Stability Criteria for Exact and Discrete Solutions of Neutral Multidelay-Integro-Differential Equations

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Keywords : : Asymptotic stability, multidelay-integro-differential equations, Runge-Kutta methods, linear multistep methods.
CR Subject Classification : 65R20, 45L05, 34K20, 34K40,
AMS(MOS) Classification : Primary : 65R20, Secondary : 45L05, 34K20, 34K40.

STABILITY CRITERIA FOR EXACT AND DISCRETE SOLUTIONS OF NEUTRAL MULTIDELAY-INTEGRO-DIFFERENTIAL EQUATIONS

CHENGJIAN ZHANG* AND STEFAN VANDEWALLE†

Abstract. This paper deals with the asymptotic stability of exact and discrete solutions of neutral multidelay-integro-differential equations. First, sufficient conditions are derived that guarantee the asymptotic stability of the continuous solutions. Then, adaptations of classical Runge-Kutta and linear multistep methods are suggested for solving these systems. Stability criteria are constructed for the asymptotic stability of these numerical methods and compared to the stability criteria derived for the continuous problem. We find that, under suitable conditions, these two classes of numerical methods retain the stability of the continuous systems, at least, for problems with commensurate delays. Finally, some numerical examples are given that illustrate the theoretical results.

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1. Introduction. Consider the following complex p-dimensional system of neutral multidelay-integro-differential equations (NMIDEs) with constant delays $\tau_q > 0$,

$$\begin{cases} \dfrac{d}{dt}[y(t) - \displaystyle\sum_{q=1}^{d} N_q y(t-\tau_q)] = Ly(t) + \displaystyle\sum_{q=1}^{d} M_q y(t-\tau_q) + \displaystyle\sum_{q=1}^{d} Q_q \int_{t-\tau_q}^{t} y(\theta)d\theta, \ t \geq t_0 \\ y(t) = \varphi(t), \quad t \in [t_0 - \hat{\tau}, t_0], \end{cases} \tag{1.1}$$

with matrices $L, M_q, N_q, Q_q \in \mathbb{C}^{p\times p}$ and $\hat{\tau} = \max_{1\leq q\leq d}\{\tau_q\}$. Function $\varphi(t)$ is a given p-dimensional vector-valued function, and $y(t) \in \mathbb{C}^p$ is unknown for $t > t_0$. Such equations, or special cases of these equations, arise in practical applications, e.g., in visco-elasticity, control theory, epidemiology, and population dynamics (cf. [8, 14]).

In the past, many researchers have studied the stability of special cases of (1.1). These studies usually concentrated on non-distributed delay equations, i.e., the case $Q_q = 0$, often with also $N_q = 0$, for $q = 1, 2, \ldots, d$. Their results have been presented, among others, in the following papers [4, 5, 6, 10, 11, 12, 18, 19, 20]. As one of the early papers, we wish to mention especially the paper by Hu and Mitsui [11], who considered neutral non-distributed delay problems, and studied the asymptotic stability both of exact and discrete solutions. More recently, we have noticed a growing interest in the analysis of delay-integro-differential equations (DIDEs). Baker & Ford [1] studied the asymptotic stability of a class of linear multistep (LM) methods for scalar linear DIDEs; Koto [15] dealt with the linear stability of Runge-Kutta (RK) methods for systems of DIDEs; Huang & Vandewalle [13] gave sufficient and necessary stability conditions for exact and discrete solutions of linear scalar DIDEs, and Luzyanina, Engelborghs & Roose [17] developed computational procedures for determining the stability of DIDEs. Baker & Tang [2] and Zhang & Vandewalle [21, 22] considered nonlinear systems of DIDEs.

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No results have been found in the literature that directly deal with systems as general as (1.1). Indirectly, the linear stability theory for non-distributed neutral multidelay systems of [12, 19] could be applied to (1.1). Under the assumption that $y(t)$ is continuous for $t \geq 0$, a transformation can be introduced as follows:

$$x_q(t) = \int_{t-\tau_q}^{t} y(\theta)d\theta, \quad q = 1, 2, \ldots, d. \tag{1.2}$$

This transformation converts (1.1) into a non-distributed system of the following form:

$$\begin{cases} \dfrac{d}{dt}[Y(t) - \displaystyle\sum_{q=1}^{d} \breve{N}_q Y(t-\tau_q)] = \breve{L}Y(t) + \displaystyle\sum_{q=1}^{d} \breve{M}_q Y(t-\tau_q), \ t \geq t_0, \\ Y(t) = [\varphi(t)^T, \varphi_1(t)^T, \ldots, \varphi_d(t)^T]^T, \quad t \in [t_0 - \hat{\tau}, t_0]. \end{cases} \tag{1.3}$$

Here, we have matrices $Y(t) \in \mathbb{C}^{(d+1)p}$, $\breve{L} \in \mathbb{C}^{(d+1)p\times(d+1)p}$, $\breve{M}_q \in \mathbb{C}^{(d+1)p\times(d+1)p}$, and $\breve{N}_q \in \mathbb{C}^{(d+1)p\times(d+1)p}$ given by

$$Y(t) = \begin{pmatrix} y(t) \\ x_1(t) \\ \vdots \\ x_d(t) \end{pmatrix}, \qquad \breve{L} = \begin{pmatrix} L & Q_1 & \ldots & Q_d \\ I_p & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ I_p & 0 & \ldots & 0 \end{pmatrix},$$

$$\breve{M}_q = \begin{pmatrix} M_q & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \\ I_p & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix}, \qquad \breve{N}_q = \begin{pmatrix} N_q & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix},$$

and $\varphi_q(t) = \int_{t-\tau_q}^{t} \varphi(\theta)d\theta$ for $t_0 - \hat{\tau} \leq t \leq t_0$. The $p \times p$ identity matrix I_p in block matrix $\breve{M}_q$ is located at (block-)position $(q+1, 1)$. The symbol "0" is used to denote a zero matrix or vector of the appropriate size. With the linear stability theory of [12, 19] some *delay-independent stability* results can be obtained for (1.3). This approach is indirect, however, and comes at the price of changing the inherent structure of the system. Here, we prefer to take the more direct route of studying (1.1) immediately. This approach will also enable us to obtain *delay-dependent stability* results.

The paper is structured as follows. In §2 we give asymptotic stability criteria for exact solutions of system (1.1). Some examples are given to illustrate the applicability of the criteria. In §3 we suggest an adaptation of the classical Linear Multistep and Runge-Kutta methods for solving (1.1). These adaptations are based on the use of a linear compound quadrature rule to discretize the integrals in the system. In §4 and §5, we deal with the asymptotic stability of Runge-Kutta and Linear Multistep methods, respectively. The stability criteria presented there can be considered discrete versions of the stability criteria derived in §2. The theoretical results are illustrated by some numerical examples.

2. Stability criteria for the exact solution. Before starting the stability analysis, we introduce a complex function

$$\eta(z) = \begin{cases} \frac{1-z}{\ln z}, & z \in \mathbb{C}\backslash\{0,1\}, \\ 0, & z = 0, \\ -1, & z = 1, \end{cases} \tag{2.1}$$

which will appear in the characteristic equation of (1.1). It possesses the properties given in the lemma below. The notation $\mathbb{R}_0^-$ stands for the set $\{x \in \mathbb{R} : x \leq 0\}$.

LEMMA 2.1. *Function $\eta(z)$ is continuous in $\mathbb{C}$; it is analytic in $\mathbb{C}\backslash\mathbb{R}_0^-$, and satisfies $|\eta(z)| \leq 1$ for $|z| \leq 1$.*

Proof. The proof of continuity in $\mathbb{C}$ and analyticity in $\mathbb{C}\backslash\{\mathbb{R}_0^- \bigcup\{1\}\}$ is straightforward. To show analyticity in $\mathbb{C}\backslash\mathbb{R}_0^-$, we only need to verify that $\eta'(1)$ exists. This follows from l'Hospital's rule:

$$\eta'(1) = \lim_{\xi\to1} \frac{\frac{1-\xi}{\ln\xi} - (-1)}{\xi - 1} = -\frac{1}{2}.$$

It remains to prove the bound on $|\eta(z)|$. Since $\eta(z)$ is continuous on the unit disc $\mathbb{D}$ and analytic on the open set $\mathbb{D}\backslash\mathbb{B}$ with $\mathbb{B} = [-1,0]\bigcup\{z : |z| = 1\}$, we can apply the Maximum Modulus Theorem. This theorem implies

$$\max_{z\in\mathbb{D}}\{|\eta(z)|\} = \max_{z\in\mathbb{B}}\{|\eta(z)|\}.$$

For $-1 < z < 0$, setting $z = -x$, we find

$$|\eta(z)| = \left|\frac{1+x}{\ln(-x)}\right| = \frac{1+x}{\sqrt{\ln^2 x + \pi^2}} \leq \frac{2}{\pi} < 1.$$

For $z \in \{z : |z| = 1\}\backslash\{1\}$, setting $z = \exp(i\theta)$ with $\theta \in [-\pi,\pi]\backslash\{0\}$, we have

$$|\eta(z)| = \left|\frac{1-\exp(i\theta)}{i\theta}\right| = \frac{\sqrt{(1-\cos\theta)^2 + \sin^2\theta}}{|\theta|} = \frac{2|\sin\frac{\theta}{2}|}{|\theta|} \leq \frac{2|\frac{\theta}{2}|}{|\theta|} = 1.$$

Also, by (2.1) we have $|\eta(0)| = 0$ and $|\eta(1)| = 1$. Hence, $\max_{z\in\mathbb{B}}\{|\eta(z)|\} = 1$. □

Specializing Corollary 3.1 in Hale & Verduyn [8, Ch.9] to the case of system (1.1) immediately yields the following lemma.

LEMMA 2.2. *Assume*

$$\sup\{\Re(\lambda) : \; P(\lambda) = 0\} < 0, \tag{2.2}$$

where

$$P(\lambda) := \det[\lambda I_p - \sum_{q=1}^{d} e^{-\lambda\tau_q} N_q - L - \sum_{q=1}^{d} e^{-\lambda\tau_q} M_q + \sum_{q=1}^{d} \eta(e^{-\lambda\tau_q})\tau_q Q_q] = 0 \tag{2.3}$$

is the characteristic equation of (1.1)*. Then, system* (1.1) *is asymptotically stable.*

In the following, we denote the determinant, the $l-$th eigenvalue and the spectrum of a square matrix $\mathcal{A}$ by $\det(\mathcal{A}), \lambda_l(\mathcal{A})$ and $\sigma(\mathcal{A})$, respectively, and introduce the sets of the real and imaginary parts $\Re[\sigma(\mathcal{A})] = \{\Re(\lambda) : \lambda \in \sigma(\mathcal{A})\}$, resp. $\Im[\sigma(\mathcal{A})] = \{\Im(\lambda) : \lambda \in \sigma(\mathcal{A})\}$, and the set $\mathbb{C}^- = \{z \in \mathbb{C} : \Re(z) < 0\}$.

THEOREM 2.3. *System* (1.1) *is asymptotically stable if, for* $q = 1, 2, \ldots, d$:

(a) $\det[I_p - \sum_{q=1}^{d} \xi_q N_q] \neq 0$ *for* $|\xi_q| \leq 1$

(b) $\Re[\sigma(G(\xi))] \subseteq \mathbb{C}^-$ *for* $\xi = (\xi_1, \xi_2, \ldots, \xi_d)^T$ *with* $|\xi_q| \leq 1$,

where $G(\xi) = (I_p - \sum_{q=1}^{d} \xi_q N_q)^{-1}(L + \sum_{q=1}^{d} \xi_q M_q - \sum_{q=1}^{d} \eta(\xi_q)\tau_q Q_q)$.

Proof. The proof is based on the ideas in paper [19], in which a neutral multidelay system without distributed delays is considered. Here, the proof in that paper is extended towards distributed delays.

When $|\xi_q| \leq 1$ for $q = 1, 2, \ldots, d$ and $\Re(\lambda) \geq 0$, conditions (a) and (b) lead to

$$\begin{aligned}\hat{P}(\lambda, \xi_1, \xi_2, \ldots, \xi_d) &:= \det[\lambda I_p - \sum_{q=1}^{d} \xi_q N_q - L - \sum_{q=1}^{d} \xi_q M_q + \sum_{q=1}^{d} \eta(\xi_q)\tau_q Q_q] \\ &= \det[I_p - \sum_{q=1}^{d} \xi_q N_q] \det[\lambda I_p - G(\xi)] \\ &= \det[I_p - \sum_{q=1}^{d} \xi_q N_q] \prod_{l=1}^{d} [\lambda - \lambda_l(G(\xi))] \neq 0.\end{aligned}$$

This implies $P(\lambda) = \hat{P}(\lambda, e^{-\lambda\tau_1}, e^{-\lambda\tau_2}, \ldots, e^{-\lambda\tau_d}) \neq 0$ for $\Re(\lambda) \geq 0$. This means that the set $\{\lambda : P(\lambda) = 0\} \subseteq \mathbb{C}^-$. Next, we show that the strict inequality in (2.2) holds. Otherwise, there is a sequence $\{\Lambda_n\} \subseteq \{\lambda \in \mathbb{C}^- : P(\lambda) = 0\}$ such that

$$\lim_{n\to\infty} \Re(\Lambda_n) = 0. \tag{2.4}$$

In view of continuity of the eigenvalues $\lambda_l(G(\xi))$, together with condition (a) and Lemma 2.1, we conclude that the functions $\mathcal{G}_l(\xi) := \Re[\lambda_l(G(\xi))]$ $(1 \leq l \leq p)$ are continuous on the compact set $\mathbb{E} := \{\xi \in \mathbb{C}^d : |\xi_q| \leq 1 \ (q = 1, 2, \ldots, d)\}$. Hence, they attain their respective maxima on $\mathbb{E}$. Thus, it follows from condition (b) that there exists a constant $\delta > 0$ such that

$$\max_{\xi\in\mathbb{E}}\{\mathcal{G}_l(\xi)\} \leq -\delta \quad \text{for all} \quad l.$$

Also, because the functions $\mathcal{G}_l(\xi)$ are continuous except on the set

$$\mathbb{J} := \{\xi \in \mathbb{C}^d : \ \det[I_p - \sum_{q=1}^{d} \xi_q N_q] = 0\}$$

and $\mathbb{J}$ is closed, there exist positive constants κ and δ_κ small enough such that

$$\det[I_p - \sum_{q=1}^{d} \xi_q N_q] \neq 0 \quad \text{on} \quad \mathbb{E}_\kappa := \{\xi \in \mathbb{C}^d : \ |\xi_q| \leq 1 + \kappa \ \ (q = 1, 2, \ldots, d)\} \tag{2.5}$$

and

$$\max_{\xi\in\mathbb{E}_\kappa}\{\mathcal{G}_l(\xi)\} \leq -\delta_\kappa \quad \text{for all} \quad l. \tag{2.6}$$

Moreover, (2.4) implies that there is a positive integer n_0 such that when $n > n_0$,

$$|e^{-\Lambda_n \tau_q}| \leq 1+\kappa, \quad q = 1,2,\dots,d,$$

where Λ_n satisfies $P(\Lambda_n) = 0$. This, together with (2.5) and (2.6), infers that

$$\det[I_p - \sum_{q=1}^{d} e^{-\Lambda_n \tau_q} N_q] \neq 0 \ \text{ for } \ n > n_0 \tag{2.7}$$

and

$$\Re[\lambda_l(G(e^{-\Lambda_n \tau_q}))] < -\delta_\kappa \ \text{ for all } \ l \ \text{ and } \ n > n_0. \tag{2.8}$$

Hence, by (2.7) and the equalities

$$P(\Lambda_n) = \det[I_p - \sum_{q=1}^{d} e^{-\Lambda_n \tau_q} N_q] \prod_{l=1}^{p} \{\Lambda_n - \lambda_l[G(e^{-\Lambda_n \tau_q})]\} = 0, \quad n > n_0,$$

we have

$$\prod_{l=1}^{p} \{\Lambda_n - \lambda_l[G(e^{-\Lambda_n \tau_q})]\} = 0 \ \text{ for } \ n > n_0.$$

Combining this with (2.8), leads us to conclude that there is an l such that

$$\Re(\Lambda_n) = \Re\{\lambda_l[G(e^{-\Lambda_n \tau_q})]\} \leq -\delta_\kappa \ \text{ for all } \ n > n_0.$$

This contradicts (2.4), and, hence, completes the proof. □

In order to be able to derive our next stability criteria, we need to introduce some notations and propositions. Throughout this paper, $\rho(\cdot)$ will denote the spectral radius of a matrix; $\mu(\cdot)$ is the logarithmic norm subject to a given matrix norm $\|\cdot\|$; matrix $|M|$ satisfies $|M| = (|m_{ij}|)$ when $M = (m_{ij})$, and $M \leq \tilde{M}$ means $m_{ij} \leq \tilde{m}_{ij}$ for $M = (m_{ij})$ and $\tilde{M} = (\tilde{m}_{ij}) \in \mathbb{C}^{n\times n}$.

Proposition 2.4. *(cf. [16]) Assume matrices $\mathcal{G}, \mathcal{H} \in \mathbb{C}^{n\times n}$. Then,*

(1) $\rho(\mathcal{G}) \leq \rho(\mathcal{H})$ whenever $|\mathcal{G}| \leq \mathcal{H}$;

(2) matrix $I_n - \mathcal{G}$ is invertible and $(I_n - \mathcal{G})^{-1} = I_n + \sum_{i=1}^{\infty} \mathcal{G}^i$ whenever there exists a matrix norm $\|\cdot\|$ such that $\|\mathcal{G}\| < 1$.

Proposition 2.5. *(cf. [3]) Assume matrices $\mathcal{G}, \mathcal{H} \in \mathbb{C}^{n\times n}$. Then, for an any given matrix norm $\|\cdot\|$, the following inequalities hold:*

(I) $\Re(\lambda_l(\mathcal{G})) \leq \mu(\mathcal{G}) \leq \|\mathcal{G}\|$ for all l;

(II) $\mu(\mathcal{G}+\mathcal{H}) \leq \mu(\mathcal{G}) + \mu(\mathcal{H})$;

(III) $\mu(a\mathcal{G}) = a\mu(\mathcal{G}))$, for $a \geq 0$.

By statement (1) in Prop. 2.4 and by using classical spectral radius and norm properties, we can prove for any matrix norm $\|\cdot\|$ that the following inequalities hold:

$$\rho(\sum_{q=1}^{d} \xi_q N_q) \leq \rho(\sum_{q=1}^{d} |N_q|) \quad \text{and} \quad \rho(\sum_{q=1}^{d} \xi_q N_q) \leq \sum_{q=1}^{d} \|N_q\| \quad \text{for} \ \ |\xi_q| \leq 1.$$

With this information we can now reformulate Theorem 2.3.

THEOREM 2.6. *System* (1.1) *is asymptotically stable whenever condition (b) from Theorem 2.3 holds and whenever there exists a matrix norm* $\|\cdot\|$ *such that*

$$\min\{\rho(\sum_{q=1}^{d}|N_q|),\sum_{q=1}^{d}\|N_q\|\}<1. \tag{2.9}$$

For example, for a scalar multidelay problem, we have the following corollary.

COROLLARY 2.7. *System* (1.1) *with* $p=1$ *is asymptotically stable whenever*

$$\sum_{q=1}^{d}|N_q|<1 \quad \text{and} \quad \Re\left(\frac{L+\sum_{q=1}^{d}\xi_q M_q-\sum_{q=1}^{d}\eta(\xi_q)\tau_q Q_q}{1-\sum_{q=1}^{d}\xi_q N_q}\right)<0 \quad \text{for} \quad |\xi_q|\le 1.$$

For the p-dimensional one-delay system, i.e., with $d=1$,

$$\begin{cases} \frac{d}{dt}[y(t)-Ny(t-\tau)]=Ly(t)+My(t-\tau)+Q\int_{t-\tau}^{t}y(\theta)d\theta, \; t\ge t_0, \\ y(t)=\varphi(t), \quad t\in[t_0-\tau,t_0], \end{cases} \tag{2.10}$$

condition (2.9) can be weakened into $\rho(N)<1$. Hence we have the following corollary.

COROLLARY 2.8. *System* (2.10) *is asymptotically stable whenever*

$$\rho(N)<1 \quad \text{and} \quad \Re[\sigma(G(\xi))]\subseteq\mathbb{C}^- \quad \text{for} \quad |\xi|\le 1, \tag{2.11}$$

with $G(\xi)=(I_p-\xi N)^{-1}[L+\xi M-\eta(\xi)\tau Q]$.

EXAMPLE 2.1. Consider system (2.10) with $\tau=1/10$, and with

$$L=\begin{pmatrix}-10&2&2\\2&-8&-3\\1&5&-9\end{pmatrix}, \qquad M=\begin{pmatrix}2&-4&1\\0&1&-2\\0&-1&-4\end{pmatrix},$$

$$N=\frac{1}{10}\begin{pmatrix}-5&1&2\\6&0&-3\\0&3&0\end{pmatrix}, \qquad Q=\begin{pmatrix}1&0&2\\4&1&0\\2&0&1\end{pmatrix},$$

and any initial function $\varphi(t)$. A direct calculation yields $\rho(N)\cong 0.4767<1$. Also, using a MATLAB code, we numerically computed that

$$\max_{|\xi|\le 1}\{\Re(\lambda):\ \lambda\in\sigma[G(\xi)]\}\cong -4.7104<0.$$

This implies that $\Re[\sigma(G(\xi))]\subseteq\mathbb{C}^-$ for $|\xi|\le 1$. Thus, by Corollary 2.8 we may conclude that this system is asymptotically stable.

Another corollary deals with non-neutral equations, i.e., $N=0$, of the form (2.10).

COROLLARY 2.9. *System* (2.10) *with* $N=0$ *is asymptotically stable whenever*

$$\Re[\sigma(L+\xi M-\eta(\xi)\tau Q)]\subseteq\mathbb{C}^- \quad \text{for} \quad |\xi|\le 1.$$

Obviously, it is quite difficult in the multidelay case to check the conditions of Theorem 2.3. To deal with this problem, we derive some sufficient conditions that are more easily verified. For this, we employ a reasoning technique used in Hu and Cahlon [12] for multidelay systems without distributed delays.

THEOREM 2.10. *System* (1.1) *is asymptotically stable if there exists a matrix norm* $\|\cdot\|$ *such that*

($\hat{a}$) $\sum_{q=1}^{d} \|N_q\| < 1$,

($\hat{b}$) $\mu(L)+\sum_{q=1}^{d}\|M_q\| + \sum_{q=1}^{d}\tau_q\|Q_q\| + \dfrac{\sum_{q=1}^{d}[\|N_qL\| + \sum_{\hat{q}=1}^{d}(\|N_qM_{\hat{q}}\| + \tau_{\hat{q}}\|N_qQ_{\hat{q}}\|)]}{1-\sum_{q=1}^{d}\|N_q\|} < 0.$

Proof. By Theorem 2.6 and condition ($\hat{a}$), it suffices to show that condition (b) from Theorem 2.3 holds. In fact, with Props. 2.4, 2.5, Lemma 2.1 and condition ($\hat{a}$), one can infer the bound given below for all l and $|\xi_q| \le 1$. Using the notation $H := \sum_{q=1}^{d} \xi_q N_q$, we have

$$
\begin{aligned}
&\Re[\lambda_l(G(\xi))] \le \mu[G(\xi)] \\
&= \mu[(I_p + \sum_{i=1}^{\infty} H^i)(L + \sum_{q=1}^{d}\xi_q M_q - \sum_{q=1}^{d}\eta(\xi_q)\tau_q Q_q)] \\
&= \mu[L + \sum_{q=1}^{d}\xi_q M_q - \sum_{q=1}^{d}\eta(\xi_q)\tau_q Q_q + (I_p + \sum_{i=1}^{\infty} H^i)\cdot \\
&\qquad (HL + H\sum_{q=1}^{d}\xi_q M_q - H\sum_{q=1}^{d}\eta(\xi_q)\tau_q Q_q)] \\
&\le \mu(L) + \sum_{q=1}^{d}\|M_q\| + \sum_{q=1}^{d}\tau_q\|Q_q\| + [1 + \sum_{i=1}^{\infty}(\sum_{q=1}^{d}\|N_q\|)^i]\cdot \\
&\qquad [\sum_{q=1}^{d}\|N_qL\| + \sum_{q=1}^{d}\sum_{\hat{q}=1}^{d}\|N_qM_{\hat{q}}\| + \sum_{q=1}^{d}\sum_{\hat{q}=1}^{d}\tau_{\hat{q}}\|N_qQ_{\hat{q}}\|] \qquad (2.12)
\end{aligned}
$$

Hence condition ($\hat{b}$) guarantees $\Re[\lambda_l(G(\xi))] < 0$, and condition ($b$) in Th. 2.3 holds. □

EXAMPLE 2.2. Consider a system of the form (1.1) with

$$
L = \begin{pmatrix} -68 & 0 & 2 \\ 1 & -79 & -3 \\ 1 & -4 & -82 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -2 & 2 \\ 3 & -2 & 5 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 2 & -1 & 1 \\ 1 & -1 & 3 \\ 0 & 2 & 1 \end{pmatrix},
$$

$$
N_1 = \frac{1}{25}\begin{pmatrix} 1 & -3 & 0 \\ 2 & 0 & 2 \\ -1 & 3 & 2 \end{pmatrix}, \; N_2 = \frac{1}{20}\begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 3 & 0 & 0 \end{pmatrix}, \; Q_1 = \begin{pmatrix} -1 & 2 & 4 \\ 2 & -1 & 3 \\ 1 & 0 & -2 \end{pmatrix},
$$

$$Q_2 = \begin{pmatrix} 2 & 0 & 1 \\ -1 & -3 & 1 \\ -3 & 2 & 1 \end{pmatrix}, \quad \tau_1 = \frac{1}{20} \quad \tau_2 = \frac{1}{10}.$$

With the 2-norm $\|\mathcal{A}\|_2 = \sqrt{\rho(\mathcal{A}\mathcal{A}^T)}$ and its induced logarithmic norm $\mu_2(\mathcal{A}) = \max\{\sigma(\frac{\mathcal{A}+\mathcal{A}^T}{2})\}$, a simple computation yields $\sum_{q=1}^{2} \|N_q\|_2 \cong 0.3769 < 1$, $\Omega \cong -5.6623 < 0$, with Ω denoting the left-hand side of the inequality in condition $(\hat{b})$. Hence, this system is asymptotically stable.

Theorem 2.10 can be simplified for the p-dimensional non-neutral system

$$\begin{cases} y'(t) = Ly(t) + \sum_{q=1}^{d} M_q y(t-\tau_q) + \sum_{q=1}^{d} Q_q \int_{t-\tau_q}^{t} y(\theta)d\theta, \ t \geq t_0, \\ y(t) = \varphi(t), \quad t \in [t_0 - \hat{\tau}, t_0], \ \hat{\tau} = \max_{1\leq q\leq d} \tau_q. \end{cases} \tag{2.13}$$

COROLLARY 2.11. *System* (2.13) *is asymptotically stable whenever there exists a matrix norm* $\|\cdot\|$ *such that*

$$\mu(L) + \sum_{q=1}^{d} \|M_q\| + \sum_{q=1}^{d} \tau_q \|Q_q\| < 0. \tag{2.14}$$

When $d = 1$, this result is consistent with a corresponding conclusion in paper [21]. For scalar NMIDEs, a slight modification to the proof of Theorem 2.10, consisting of substituting $\Re(\cdot)$ for $\mu(\cdot)$ in (2.12), leads to the final corollary of this section.

COROLLARY 2.12. *System* (1.1) *with* $p = 1$ *is asymptotically stable whenever*

$$\sum_{q=1}^{d} |N_q| < 1 \quad and \quad \Re(L) + \frac{\sum_{q=1}^{d} (|M_q| + |N_q L| + \tau_q |Q_q|)}{1 - \sum_{q=1}^{d} |N_q|} < 0. \tag{2.15}$$

3. Runge-Kutta and linear multistep methods for NMIDEs.

In this section, we will confine our discussion to systems of NMIDEs with commensurate delays, i.e., systems of the form (1.1) with $\tau_q = q\tau$:

$$\begin{cases} \frac{d}{dt}[y(t) - \sum_{q=1}^{d} N_q y(t-q\tau)] = Ly(t) + \sum_{q=1}^{d} M_q y(t-q\tau) + \sum_{q=1}^{d} Q_q \int_{t-q\tau}^{t} y(\theta)d\theta, \ t \geq t_0, \\ y(t) = \varphi(t), \quad t \in [t_0 - d\tau, t_0], \end{cases} \tag{3.1}$$

where $\tau > 0$ is a constant and $L, M_q, N_q, Q_q \in \mathbb{C}^{p\times p}$. Before constructing discrete schemes for this system, we review classical Runge-Kutta (RK) methods, Linear Multistep (LM) methods and related concepts. This is done mainly for setting some notation and for future reference. For solving ODEs systems of the form

$$\begin{cases} y'(t) = f(t, y(t)), \ t \geq t_0, \\ y(t_0) = y_0, \end{cases} \tag{3.2}$$

there are two classes of classical methods. RK methods are of the form

$$\begin{cases} y_i^{(n)} = y_n + h \sum_{j=1}^{s} a_{ij} f(t_n + c_j h, y_j^{(n)}), \quad i = 1, 2, ..., s, \\ y_{n+1} = y_n + h \sum_{j=1}^{s} b_j f(t_n + c_j h, y_j^{(n)}), \quad n \geq 0, \end{cases} \tag{3.3}$$

where the abscissae c_j, the weights b_j and the coefficients a_{ij} are characteristics of the method; h denotes the stepsize; $t_n = t_0 + nh$, and $y_i^{(n)}$ and y_n are approximations to $y(t_n + c_i h)$ and $y(t_n)$, respectively. The other class of methods are LM methods, compactly denoted as

$$\mathcal{P}(E) y_n = h \mathcal{Q}(E) f_n, \tag{3.4}$$

where E is the shift operator, $\mathcal{P}(\xi)$ and $\mathcal{Q}(\xi)$ are irreducible polynomials of degree k,

$$\mathcal{P}(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j \quad \textit{and} \quad \mathcal{Q}(\xi) = \sum_{j=0}^{k} \beta_j \xi^j,$$

and y_n and f_n are approximations to $y(t_n)$ and $f(t_n, y(t_n))$, respectively. The stability regions (cf. [7]) of these methods are given by the following sets:

$$\mathbb{S}_{\mathbb{RK}} := \{\zeta \in \mathbb{C} : \ |1 + \zeta b^T (I_s - \zeta A)^{-1} e| < 1\}$$
$$\mathbb{S}_{\mathbb{LM}} := \{\zeta \in \mathbb{C} : \ \mathcal{P}(z) = \zeta \mathcal{Q}(z) \Rightarrow |z| < 1\},$$

where $b = (b_1, b_2, \ldots, b_s)^T, A = (a_{ij})$ and $e = (1, 1, \ldots, 1)^T \in \mathbb{R}^s$. If there exists an $\alpha \in (0, \frac{\pi}{2}]$ such that the stability region contains the sector $\mathbb{S}_\alpha := \{\zeta \in \mathbb{C} : \ |\arg(-\zeta)| < \alpha\}$, the method is called $A(\alpha)$-stable. In particular, an $A(\frac{\pi}{2})$-stable method is called A-stable.

Adapting (3.3) to (3.1), for a stepsize $h = \frac{\tau}{m}$, with m a positive integer, yields

$$\begin{cases} f_i^{(n)} = L y_i^{(n)} + \sum_{q=1}^{d} M_q y_i^{(n-qm)} + \sum_{q=1}^{d} N_q f_i^{(n-qm)} + \sum_{q=1}^{d} Q_q y_{i,q}^{(n)}, \quad i = 1, 2, ..., s, \\ y_{i,q}^{(n)} = h \sum_{r=1}^{qm} [v y_i^{(n-r+1)} + (1 - v) y_i^{(n-r)}], \quad i = 1, 2, \ldots, s; \quad q = 1, 2, \ldots, d, \\ y_i^{(n)} = y_n + h \sum_{j=1}^{s} a_{ij} f_j^{(n)}, \quad i = 1, 2, ..., s, \\ y_{n+1} = y_n + h \sum_{j=1}^{s} b_j f_j^{(n)}, \quad n \geq 0 \end{cases} \tag{3.5}$$

where, v is a parameter on the interval $[0, 1]$, and $y_n, y_i^{(n)}, f_i^{(n)}$ and $y_{i,q}^{(n)}$ are approximations to $y(t_n), y(t_n + c_i h), y'(t_n + c_i h)$ and

$$y_q(t_n + c_i h) := \int_{t_n + c_i h - q\tau}^{t_n + c_i h} y(\theta) d\theta = \sum_{r=1}^{qm} \int_{t_{n-r} + c_i h}^{t_{n-r+1} + c_i h} y(\theta) d\theta,$$

respectively. For the discretization of $y_q(t_n + c_i h)$, a linear compound quadrature formula has been used. Similarly, applying the LM method (3.4) gives, for $n \geq 0$,

$$\begin{cases} \sum_{j=0}^{k} \alpha_j y_{n+j-qm} = h \sum_{j=0}^{k} \beta_j f_{n+j-qm}, \quad q = 1, 2, \ldots, d, \\ y_{n+j,q} = h \sum_{r=1}^{qm} [v y_{n+j-r+1} + (1 - v) y_{n+j-r}], \quad j = 1, \ldots, k; \ q = 1, 2, \ldots, d, \\ \sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j [L y_{n+j} + \sum_{q=1}^{d} M_q y_{n+j-qm} + \sum_{q=1}^{d} N_q f_{n+j-qm} + \sum_{q=1}^{d} Q_q y_{n+j,q}], \end{cases} \tag{3.6}$$

where y_n, f_n and $y_{n,\ q}$ are approximations to $y(t_n), y'(t_n)$ and

$$y_q(t_n) := \int_{t_n - q\tau}^{t_n} y(\theta) d\theta = \sum_{r=1}^{qm} \int_{t_{n-r}}^{t_{n-r+1}} y(\theta) d\theta.$$

In the subsequent sections we will study the asymptotic stability of schemes (3.5) and (3.6), i.e., we investigate the conditions under which

4. Stability criteria of the adapted RK methods. In the analysis, the following properties of Kronecker products will come in handy, see, e.g., [9].

PROPOSITION 4.1. *Let* $\mathcal{A} \in \mathbb{C}^{i_1 \times i_2}, \mathcal{B} \in \mathbb{C}^{j_1 \times j_2}, C \in \mathbb{C}^{k_1 \times k_2},\ D \in \mathbb{C}^{l_1 \times l_2}$*. Then*

$$(\mathcal{A} \otimes \mathcal{B})(\mathcal{C} \otimes \mathcal{D}) = (\mathcal{AC}) \otimes (\mathcal{BD}). \tag{4.1}$$

When $\mathcal{A} \in \mathbb{C}^{i_1 \times i_1}$*, and* $\mathcal{B} \in \mathbb{C}^{j_1 \times j_1}$*, then*

$$\det(\mathcal{A} \otimes \mathcal{B}) = \det(\mathcal{B} \otimes \mathcal{A}). \tag{4.2}$$

When, additionally, they are nonsingular, then so is $\mathcal{A} \otimes \mathcal{B}$*, and*

$$(\mathcal{A} \otimes \mathcal{B})^{-1} = \mathcal{A}^{-1} \otimes \mathcal{B}^{-1}. \tag{4.3}$$

Let $F^{(n)} = \left(f_1^{(n)^T}, f_2^{(n)^T}, \ldots, f_s^{(n)^T}\right)^T$ and $Y^{(n)} = \left(y_1^{(n)^T}, y_2^{(n)^T}, \ldots, y_s^{(n)^T}\right)^T$. Then method (3.5) can be written more compactly as

$$\begin{cases} F^{(n)} = (I_s \otimes L)Y^{(n)} + \sum\limits_{q=1}^{d} (I_s \otimes M_q) Y^{(n-qm)} + \sum\limits_{q=1}^{d} (I_s \otimes N_q) F^{(n-qm)} \\ \qquad\quad + h \sum\limits_{q=1}^{d} (I_s \otimes Q_q) \sum\limits_{r=1}^{qm} [v Y^{(n-r+1)} + (1-v) Y^{(n-r)}], \\ Y^{(n)} = (e \otimes I_p) y_n + h(A \otimes I_p) F^{(n)}, \\ y_{n+1} = y_n + h(b^T \otimes I_p) F^{(n)}. \end{cases} \tag{4.4}$$

Substituting the second equality into the first one in order to eliminate the stage values and using properties from Proposition 4.1 gives

$$\begin{cases} F^{(n)} = (e \otimes L) y_n + h(A \otimes L) F^{(n)} + \sum\limits_{q=1}^{d} [(e \otimes M_q) y_{n-qm} + h(A \otimes M_q) F^{(n-qm)}] \\ \quad + \sum\limits_{q=1}^{d} (I_s \otimes N_q) F^{(n-qm)} + h \sum\limits_{q=1}^{d} \sum\limits_{r=1}^{qm} [(e \otimes Q_q)(v y_{n-r+1} + (1-v) y_{n-r}) \\ \quad + h(A \otimes Q_q)(v F^{(n-r+1)} + (1-v) F^{(n-r)})], \\ y_{n+1} = y_n + h(b^T \otimes I_p) F^{(n)}. \end{cases} \tag{4.5}$$

Define

$$Y_{n+1} = \left(F^{(n)^T}, y_{n+1}^T\right)^T, \quad \bar{L} = hL, \quad \bar{M}_q = hM_q, \quad \bar{Q}_q = hQ_q \ \ \textit{and} \ \ \bar{\bar{Q}}_q = h^2 Q_q.$$

Then, equations (4.5) can be transformed into the following difference equation:

$$\begin{pmatrix} I_s\otimes I_p - A\otimes\bar{L} & 0 \\ -h(b^T\otimes I_p) & I_p \end{pmatrix} Y_{n+1} = \begin{pmatrix} 0 & e\otimes L \\ 0 & I_p \end{pmatrix} Y_n + \sum_{q=1}^{d} \begin{pmatrix} 0 & e\otimes M_q \\ 0 & 0 \end{pmatrix} Y_{n-qm}$$

$$+ \sum_{q=1}^{d} \begin{pmatrix} A\otimes\bar{M}_q + I_s\otimes N_q & 0 \\ 0 & 0 \end{pmatrix} Y_{n-qm+1} + \sum_{q=1}^{d}\sum_{r=1}^{qm} \begin{pmatrix} 0 & (1-v)(e\otimes\bar{Q}_q) \\ 0 & 0 \end{pmatrix} Y_{n-r}$$

$$+\sum_{q=1}^{d}\sum_{r=1}^{qm} \begin{pmatrix} (1-v)(A\otimes\bar{\bar{Q}}_q) & v(e\otimes\bar{Q}_q) \\ 0 & 0 \end{pmatrix} Y_{n-r+1} + \sum_{q=1}^{d}\sum_{r=1}^{qm} \begin{pmatrix} v(A\otimes\bar{\bar{Q}}_q) & 0 \\ 0 & 0 \end{pmatrix} Y_{n-r+2}.$$

Its characteristic equation is given by

$$\det \begin{bmatrix} T_1(z) & T_2(z) \\ T_3(z) & T_4(z) \end{bmatrix} = 0, \qquad z\in\mathbb{C} \tag{4.6}$$

with $m_0 = md$, with $T_1(z)$ given by the expression

$$z^{m_0+1}[I_s\otimes(I_p - \sum_{q=1}^{d} z^{-qm}N_q) - A\otimes(\bar{L} + \sum_{q=1}^{d} z^{-qm}\bar{M}_q + \sum_{q=1}^{d}\sum_{r=1}^{qm}(vz^{1-r} + (1-v)z^{-r})\bar{\bar{Q}}_q)],$$

and with

$$T_2(z) = -z^{m_0}[e\otimes(L + \sum_{q=1}^{d} z^{-qm}M_q + \sum_{q=1}^{d}\sum_{r=1}^{qm}(vz^{1-r} + (1-v)z^{-r})\bar{Q}_q)],$$

$$T_3(z) = -z^{m_0+1}h(b^T\otimes I_p), \qquad T_4(z) = z^{m_0}(z-1)I_p.$$

It follows from the theory on difference equations (cf. [16]) that $\lim_{n\to\infty} Y_n = 0$ if and only if all the zeros of equation (4.6) satisfy $|z| < 1$. Hence, we can formulate the following lemma.

LEMMA 4.2. *Numerical method* (4.5) *satisfies* $\lim_{n\to\infty} y_n = 0$ *whenever all the zeros of equation* (4.6) *satisfy* $|z| < 1$.

LEMMA 4.3. *Assume that condition (a) from Theorem 2.3 holds and assume that matrices* $I_s - \lambda_l(r(z))A$ $(1\le l\le p)$ *are invertible for* $|z|\ge 1$*, where*

$$r(z) = (I_p - \sum_{q=1}^{d} z^{-qm}N_q)^{-1}(\bar{L} + \sum_{q=1}^{d} z^{-qm}\bar{M}_q + \sum_{q=1}^{d}\sum_{r=1}^{qm}(vz^{1-r} + (1-v)z^{-r})\bar{\bar{Q}}_q). \tag{4.7}$$

Then, $\det[T_1(z)] \ne 0$.

Proof. Condition (a) in Theorem 2.3 implies that matrix $I_p - \sum_{q=1}^{d} z^{-qm}N_q$ is invertible for $|z|\ge 1$. It follows from a simple computation and Proposition 4.1 that

$$T_1(z) = z^{m_0+1}[I_s\otimes(I_p - \sum_{q=1}^{d} z^{-qm}N_q)][I_s\otimes I_p - A\otimes r(z)] \quad \text{for} \quad |z|\ge 1.$$

With this, we have for $|z| \geq 1$ that

$$\begin{aligned}\det[T_1(z)] &= (z^{m_0+1})^{sp}[\det(I_p - \sum_{q=1}^{d} z^{-qm} N_q)]^s \det[I_s \otimes I_p - A \otimes r(z)] \\ &= (z^{m_0+1})^{sp}[\det(I_p - \sum_{q=1}^{d} z^{-qm} N_q)]^s \prod_{l=1}^{p}\prod_{j=1}^{s}[1 - \lambda_l(r(z))\lambda_j(A)]. \quad (4.8)\end{aligned}$$

Also, the invertibility of matrices $I_s - \lambda_l(r(z))A$ means that $\lambda_l(r(z))\lambda_j(A) \neq 1$ for all l, j. Hence, $\det[T_1(z)] \neq 0$ for $|z| \geq 1$. ☐

Theorem 4.4. *Method* (3.5) *is asymptotically stable if condition (a) in Theorem 2.3 holds and* $\sigma[r(z)] \subseteq \mathbb{S}_{\mathbb{RK}}$ *for* $|z| \geq 1$.

Proof. By Lemma 4.2, we need to prove that all the zeros of equation (4.6) satisfy $|z| < 1$. If this were untrue, there would exist a $z_0 \in \mathbb{C} : |z_0| \geq 1$ such that

$$\det \begin{bmatrix} T_1(z_0) & T_2(z_0) \\ T_3(z_0) & T_4(z_0) \end{bmatrix} = 0. \quad (4.9)$$

Since by Lemma 4.3 one has that $\det[T_1(z_0)] \neq 0$, equality (4.9) is equivalent to

$$\det[T_4(z_0) - T_3(z_0)T_1^{-1}(z_0)T_2(z_0)] = 0. \quad (4.10)$$

Using Proposition 4.1 and the Jordan canonical form $J(z_0)$ of $r(z_0)$, we find that

$$\begin{aligned}&\det[T_4(z_0) - T_3(z_0)T_1^{-1}(z_0)T_2(z_0)] \\ &= z_0^{m_0 p} \det\{z_0 I_p - [I_p + (b^T \otimes I_p)(I_s \otimes I_p - A \otimes r(z_0))^{-1}(e \otimes r(z_0))]\} \\ &= z_0^{m_0 p} \det\{z_0 I_p - [I_p + (I_p \otimes b^T)(I_p \otimes I_s - r(z_0) \otimes A)^{-1}(r(z_0) \otimes e)]\} \\ &= z_0^{m_0 p} \det\{z_0 I_p - [I_p + (I_p \otimes b^T)(I_p \otimes I_s - J(z_0) \otimes A)^{-1}(J(z_0) \otimes e)]\} \\ &= z_0^{m_0 p} \prod_{l=1}^{p}\{z_0 - [1 + \lambda_l(r(z_0))b^T(I_s - \lambda_l(r(z_0))A)^{-1}e]\}. \quad (4.11)\end{aligned}$$

Combining (4.11) with (4.10) shows that there is an l such that

$$|1 + \lambda_l(r(z_0))b^T(I_s - \lambda_l(r(z_0))A)^{-1}e| = |z_0| \geq 1.$$

This implies $\lambda_l(r(z_0)) \notin \mathbb{S}_{\mathbb{RK}}$, which contradicts the earlier assumption $\sigma[r(z)] \subseteq \mathbb{S}_{\mathbb{RK}}$ for $|z| \geq 1$. Hence, the theorem is proven. ☐

Theorem 4.5. *Assume that the classical RK method* (3.3) *is $A(\alpha)$-stable and system* (3.1) *satisfies* (2.9), *and assume that* $\sigma[r(z)] \subseteq \mathbb{S}_\alpha$ *for* $|z| \geq 1$. *Then, the induced method* (3.5) *is asymptotically stable.*

Moreover, with Theorem 4.5 and a similar derivation as the one in the proof of Theorem 2.10, we can show the correctness of the following theorem.

Theorem 4.6. *Assume that the classical RK method* (3.3) *is A-stable and that system* (3.1) *satisfies conditions* ($\hat{a}$) *and* ($\hat{b}$) *from Theorem 2.10. Then, the induced method* (3.5) *is asymptotically stable.*

Theorem 4.5 and Theorem 4.6 can be viewed as the discrete counterparts of Theorem 2.6 and Theorem 2.10. For some special cases of system (3.1), such as non-neutral multidelay systems, neutral one-delay systems and scalar systems, the stability conditions can again be simplified, similar to what was done in §2 for the continuous problem.

It is well-known that the classical Radau IA, Radau IIA, Gauss, Lobatto IIIA, Lobatto IIIB and Lobatto IIIC methods are all A-stable (cf. [7]). Combining this with Theorem 4.5 and with Theorem 4.6 yields the next two corollaries.

COROLLARY 4.7. *Assume that system* (3.1) *satisfies* (2.9) *and assume that* $\sigma[r(z)] \subseteq \mathbb{C}^-$ *for* $|z| \geq 1$. *Then, method* (3.5) *based on a Radau IA, Radau IIA, Gauss, Lobatto IIIA, Lobatto IIIB or Lobatto IIIC method is asymptotically stable.*

COROLLARY 4.8. *Assume that system* (3.1) *satisfies conditions* $(\hat{a})$-$(\hat{b})$ *from Theorem 2.10. Then, method* (3.5) *based on a Radau IA, Radau IIA, Gauss, Lobatto IIIA, Lobatto IIIB or Lobatto IIIC method is asymptotically stable.*

As a numerical illustration, we applied methods (3.5) with $m = 10$ and $v = 1/2$, and based respectively on the classical 2-stage Radau IA, Radau IIA, Gauss, Lobatto IIIA, Lobatto IIIB and Lobatto IIIC methods, to the system of Example 2.1 with initial condition $\varphi(t) = (-2, 2, 4)^T$ for $t \in [-1/10, 0]$). We numerically verified that

$$\max_{|z|\geq 1}\{\Re(\lambda) : \ \lambda \in \sigma[r(z)]\} \cong -0.0471 < 0,$$

$$\max_{|z|\geq 1}\{|\arg(-\lambda)| : \ \lambda \in \sigma[r(z)]\} \cong 54.90^\circ.$$

The first equality shows that $\sigma[r(z)] \subseteq \mathbb{C}^-$ for $|z| \geq 1$. (The second equality will be used in the next section.) By Corollary 4.7 this will produce asymptotically stable numerical solutions for each of the methods mentioned above. The numerical solution obtained with the adapted 2-stage Radau IA methods is shown in Figure 4.1. The numerical results obtained with the other methods are visually identical.

For a numerical illustration of Corollary 4.8, we consider the methods with $m = 50$ and $v = \frac{1}{2}$ induced respectively by the classical 3-stage Radau IA, Radau IIA, Gauss, Lobatto IIIA, Lobatto IIIB and Lobatto IIIC methods, to the system in Example 2.2, with initial condition $\varphi(t) = (-1, 0.5, 1)^T$ for $t \in [-1/10, 0]$. We have shown in Example 2.2 that this system satisfies conditions $(\hat{a})$-$(\hat{b})$ of Theorem 2.10 with $\tau_q = q/20$ $(q = 1, 2)$. Thus, it follows from Corollary 4.8 that the numerical solutions are asymptotically stable. The solution obtained by the method based on 3-stage Radau IIA is shown in Figure 4.2. The other numerical results are visually identical.

5. Stability criteria of the adapted LM methods. This section will deal with the asymptotic stability of the adapted LM methods (3.6). In the following analysis, the symbols $\bar{L}, \bar{M}$, $\bar{Q}$ and $\bar{\bar{Q}}$, introduced in §4, will be used again. Substituting the first and the second equality into the third one of (3.6) leads to

$$\sum_{j=0}^{k}\alpha_j[y_{n+j} - \sum_{q=1}^{d} N_q y_{n+j-qm}]$$
$$= \sum_{j=0}^{k}\beta_j[\bar{L}y_{n+j} + \sum_{q=1}^{d}\bar{M}_q y_{n+j-qm} + \sum_{q=1}^{d}\bar{\bar{Q}}_q \sum_{r=1}^{qm}(v y_{n+j-r+1} + (1-v) y_{n+j-r})]. \quad (5.1)$$

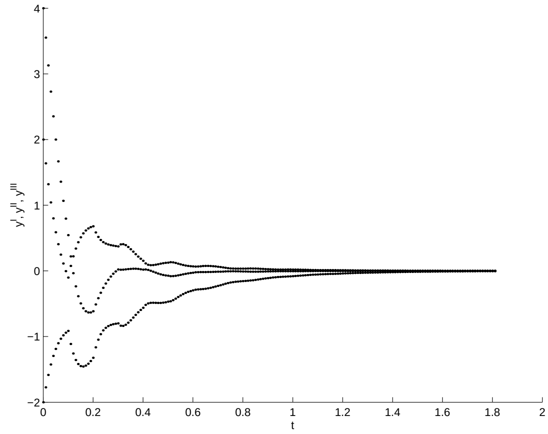

FIG. 4.1. *The numerical solution* $y = (y^I, y^{II}, y^{III})^T$ *obtained with the adapted two-stage Radau IA method with* $m = 10$ *and* $v = \frac{1}{2}$ *for the system in Example 2.1 with initial condition* $\varphi(t) = (-2, 2, 4)^T$ *for* $t \in [-\frac{1}{10}, 0]$.

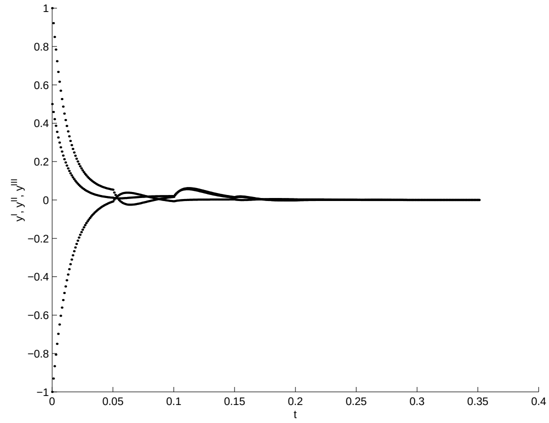

FIG. 4.2. *The numerical solution* $y = (y^I, y^{II}, y^{III})^T$ *obtained with the adapted three-stage Radau IIA method with* $m = 50$ *and* $v = \frac{1}{2}$ *for the system in Example 2.2 with the initial condition* $\varphi(t) = (-1, 0.5, 1)^T$ *for* $t \in [-\frac{1}{10}, 0]$.

The characteristic equation of this difference equation for $z \in \mathbb{C}$ is given by

$$\det[\mathcal{P}(z)(I_p - \sum_{q=1}^{d} z^{-qm} N_q) - \mathcal{Q}(z)(\bar{L} + \sum_{q=1}^{d} z^{-qm} \bar{M} + \sum_{q=1}^{d} \sum_{r=1}^{qm} (vz^{1-r} + (1-v)z^{-r})\bar{\bar{Q}}_q)] = 0. \quad (5.2)$$

From the theory on difference equations (cf. [16]), one has the following result.

LEMMA 5.1. *Numerical method* (5.1) *satisfies* $\lim_{n\to\infty} y_n = 0$ *whenever all the zeros of equation* (5.2) *satisfy* $|z| < 1$.

THEOREM 5.2. *The method* (3.6) *is asymptotically stable if condition (a) in Theorem 2.3 holds and* $\sigma[r(z)] \subseteq \mathbb{S}_{\mathbb{LM}}$ *for* $|z| \geq 1$*, where* $r(z)$ *is given by* (4.7).

Proof. By Lemma 5.1, we need to prove that all the zeros of (5.2) satisfy $|z| < 1$. Assume that there exists a z_0 with $|z_0| \geq 1$ such that (5.2) holds with $z = z_0$. Since matrix $I_p - \sum_{q=1}^{d} z_0^{-qm} N_q$ is assumed to be invertible, (5.2) is then equivalent to

$$\det[\mathcal{P}(z_0) I_p - \mathcal{Q}(z_0) r(z_0)] = 0,$$

which implies that there exists an l such that

$$\mathcal{P}(z_0) = \lambda_l(r(z_0))\mathcal{Q}(z_0).$$

Since $|z_0| \geq 1$ one has $\lambda_l(r(z_0)) \notin \mathbb{S}_{\mathbb{LM}}$. This contradicts the earlier assumption $\sigma[r(z)] \subseteq \mathbb{S}_{\mathbb{LM}}$ for $|z| \geq 1$. Hence, the theorem holds. □

Combining Theorem 5.2 with the definition of $A(\alpha)$-stability and condition (2.9), we obtain the following results.

THEOREM 5.3. *Assume that the LM method* (3.4) *is* $A(\alpha)$*-stable and system* (3.1) *satisfies* (2.9)*, and assume that* $\sigma[r(z)] \subseteq \mathbb{S}_\alpha$ *for* $|z| \geq 1$*. Then, the induced method* (3.6) *is asymptotically stable.*

THEOREM 5.4. *Assume that the LM method* (3.4) *is A-stable and system* (3.1) *satisfies conditions* $(\hat{a})$-$(\hat{b})$ *in Theorem 2.10. Then, the induced method* (3.6) *is asymptotically stable.*

We may regard Theorem 5.3 and Theorem 5.4 as discrete versions of Theorem 2.6 and Theorem 2.10, respectively. Again, the stability conditions can be simplified for non-neutral multidelay systems, neutral one-delay systems and scalar systems.

It is well-known that the classical BDF methods of order 1 and 2 are A-stable, those of order 3 and 4 are $A(86.03°)$- and $A(73.35°)$-stable, respectively. For the system in Example 2.1, we have obtained that $\rho(N) \cong 0.4767 < 1$ and we already verified for $m = 1/10$ and $v = 1/2$ that this system satisfies the conditions in Theorem 5.3. Hence, method (3.6) will produce asymptotically stable numerical solution when based on the classical BDF methods of order $1, 2, 3$ or 4. As an illustration of Theorem 5.4, we consider method (3.6) (with $m = 50$ and $v = 1/2$) induced by the classical first or second order BDF method. Both methods are A-stable and the system in Example 2.2 has been proven to satisfy conditions $(\hat{a})$-$(\hat{b})$ for $\tau_q = q/20$ $(q = 1, 2)$. Hence, when applying the adapted BDF methods, we will obtain numerical solutions with asymptotic stability.

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