem.

For a lattice rule to integrate exactly all polynomials of trigonometric degree $\leq d$, the right-hand side of (3) must vanish whenever f is such a polynomial. This requirement reduces to the condition that $\Lambda^\perp$ has no elements other than the origin itself in the region $\|\mathbf{h}\|_1 \leq d$. One can define the trigonometric degree of a lattice rule in this way. It is more convenient to work with the enhanced trigonometric degree [1]. We use the following definitions:

Definition 1.1. *A lattice rule Q is of enhanced trigonometric degree $\delta := d+1$ if and only if for all $\mathbf{h} \in \Lambda^\perp$, other than $\mathbf{h} = 0$:*

$$\|\mathbf{h}\|_1 := |h_1| + |h_2| + \ldots + |h_s| \geq \delta.$$

The lattice rule Q is of strict enhanced trigonometric degree δ if and only if it is not also of enhanced trigonometric degree $\delta + 1$.

This definition implies that for a lattice rule of enhanced degree δ no element in $\Lambda^{\perp}$ (other than the origin) lies within the s-dimensional crosspolytope

$$\Omega(s,\delta) := \left\{ \mathbf{x} \in \mathbb{R}^s : \sum_{j=1}^{s} |x_j| \leq \delta \right\}$$

(a generalization of a regular three-dimensional octahedron).

A measure for the efficiency of a lattice rule is the *rho-index*

$$\rho(N) := \frac{\delta^s}{s!N}.$$

It was introduced in [1] and discussed in detail in, e.g., [5]. The rho-index is bounded,

$$\forall s,\delta : 0 < \rho(N) \leq 1,$$

and that makes it a convenient instrument for visual presentations.

Definition 1.2. *A cubature formula (1) for the integral (2) is shift symmetric if whenever $(x_1,\ldots,x_s)$ is a point of the formula so is $\{(x_1+\frac{1}{2},\ldots,x_s+\frac{1}{2})\}$, with both points having the same weight.*

The term shift symmetric for cubature formulas of trigonometric degree was introduced in [2]. It plays the same role for cubature formulas of trigonometric degree as central symmetry does for cubature formulas of algebraic degree: i.e. it guarantees that odd functions integrate to zero automatically. One usually verifies the degree of a cubature formula by testing if it is exact for all monomials up to that degree. If a cubature formula is known to be shift symmetric, the amount of time for this verification is reduced by about 50 %.

Observe that a lattice rule is shift symmetric if and only if $(\frac{1}{2},\ldots,\frac{1}{2}) \in \Lambda$.

1.3 K-optimal lattice rules

Cools and Lyness [1] argue that it is reasonable to believe that the lattice Λ of an optimal lattice rule of enhanced trigonometric degree δ has a dual $\Lambda^{\perp}$ with many elements on the boundary of the crosspolytope $\Omega(s,\delta)$. They defined a set $K(s,\delta)$ which comprises all s-dimensional lattices which may be generated by s point pairs, each of which belongs to a distinct facet-pair of the s-crosspolytope $\Omega(s,\delta)$. (A facet-pair of an s-crosspolytope is the s-dimensional generalization of a 2-dimensional pair of opposite faces of a regular 3-dimensional octahedron.) The lattice rules corresponding to lattices from the set $K(s,\delta)$ with a minimum number of abscissa, are called $K(s,\delta)$-optimal lattice rules.

Their search for $K(s,\delta)$-optimal lattice rules has a high computational cost. The search is so expensive that for higher values of δ they were obliged to treat only sub-categories of $K(s,\delta)$. The fundamental problem with this search is that the amount of time required grows as δ^{s^2}.

Lyness and Sørevik [4] noticed that some but not all of the four-dimensional $K(s,\delta)$-optimal lattice rules in [1] could be generated by a skew-circulant matrix. This led them to define and investigate the class of skew-circulant lattice rules. Restricting the search to this smaller class of skew-circulant lattice rules reduces the amount of time to $\mathcal{O}(\delta^{2s-2})$ making this approach feasible in higher dimensions. The drawback is of course that by reducing the population which is searched one does not necessarily obtain the lattice rule with the lowest possible number of points.

In the following section, we give a brief description of the theory of circulant and skew-circulant matrices, as far as it is relevant for our work. In Section 3, we describe the implementation of our search in 5 and 6 dimensions. The results are described in Sections 4 and 5.

2 Circulant and skew-circulant lattice rules

Definition 2.1. *A* circulant matrix *is a matrix of the form*

$$\mathcal{C}irc(\mathbf{b}) = \mathcal{C}irc(b_0, b_1, \ldots, b_{s-1}) := \begin{pmatrix} b_0 & b_1 & b_2 & \ldots & b_{s-1} \\ b_{s-1} & b_0 & b_1 & \ldots & b_{s-2} \\ \vdots & \vdots & \vdots & & \vdots \\ b_2 & b_3 & b_4 & \ldots & b_1 \\ b_1 & b_2 & b_3 & \ldots & b_0 \end{pmatrix}.$$

Definition 2.2. *A* skew-circulant matrix *is a matrix of the form*

$$\begin{aligned} \mathcal{SC}irc(\mathbf{b}) &= \\ \mathcal{SC}irc(b_0, b_1, \ldots, b_{s-1}) &:= \begin{pmatrix} b_0 & b_1 & b_2 & \ldots & b_{s-1} \\ -b_{s-1} & b_0 & b_1 & \ldots & b_{s-2} \\ \vdots & \vdots & \vdots & & \vdots \\ -b_2 & -b_3 & -b_4 & \ldots & b_1 \\ -b_1 & -b_2 & -b_3 & \ldots & b_0 \end{pmatrix}. \end{aligned}$$

Definition 2.3. *A* circulant lattice rule *is one whose integration lattice can be generated by a circulant matrix. A* skew-circulant lattice rule *is one whose integration lattice can be generated by a skew-circulant matrix.*

Observe that a (skew-)circulant matrix is completely determined by its first row. Furthermore, the enhanced degree of a (skew-)circulant lattice rule satisfies $\delta \leq \|\mathbf{b}\|_1$, where $\mathbf{b}$ is a row of a (skew-)circulant generator matrix of the dual lattice. For our purposes it is useful to know that if B is a (skew-)circulant matrix, then both B^T and B^{-1} are also (skew-)circulant. For more information about (skew-)circulant matrices, we refer to [3].

Although in retrospect this result looks trivial, we will now prove that all (skew-)circulant lattice rules of even strict enhanced degree are shift symmetric. This is actually a special case of the following theorem.

Theorem 2.1. *If the dual lattice $\Lambda^\perp \subset \mathbb{Z}^s$ is generated by vectors $\mathbf{b}_j, j = 1, \ldots, s$ such that $\|\mathbf{b}_j\|_1$ is even, then all points $\mathbf{x} \in \Lambda^\perp$ have $\|\mathbf{x}\|_1$ even and Λ is shift symmetric.*

Proof. Following the definition of a dual lattice, it is sufficient to prove that each $\mathbf{p} \in \Lambda^\perp$ and $\mathbf{x} \in \Lambda$ satisfy

$$\mathbf{p} \cdot \left(x_1 + \frac{1}{2}, \ldots, x_s + \frac{1}{2} \right) \in \mathbb{Z}. \tag{4}$$

This is true for each generator $\mathbf{b}_j$ of $\Lambda^\perp$ because $\|\mathbf{b}_j\|_1$ is an even number. Because each $\mathbf{p} \in \Lambda^\perp$ can be written as an integer linear combination of generators of $\Lambda^\perp$, relation (4) holds for each $\mathbf{p}$. □

From this it follows immediately that

Corollary 2.1. *Every (skew-)circulant lattice rule, $\mathcal{C}irc(\mathbf{b})$ or $\mathcal{SC}irc(\mathbf{b})$, with $\|\mathbf{b}\|_1$ even is shift symmetric.*

From Theorem 2.1 we can conclude that also the K-optimal rules of even δ are shift symmetric. The fact that for these rules the odd degree monomials must not be verified was already used in the computations reported in [1], and probably also in earlier computations. To the best of our knowledge Theorem 2.1 is the first to make the connection with the structure of the lattice rule explicitly.

3 The search program

3.1 General description

We have written a program to find all optimal circulant and skew-circulant lattice rules of strict enhanced degree δ in 5 and 6 dimensions. Our search method is an extension of the method developed in [1, 4].

The search program works in four steps. In the first step, all integer vectors $\mathbf{b}$ with non-negative components such that $\|\mathbf{b}\|_1 = \delta$ are constructed. In the second step, $\mathbf{b}$ is modified with a sign-pattern that is specified by the user. The purpose of this is explained in the next subsection. With this modified integer vector $\mathbf{b}$ either a circulant or skew-circulant matrix B is built in the third step. In the last step, we check whether this matrix B satisfies Definition 1.1, and thus is a generator matrix of $\Lambda^\perp$ so that the associated lattice rule $Q[f]$ has the specified enhanced degree δ.

3.2 5-dimensional sign-patterns.

A circulant matrix is fully specified by its first row. We start from an integer vector $\mathbf{b}$ with non-negative components and modify it with a sign-pattern. In this section we will show that from the 2^5 sign-patterns, only 3 have to be investigated.

Definition 3.1. *Let b_0, b_1, b_2, b_3, b_4 be non-negative integers ($b_i \in \mathbb{N}^+$). Define classes of circulant lattices as follows:*

$$
\begin{aligned}
\mathcal{B}_1 &:= \{\mathcal{C}irc(b_0, b_1, b_2, b_3, b_4) : b_i \in \mathbb{N}^+, i = 0, \ldots, 4\} \\
\mathcal{B}_2 &:= \{\mathcal{C}irc(-b_0, b_1, b_2, b_3, b_4) : b_i \in \mathbb{N}^+, i = 0, \ldots, 4\} \\
\mathcal{B}_3 &:= \{\mathcal{C}irc(-b_0, -b_1, b_2, b_3, b_4) : b_i \in \mathbb{N}^+, i = 0, \ldots, 4\} \\
\mathcal{B}_4 &:= \{\mathcal{C}irc(-b_0, b_1, -b_2, b_3, b_4) : b_i \in \mathbb{N}^+, i = 0, \ldots, 4\}
\end{aligned}
$$

These classes obviously overlap because b_i is allowed to be zero, but possibly there is even more overlap.

Theorem 3.1. *Any circulant lattice rule in 5 dimensions is in at least one of the classes $\mathcal{B}_i, i = 1, \ldots, 4$.*

Proof. Because a lattice is a discrete set closed under addition and subtraction and generated by the rows of the generator matrix, it is unchanged by multiplying a row of the generator matrix by -1 or interchanging two rows. Using these properties, one can see that, for example, $\mathcal{C}irc(-b_0, -b_1, -b_2, b_3, b_4)$, $b_i \in \mathbb{N}^+, i = 0, \ldots, 4$ and $\mathcal{C}irc(-b_3, -b_4, b_0, b_1, b_2)$ generate the same lattice belonging to class $\mathcal{B}_3$. All other sign patterns can be verified similarly. □

Theorem 3.2. *Every lattice in $\mathcal{B}_3$ has a symmetric copy in $\mathcal{B}_4$.*

Proof. A matrix of class $\mathcal{B}_3$ can be transformed into a matrix of class $\mathcal{B}_4$:

$$
\begin{pmatrix}
-b_0 & -b_2 & b_4 & b_1 & b_3 \\
b_3 & -b_0 & -b_2 & b_4 & b_1 \\
b_1 & b_3 & -b_0 & -b_2 & b_4 \\
b_4 & b_1 & b_3 & -b_0 & -b_2 \\
-b_2 & b_4 & b_1 & b_3 & -b_0
\end{pmatrix}
$$

$$
= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\cdot
\begin{pmatrix}
-b_0 & b_1 & -b_2 & b_3 & b_4 \\
b_4 & -b_0 & b_1 & -b_2 & b_3 \\
b_3 & b_4 & -b_0 & b_1 & -b_2 \\
-b_2 & b_3 & b_4 & -b_0 & b_1 \\
b_1 & -b_2 & b_3 & b_4 & -b_0
\end{pmatrix}
\cdot
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0
\end{pmatrix}
$$

□

So we are left with the three classes: $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$. Also, in 5 dimensions we do not have to consider skew-circulant lattices.

Theorem 3.3. *Every circulant lattice in 5 dimensions has a skew-circulant symmetric copy.*

Proof. This can easily be verified by taking $T = \mathrm{diag}(1, -1, 1, -1, 1)$ and calculating $T \cdot \mathcal{C}irc(b_0, b_1, b_2, b_3, b_4) \cdot T$, with $b_i \in \mathbb{N}, i = 0, \ldots, 4$. (Here signs do not play a role.) One immediately obtains

$$T \cdot \mathcal{C}irc(b_0, b_1, b_2, b_3, b_4) \cdot T = \mathcal{SC}irc(b_0, -b_1, b_2, -b_3, b_4)$$

□

It was already pointed out in [4] that in an *odd*-dimensional context it can be shown that every skew-circulant rule has a symmetrically equivalent circulant one and vice versa. This is not true in an *even*-dimensional context.

3.3 6-dimensional sign-patterns

In 6 dimensions there are more cases to consider. The technical details are similar to those in the 5-dimensional case and are omitted.

Definition 3.2. *Let $b_0, b_1, b_2, b_3, b_4, b_5$ be non-negative integers. Define the following classes of circulant and skew-circulant lattices:*

$$\begin{aligned}
\mathcal{B}_1 &:= \{\mathcal{C}irc(b_0, b_1, b_2, b_3, b_4, b_5) : b_i \in \mathbb{N}^+, i = 0, \ldots, 5\} \\
\mathcal{B}_2 &:= \{\mathcal{C}irc(-b_0, b_1, b_2, b_3, b_4, b_5) : b_i \in \mathbb{N}^+, i = 0, \ldots, 5\} \\
\mathcal{B}_3 &:= \{\mathcal{C}irc(-b_0, -b_1, b_2, b_3, b_4, b_5) : b_i \in \mathbb{N}^+, i = 0, \ldots, 5\} \\
\mathcal{B}_4 &:= \{\mathcal{C}irc(-b_0, b_1, b_2, -b_3, b_4, b_5) : b_i \in \mathbb{N}^+, i = 0, \ldots, 5\} \\
\mathcal{B}'_1 &:= \{\mathcal{SC}irc(b_0, b_1, b_2, b_3, b_4, b_5) : b_i \in \mathbb{N}^+, i = 0, \ldots, 5\} \\
\mathcal{B}'_2 &:= \{\mathcal{SC}irc(b_0, -b_1, b_2, b_3, b_4, b_5) : b_i \in \mathbb{N}^+, i = 0, \ldots, 5\} \\
\mathcal{B}'_3 &:= \{\mathcal{SC}irc(b_0, b_1, -b_2, b_3, b_4, b_5) : b_i \in \mathbb{N}^+, i = 0, \ldots, 5\} \\
\mathcal{B}'_4 &:= \{\mathcal{SC}irc(b_0, b_1, -b_2, -b_3, b_4, b_5) : b_i \in \mathbb{N}^+, i = 0, \ldots, 5\}
\end{aligned}$$

Theorem 3.4. *Any circulant lattice rule in 6 dimensions has a symmetric copy in at least one of the classes $\mathcal{B}_i$, i=1,...,4. Any skew-circulant lattice rule in 6 dimensions has a symmetric copy in at least one of the classes $\mathcal{B}'_i$, i=1,...,4.*

4 Five-dimensional results

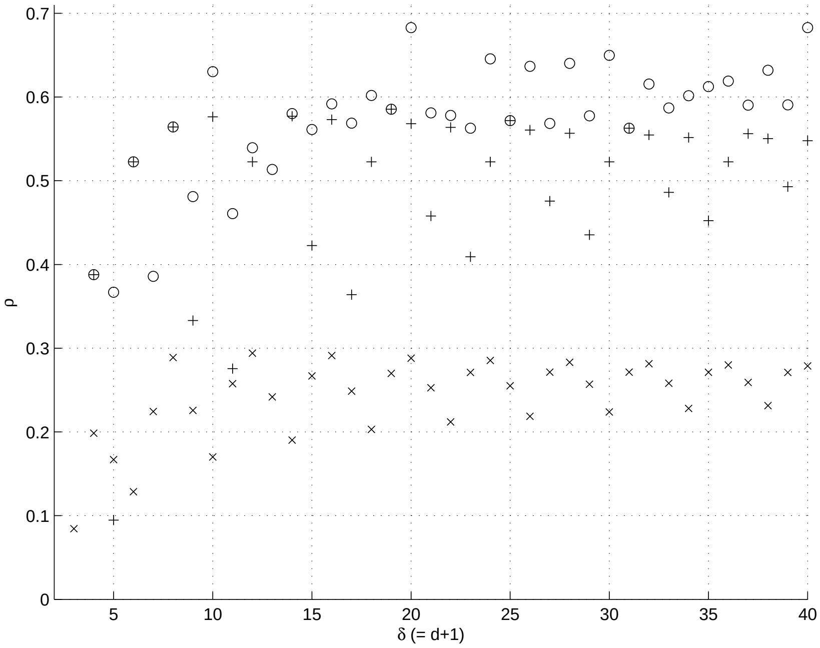
4.1 Computational results

In Figure 1 the rho-index of the optimal circulant lattice rules of degree $\delta \leq 40$ that we obtained, is represented by the symbol $\circ$. In Table 1 we list the first row of the circulant generator matrices for the best lattices we obtained for $\delta \leq 26$.

For $\delta \leq 40$ lattice rules of class $\mathcal{B}_3$ provide the most efficient lattice rules for the indicated δ except in the case of $\delta = 11$. (One can see this in Table 1 for $\delta \leq 26$.) Many lattices of class $\mathcal{B}_2$ provide lattice rules that are as good as those of class $\mathcal{B}_3$. For more details about each class, and the data represented in Figure 1, we refer to the Appendix.

The highest rho-index for a 5-dimensional lattice rule known to us is $\rho = 1600/2343 \approx 0.68289$ and corresponds to the rule of $\delta = 20$ listed in Table 1.

Figure 1: The rho-index of 5-dimensional lattice rules



Optimal circulant rules (○), $Q_2(5, \delta)$ (+) and rules by Semenova [7] (×)

4.2 Sequences of lattice rules

We observed a regularity in the generators $(b_0, b_1, b_2, b_3, b_4)$ of the classes $\mathcal{B}_i$ which lead us to the characterization of families of generators, parametrized by degree. The best families we found in 5D are in class $\mathcal{B}_2$ and we denote them by $Q_2(5, 6k + r)$. (Here, the subscript 2 refers to the class, 5 is the dimension and $6k + r$ indicates the degree.) These families are represented in Table 2. In the upper part of that table we give a generator of $\Lambda^\perp$. In the lower part we provide the components of a circulant integer matrix $\tilde{A}_2 = (\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4)$ which is a scaled version of A_2, a circulant generator matrix of Λ. The following relation exists between $\tilde{A}_2$ and A_2:

$$\tilde{A}_2 = A_2/N_2,$$

where N_2 is the number of points of $Q_2(5, 6k+r)$. In Theorem 4.1 we will prove that for all $r = 0, 1, \ldots, 5$ and k as indicated in Table 2 the five-dimensional circulant lattice rules $Q_2(5, 6k + r)$ specified in Table 2 have degree $\delta = 6k + r$. We remark that the lattice rules generated by our families are not always equal to the optimal rules of class $\mathcal{B}_2$. The only other family of lattice rules in 5-dimensions we know are constructed by Semenova [7]. Our rules use about half the number of points. In Figure 1 the rho-index of the rules $Q_2(5, 6k + r)$ is marked by a +; those of [7] are marked by a ×.

The rest of this section is devoted to establishing Theorem 4.1. This theorem and its proof are very similar to the ones given by J. Lyness and T. Sørevik for the 4-dimensional case, in [4].

Table 1: Generators of best known 5D circulant dual lattices

δ	N	First row of circulant generator of $\Lambda^{\perp}$				
4	22	0	−1	1	1	1
5	71	−1	−1	1	0	2
6	124	0	−1	0	2	3
7	363	0	−2	1	3	1
8	484	−1	−1	1	4	1
9	1023	−1	−2	3	3	0
10	1322	−1	−3	3	2	1
11	2913	−4	1	2	3	1
12	3844	−2	−2	1	6	1
13	6025	−1	−3	2	6	1
14	7724	−1	−4	2	6	1
15	11275	−1	−4	3	6	1
16	14764	−1	−5	4	4	2
17	20801	−1	−2	2	5	7
18	26164	−2	−9	0	4	3
19	35243	−1	−2	2	6	8
20	39050	−1	−4	4	5	6
21	58575	−1	−2	2	7	9
22	74292	−2	−3	2	11	4
23	95325	−1	−3	2	7	10
24	102798	−1	−2	3	8	10
25	142307	−1	−3	2	8	11
26	155500	−1	−2	3	9	11

Theorem 4.1. *The circulant lattice rules $Q_2(5, 6k+r)$ specified in Table 2 have enhanced degree $\delta = 6k + r$.*

We shall proof this theorem by showing that for each of the dual lattices under consideration each nonzero element $\mathbf{h}$ satisfies $\|\mathbf{h}\|_1 \geq \delta$. The proof falls into two parts. In Lemma 4.2, we show that if $\|\lambda\|_1 \geq 6$, then the corresponding $\|\mathbf{h}\|_1 \geq 6k + r$. In Lemma 4.3 we simply record the result of computing $\|\mathbf{h}\|_1$ for the remaining elements of $\Lambda^{\perp}$.

We first repeat a lemma from [4] that will help us to prove Lemma 4.2.

Lemma 4.1. *Let A be the generator matrix of an integration lattice Λ, and let $B = (A^T)^{-1}$. Let $\mathbf{h} = \lambda B$ with $\lambda \in \Lambda_0^s$. Then*

$$\|\mathbf{h}\|_1 \geq \|\lambda\|_1 / \|A\|_1.$$

Lemma 4.2. *For all k, r, A and $\delta = 6k + r$ as specified in Table 2, If $\lambda \in \Lambda_0^s$ and $\|\lambda\|_1 \geq 6$, then $\|\lambda\|_1 / \|A_2\|_1 \geq \delta$.*

Table 2: Specification of circulant lattice rules $Q_2(5, 6k + r)$

r	$b_2(5, 6k + r)$	δ	k
0	$(-k, k, k, 2k, k)$	$6k$	≥ 1
1	$(-k, k, k + 1, 2k + 1, k - 1)$	$6k + 1$	≥ 3
2	$(-k - 1, k, k + 1, 2k, k)$	$6k + 2$	≥ 1
3	$(-k, k, k, 2k + 2, k + 1)$	$6k + 3$	≥ 1
4	$(-k - 1, k, k + 1, 2k + 1, k + 1)$	$6k + 4$	≥ 0
5	$(-k, k + 1, k, 2k + 3, k + 1)$	$6k + 5$	≥ 0

r			N_2
0	$-\tilde{a}_0$	$33k^4$	$124k^5$
	$\tilde{a}_1$	$23k^4$	
	$-\tilde{a}_2$	k^4	
	$\tilde{a}_3$	$27k^4$	
	$\tilde{a}_4$	$15k^4$	
1	$-\tilde{a}_0$	$33k^4 + 11k^3 - 10k^2 - 7k - 3$	$124k^5 + 55k^4 + 10k^3 + 25k^2 + 50k + 11$
	$\tilde{a}_1$	$23k^4 + 35k^3 + 2k^2 - 5k + 1$	
	$-\tilde{a}_2$	$k^4 + 23k^3 + 23k^2 - 4k - 4$	
	$\tilde{a}_3$	$27k^4 + 16k^3 + 25k^2 + 18k + 5$	
	$\tilde{a}_4$	$15k^4 - 11k^3 - 13k^2 - 18k - 2$	
2	$-\tilde{a}_0$	$33k^4 + 40k^3 + 22k^2 + 5k - 1$	$124k^5 + 160k^4 + 120k^3 + 60k^2 + 20k$
	$\tilde{a}_1$	$23k^4 + 36k^3 + 14k^2 + 3k + 1$	
	$-\tilde{a}_2$	$k^4 - 8k^3 - 14k^2 - 11k - 1$	
	$\tilde{a}_3$	$27k^4 + 8k^3 - 2k^2 - k + 1$	
	$\tilde{a}_4$	$15k^4 + 28k^3 + 26k^2 + 7k + 1$	
3	$-\tilde{a}_0$	$33k^4 + 68k^3 + 59k^2 + 24k + 2$	$124k^5 + 345k^4 + 465k^3 + 355k^2 + 155k + 33$
	$\tilde{a}_1$	$23k^4 + 46k^3 + 47k^2 + 23k + 4$	
	$-\tilde{a}_2$	$k^4 + 24k^3 + 40k^2 + 29k + 8$	
	$\tilde{a}_3$	$27k^4 + 77k^3 + 92k^2 + 57k + 16$	
	$\tilde{a}_4$	$15k^4 + 32k^3 + 29k^2 + 10k + 1$	
4	$-\tilde{a}_0$	$33k^4 + 85k^3 + 87k^2 + 41k + 7$	$124k^5 + 370k^4 + 470k^3 + 330k^2 + 130k + 22$
	$\tilde{a}_1$	$23k^4 + 59k^3 + 55k^2 + 21k + 3$	
	$-\tilde{a}_2$	$k^4 + k^3 - 11k^2 - 15k - 5$	
	$\tilde{a}_3$	$27k^4 + 49k^3 + 29k^2 + 7k + 1$	
	$\tilde{a}_4$	$15k^4 + 55k^3 + 71k^2 + 41k + 9$	
5	$-\tilde{a}_0$	$33k^4 + 115k^3 + 164k^2 + 116k + 29$	$124k^5 + 595k^4 + 1310k^3 + 1570k^2 + 995k + 275$
	$\tilde{a}_1$	$23k^4 + 79k^3 + 99k^2 + 43k + 1$	
	$-\tilde{a}_2$	$k^4 + 27k^3 + 52k^2 + 43k + 19$	
	$\tilde{a}_3$	$27k^4 + 136k^3 + 268k^2 + 241k + 86$	
	$\tilde{a}_4$	$15k^4 + 37k^3 + 39k^2 + 30k + 16$	

Proof. The elements of the circulant matrix $\tilde{A}_2$ are given in the lower part of Table 2. The elements of A_2 are $\tilde{A}_2/N_2$. The right-hand side of the inequality in Lemma 4.1 is

$$\|\lambda\|_1 N_2/D_2$$

where $D_2 = -\tilde{a}_0 + \tilde{a}_1 - \tilde{a}_2 + \tilde{a}_3 + \tilde{a}_4$ for all k indicated in Table 2. We can calculate that for each $r = 0, 1, \ldots, 5$ in turn,

$$(6k + r)D_2(r) < 6N_2(r).$$

It follows that if $\|\lambda\|_1 \geq 6$ then

$$\|\lambda\|_1 N_2/D_2 \geq 6N_2/D_2 > 6k + r$$

which completes this proof. □

Lemma 4.3. *Let B_2 be the skew-circulant matrix determined by the vector $b_2(5, 6k + r)$ for all k and $r = 0, 1, \ldots, 5$ as specified in Table 2. Then for all $\lambda \in \Lambda_0^s$ such that $\|\lambda\|_1 \leq 5$ the points $\mathbf{h} = \lambda B_2$ satisfy $\|\mathbf{h}\|_1 \geq \delta$.*

Proof. A computer program was constructed to carry out this calculation for all assignments of λ and all six values of r. □

Note that Lyness and Sørevik [4] did not give a direct proof of a similar lemma in 4 dimensions. The one by one verification seems to be the only possibility.

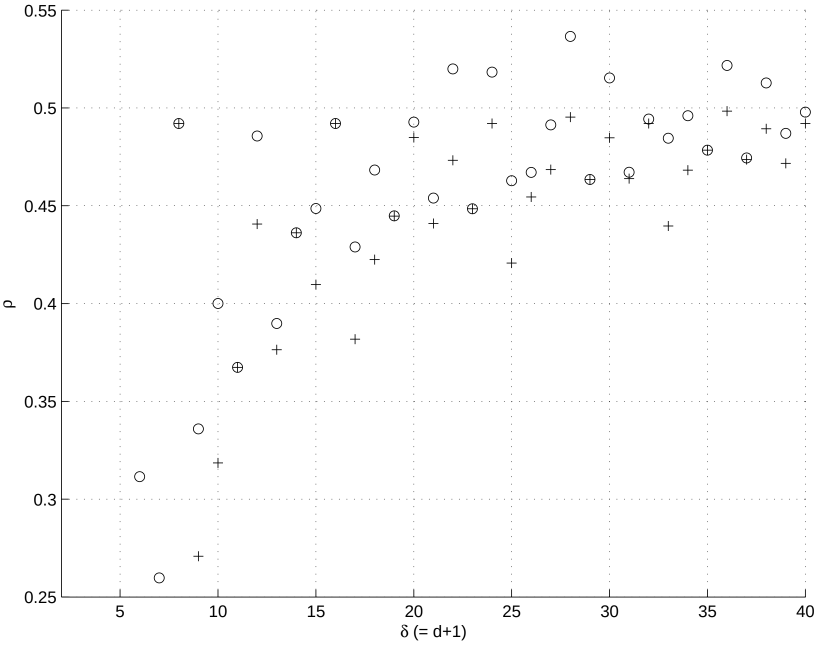
5 Six-dimensional results

5.1 Computational results

Some of our six-dimensional results are presented in Figure 2. It shows the rho-index of the optimal (skew-)circulant lattice rules of degree $\delta \leq 40$ that we obtained, marked by the symbol $\circ$. The details for each of the classes are given in the Appendix. In Table 3 we list the first row of the (skew-)circulant generator matrices for the best lattices we obtained for $\delta \leq 20$.

The highest rho-index for a 6-dimensional lattice rule known to us is $\rho = 38416/71595 \approx 0.5366$ that corresponds to a rule of $\delta = 28$. We added this exceptional rule to Table 3.

Figure 2: The rho-index of six-dimensional lattice rules



Optimal (skew-)circulant rules (○) and rules $Q_3'(6,\delta)$ (+).

Table 3: Generators of best known 6D (skew-)circulant dual lattices

δ	type	N	First row of circulant generator of $\Lambda^\perp$					
4	$\mathcal{SC}irc$	26	0	1	0	1	1	1
5	$\mathcal{C}irc$	117	0	-1	0	1	1	2
6	$\mathcal{SC}irc$	208	1	0	-2	1	1	1
7	$\mathcal{SC}irc$	629	1	0	-1	1	2	2
8	$\mathcal{SC}irc$	740	0	0	-1	1	3	3
9	$\mathcal{SC}irc$	2197	1	-1	1	2	3	1
10	$\mathcal{C}irc$	3472	0	-1	1	3	2	3
11	$\mathcal{SC}irc$	6697	1	1	-2	2	3	2
12	$\mathcal{C}irc$	8540	0	-1	1	5	3	2
13	$\mathcal{C}irc$	17199	-1	-2	1	4	2	3
14	$\mathcal{SC}irc$	23976	0	1	-1	1	5	6
15	$\mathcal{SC}irc$	35269	1	2	4	1	5	2
16	$\mathcal{SC}irc$	47360	0	0	-2	2	6	6
17	$\mathcal{SC}irc$	78165	0	3	0	4	6	4
18	$\mathcal{SC}irc$	100880	3	2	5	1	4	3
19	$\mathcal{SC}irc$	146900	3	4	-4	2	1	5
20	$\mathcal{SC}irc$	180388	1	4	8	3	1	3
28	$\mathcal{C}irc$	1247344	-3	-4	5	4	9	3

5.2 Sequences of lattice rules

A certain regularity is also noticed in the six-dimensional results and again this lead us to the characterization of familes of rules. In Table 4 we represent some families of lattice rules of class $\mathcal{B}'_3$. The same notation as in the five-dimensional case is used. In Figure 2 one can compare the rho-index of these families, marked by +, with the optimal circulant or skew-circulant rules. For a given δ, the differences between the rho-index of a lattice rule of the family with the optimal rho-index is smaller than in 5 dimensions. We see that even for many δ the optimal rho-index is equal to the rho-index of the discovered families. This means that the discovered families generate optimal skew-circulant lattice rules for those δ.

We omit the details but in exactly the same way as in 5 dimensions, it is proven that the families given in Table 4 have enhanced degree δ. Their number of points is listed in Table 5.

Table 4: Specification of skew-circulant lattice rules $Q'_3(6, 8k + r)$

r	$b'_3(6, 8k + r)$	δ	k
0	$(k, 2k, -2k, k, k, k)$	$8k$	≥ 1
1	$(k, 2k + 1, -2k, k, k - 1, k + 1)$	$8k + 1$	≥ 0
2	$(k, 2k + 1, -2k - 1, k, k + 1, k - 1)$	$8k + 2$	≥ 0
3	$(k, 2k, -2k - 1, k + 1, k + 1, k)$	$8k + 3$	≥ 0
4	$(k, 2k + 1, -2k - 1, k, k + 1, k + 1)$	$8k + 4$	≥ 0
5	$(k + 1, 2k + 2, -2k - 1, k, k, k + 1)$	$8k + 5$	≥ 0
6	$(k + 1, 2k + 2, -2k - 1, k, k, k + 2)$	$8k + 6$	≥ 0
7	$(k + 1, 2k + 2, -2k - 2, k + 1, k + 1, k)$	$8k + 7$	≥ 1

Table 5: Number of points in lattice rules $Q'_3(6, 8k + r)$

r	N
0	$740k^6$
1	$740k^6 + 840k^5 + 645k^4 + 330k^3 + 135k^2 + 30k + 5$
2	$740k^6 + 1152k^5 + 1236k^4 + 784k^3 + 336k^2 + 96k + 16$
3	$740k^6 + 1644k^5 + 2025k^4 + 1452k^3 + 648k^2 + 168k + 20$
4	$740k^6 + 2088k^5 + 2856k^4 + 2304k^3 + 1104k^2 + 288k + 32$
5	$740k^6 + 2796k^5 + 4905k^4 + 5008k^3 + 3102k^2 + 1092k + 169$
6	$740k^6 + 3264k^5 + 6408k^4 + 7064k^3 + 4584k^2 + 1656k + 260$
7	$740k^6 + 3972k^5 + 9201k^4 + 11760k^3 + 8739k^2 + 3576k + 629$

6 Conclusions

Our research continues and develops previous work by Cools and Lyness [1] and Lyness and Sørevik [4]. The previous work focuses on 3 and 4 dimensions; ours deals with 5 and 6 dimensions. Besides computational results, we have also obtained some theoretical results.

We found families of lattices that provide efficient rules of an arbitrary trigonometric degree in 5 and 6 dimensions. These are the best known families at the moment but it is not known whether they are optimal.

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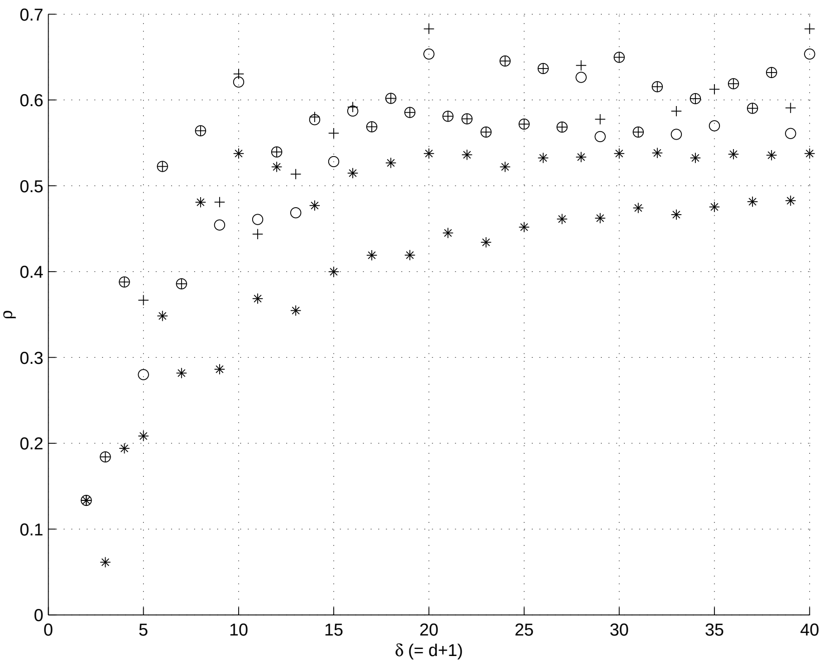
Appendix: More details of the computational results

A Five-dimensional results

In Figure 3 the rho-index of the optimal circulant lattice rules of degree $\delta \leq 40$ is represented for the three different classes (sign-patterns) given in Definition 3.1. In the first columns of Table 6 we list the number of points required by the optimal rules of each class.

In Table 6 we also give the number of points of the rules $Q_2(5,\delta)$ and of those constructed by Semenova [7].

Figure 3: The rho-index of 5-dimensional optimal circulant lattice rules



Optimal lattice rules of class $\mathcal{B}_1(*)$, $\mathcal{B}_2(\circ)$, $\mathcal{B}_3(+)$.

The optimal lattice rules of class $\mathcal{B}_1$ appear to be less efficient than the optimal rules of the other classes. This might be due to the fact that all generators of a lattice of class $\mathcal{B}_1$ are on the same facet pair. The corresponding lattice does not necessarily belong to the $K(s,\delta)$ set contrary to the rules of classes $\mathcal{B}_2$ or $\mathcal{B}_3$.

Table 6: Number of points of optimal 5D (skew-)circulant lattice rules

δ	$\mathcal{B}_1$	$\mathcal{B}_2$	$\mathcal{B}_3$	$Q_2(5,\delta)$	ref. [7]
2	2	2	2		
3	33	11	11		
4	44	22	22	22	
5	125	93	71	275	156
6	186	124	124	124	504
7	497	363	363		624
8	568	484	484	484	945
9	1719	1083	1023	1477	2180
10	1550	1342	1322	1446	4901
11	3641	2913	3025	4869	5208
12	3972	3844	3844	3968	7051
13	8723	6603	6025		12796
14	9394	7768	7724	7768	23550
15	15825	11979	11275	14971	23712
16	16976	14884	14764	15250	30001
17	28237	20801	20801	32513	47556
18	29898	26164	26164	30132	77547
19	49229	35243	35243	35243	76440
20	49600	40804	39050	46932	92571
21	76461	58575	58575	74325	134684
22	80102	74292	74292	76174	202580
23	123533	95325	95325	131087	197904
24	127104	102798	102798	126976	232513
25	180125	142307	142307	142307	318916
26	185926	155500	155500	176656	452769
27	259227	210349	210349	251389	440664
28	268828	228966	224014	257598	506395
29	369779	306679	295929	392511	665340
30	376650	311646	311646	387500	904506
31	503161	424011	424011	424011	879168
32	519392	454344	454344	504100	993441
33	699303	582383	555577	670933	1263236
34	710974	629486	629486	686422	1660295
35	920675	767993	714704	967625	1613592
36	938556	814088	814088	964224	1799371
37	1200317	978875	978875	1038875	2229916
38	1232758	1044808	1044808	1199784	2852592
39	1557699	1340289	1272831	1525527	2773680
40	1587200	1305728	1249600	1557946	3060241

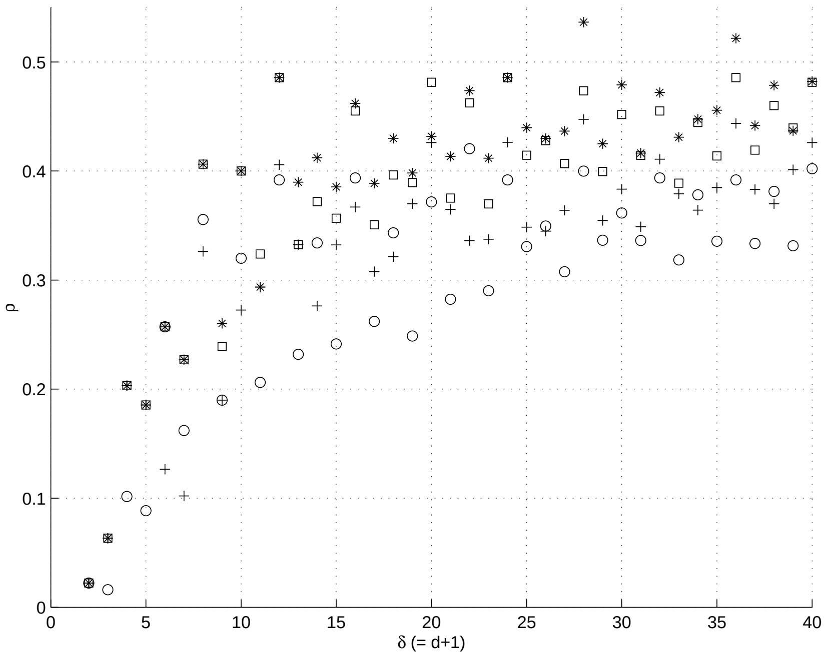
B Six-dimensional results

Our six-dimensional results are presented in Figures 4 and 5. They show the rho-index of the optimal circulant, resp. skew-circulant lattice rules of degree $\delta \leq 40$. In Tables 7 and 8 we list the underlying data. Observe in Figures 4 and 5 that all rules with $\rho > 0.5$ have $\delta > 20$.

As in the five-dimensional case, we also remark that the best results are given by those generator matrices belonging to the $K(s, \delta)$-set.

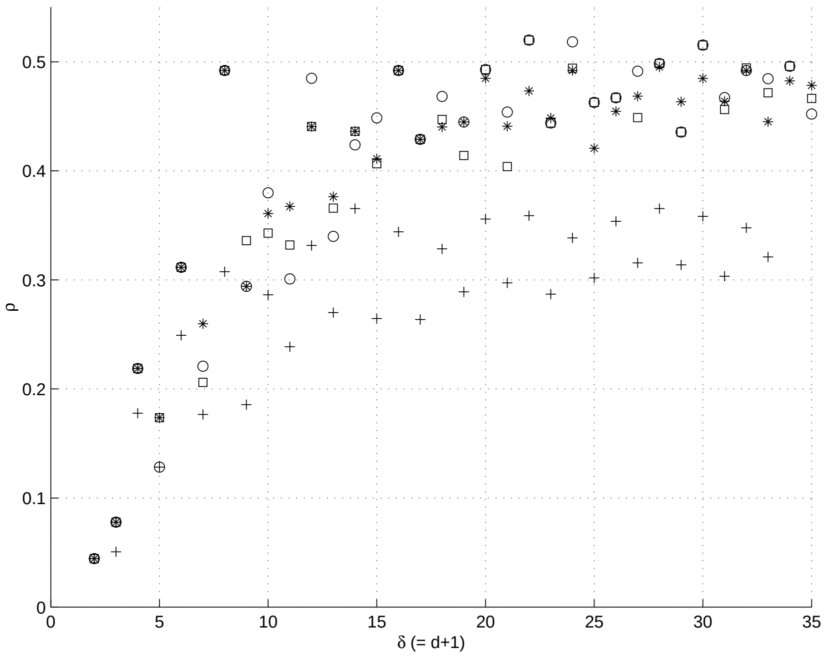
In Table 8 one can compare the number of points of the rules $Q_3'(6, 8k + r)$ with the optimal results.

Figure 4: The rho-index of six-dimensional optimal circulant lattice rules



Optimal lattice rules of class $\mathcal{B}_1(\circ)$, $\mathcal{B}_2(\Box)$, $\mathcal{B}_3(*)$, $\mathcal{B}_4(+)$.

Figure 5: The rho-index of six-dimensional optimal skew-circulant lattice rules



Optimal lattice rules of class $\mathcal{B}'_1(\circ)$, $\mathcal{B}'_2(\Box)$, $\mathcal{B}'_3(*)$, $\mathcal{B}'_4(+)$.

Table 7: Number of points of optimal 6D circulant lattice rules

δ	$\mathcal{B}_1$	$\mathcal{B}_2$	$\mathcal{B}_3$	$\mathcal{B}_4$
2	4	4	4	4
3	63	16	16	16
4	56	28	28	28
5	245	117	117	117
6	252	252	252	512
7	1008	720	720	1600
8	1024	896	896	1116
9	3888	3087	2835	3888
10	4340	3472	3472	5096
11	11935	7595	8379	8379
12	10584	8540	8540	10220
13	28899	20160	17199	20160
14	31304	28120	25376	37856
15	65520	44352	41040	47619
16	59200	51200	50440	63488
17	127925	95589	86247	108927
18	137592	119168	109872	146952
19	262675	167827	164065	176605
20	239120	184680	205884	208620
21	421785	317520	288145	326585
22	374528	340480	332424	468504
23	708400	555985	499408	609280
24	677376	546560	546560	622592
25	1025325	818045	771043	973063
26	1227564	1002848	998244	1244160
27	1749033	1322685	1232595	1478295
28	1673560	1413360	1247344	1495984
29	2455024	2068045	1944000	2329600
30	2800980	2240784	2113664	2640512
31	3665347	2974400	2959632	3532464
32	3788800	3276800	3159520	3629920
33	5632011	4612383	4161969	4731831
34	5673852	4826808	4794552	5892768
35	7606235	6169527	5602473	6633603
36	7715736	6225660	5794740	6815340
37	10684453	8502272	8069072	9301149
38	10967180	9091264	8737400	11302720
39	14742000	11121075	11187631	12179475
40	14148400	11819520	11797632	13351680

Table 8: Number of points of optimal 6D skew-circulant lattice rules

δ	$\mathcal{B}'_1$	$\mathcal{B}'_2$	$\mathcal{B}'_3$	$\mathcal{B}'_4$	$Q'_3(6,\delta)$
2	2	2	2	2	16
3	13	13	13	20	20
4	26	26	26	32	32
5	169	125	125	169	169
6	208	208	208	260	260
7	740	793	629	925	
8	740	740	740	1184	740
9	2509	2197	2509	3977	2725
10	3656	4050	3848	4850	4360
11	8177	7412	6697	10309	6697
12	8552	9412	9412	12506	9412
13	19721	18325	17812	24820	17812
14	24674	23976	23976	28616	23976
15	35269	38909	38480	59777	38617
16	47360	47360	47360	67712	47360
17	78165	78165	78165	127165	87805
18	100880	105650	107300	143828	111824
19	146900	157781	146900	226069	146932
20	180388	180388	183328	249800	183328
21	262421	294865	270137	400673	270137
22	302882	302882	332756	438746	332756
23	463165	463165	458497	716560	458497
24	512074	537192	539460	784160	539460
25	732737	732737	806045	1123357	806045
26	918632	918632	944008	1213082	944008
27	1095120	1198800	1148537	1705280	1148537
28	1342900	1342900	1351220	1831424	1351220
29	1896453	1896453	1782772	2632513	1782772
30	1964672	1964672	2088872	2826356	2088872
31	2638805	2701817	2657465	4062845	2657465
32	3031040	3016832	3031040	4289024	3031040
33	3701945	3803225	4029965	5585705	4079725
34	4325828	4325828	4446868	5944420	4583056
35	5646992	5474417	5336884	7864177	5336884
36	6100964	6100964	6066592	8338850	6066592
37	7510725	7510725	7524505	11212097	7524505
38	8155400	8155400	8528000	11749106	8546148
39	10035776	10035776	10268869	15032033	10361221
40	11427156	11544832	11427156	15841906	11562500