Stability of Runge-Kutta-Pouzet methods for Volterra integral and integro-differential equations with delays

Chengming Huang
Stefan Vandewalle

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Katholieke Universiteit Leuven
Department of Computer Science
Celestijnenlaan 200A – B-3001 Heverlee (Belgium)

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Abstract

This paper is concerned with the study of the delay-dependent stability of Runge-Kutta-Pouzet methods for Volterra integral and integro-differential equations with delays. We are interested in the comparison between analytical and numerical stability regions. It is proved that all Gauss-Pouzet methods can retain the asymptotic stability of both delay integro-differential equations with real coefficients and delay integral equations of the second kind with complex coefficients. The Lobatto IIIC-Pouzet methods violate this property.

Keywords : delay integro-differential equations, delay integral equations, asymptotic stability, Runge-Kutta-Pouzet methods
AMS(MOS) Classification : 65L20, 65R99

STABILITY OF RUNGE-KUTTA-POUZET METHODS FOR VOLTERRA INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS WITH DELAYS

CHENGMING HUANG AND STEFAN VANDEWALLE

ABSTRACT. This paper is concerned with the study of the delay-dependent stability of Runge-Kutta-Pouzet methods for Volterra integral and integro-differential equations with delays. We are interested in the comparison between analytical and numerical stability regions. It is proved that all Gauss-Pouzet methods can retain the asymptotic stability of both delay integro-differential equations with real coefficients and delay integral equations of the second kind with complex coefficients. The Lobatto IIIC-Pouzet methods violate this property.

1. INTRODUCTION

We study the asymptotic stability of Runge-Kutta methods for the Volterra delay integro-differential equation

$$y'(t) = \alpha y(t) + \beta y(t-\tau) + \gamma \int_{t-\tau}^{t} y(\nu)d\nu, \tag{1.1}$$

and the Volterra delay integral equation of the second kind

$$y(t) = (\alpha + i\beta) \int_{t-\tau}^{t} y(\nu)d\nu, \tag{1.2}$$

with initial condition $y(t) = \psi(t), t \in [-\tau, 0]$, where $\alpha, \beta, \gamma \in \mathbb{R}$ and τ is a fixed positive number. We are mainly interested in the question whether certain numerical methods can completely retain the asymptotic stability of the analytical solution, i.e. whether the analytical stability region is a subset of the numerical stability region.

A lot of work in this field has been done for differential equations with a discrete delay (see, for example, [4, 5, 7, 8, 9, 10, 11, 18, 19]). In particular, Guglielmi and Hairer [10] have recently developed a relationship between stability analysis and order stars, which has led to a series of elegant results on symmetric methods of high order. Relevant to the case of distributed

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Key words and phrases. delay integro-differential equations, delay integral equations, asymptotic stability, Runge-Kutta-Pouzet methods .

Katholieke Universiteit Leuven, Department of Computer Science, Celestijnenlaan 200A, B3001 Leuven, Belgium (Chengming.Huang@cs.kuleuven.ac.be).

Katholieke Universiteit Leuven, Department of Computer Science, Celestijnenlaan 200A, B3001 Leuven, Belgium (Stefan.Vandewalle@cs.kuleuven.ac.be).

delay (i.e., $\gamma \neq 0$), however, only few results on numerical stability have been published. Baker and Ford [2] investigated the analytical stability region of (1.1) in the case of $\beta = 0$ and numerically showed that some linear multistep methods combined with an appropriate quadrature rule are highly stable (i.e. the stability region is unbounded and approximates the analytical stability region). Koto [15] introduced Runge-Kutta methods of Pouzet type for a system of equations of the form (1.1) and proved that every A-stable natural Runge-Kutta method can preserve the *delay-independent* stability of the underlying system. Luzyanina, Engelborghs and Roose [17] gave some stability results of multistep methods for a variant of (1.1) with complex coefficients under a sufficient condition for the stability of the analytical solution. Huang and Vandewalle [14] studied the analytical stability region of (1.1) as well as the numerical stability region of the repeated trapezium rule. They proved that the scheme can completely retain the asymptotic stability of (1.1).

In order to get insight into the stability of numerical methods for delay integral equations of the second kind, Baker and Ford [1] introduced the model equation (1.2). In that paper they derived the analytical stability region. They proved that the repeated trapezium rule and repeated implicit Euler rule completely retain the asymptotic stability of the analytical solution. We also mention the paper by Vermiglio [20], where the test equation is

$$y(t) = 1 + a \int_0^t y(\nu) d\nu + b \int_0^{t-\tau} y(\nu) d\nu. \tag{1.3}$$

She presented results on Runge-Kutta methods under the assumption that $|b| < -\mathrm{Re}(a)$, which is a sufficient condition for the stability of the analytical solution.

In this paper, we consider the *delay-dependent* stability of Runge-Kutta-Pouzet methods of high order for (1.1) and (1.2) and pursue a theoretical analysis. We prove that all Gauss-Pouzet methods retain the asymptotic stability of both (1.1) and (1.2). We also show that the implicit Euler-Pouzet rule does not retain the stability of (1.1). It is known that the Lobatto IIIC methods do not retain the stability for (1.1). We now show a similarly negative result for (1.2).

This paper is organized as follows. In section 2, we study the case of delay integro-differential equations. In section 3, we investigate the numerical stability of delay integral equations.

2. Stability of delay integro-differential equations

2.1. The analytical stability region. Without loss of generality, we fix the delay τ equal to 1 and consider the model equation

$$y'(t) = \alpha y(t) + \beta y(t-1) + \gamma \int_{-1}^{0} y(t+\nu) d\nu \ , \quad t > 0 \ , \tag{2.1}$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and $y(t) = \psi(t)$ on $[-1, 0]$. By looking at solutions of the form $y(t) = \exp(\lambda t)$ we are led to the characteristic equation

$$\lambda = \alpha + \beta \exp(-\lambda) + \gamma \int_{-1}^{0} \exp(\lambda\nu) d\nu \ . \tag{2.2}$$

In correspondence with the infinite-dimensional nature of equation (2.1), (2.2) normally has an infinite number of roots $\lambda \in \mathbb{C}$. Yet, the number of roots in any right half plane $\mathrm{Re}\lambda > \eta$, for $\eta \in \mathbb{R}$, is finite (cf. [13]). The asymptotic stability of the zero solution to (2.1) is equivalent to the condition that all the roots of algebraic equation (2.2) have negative real part. The analytical stability region S_* is given by

$$S_* = \{(\alpha, \beta, \gamma) : \text{ all roots of (2.2) satisfy } \mathrm{Re}\lambda < 0\}.$$

We use the notation $S_*(\beta)$ to denote the stability region in the (α, γ)-plane for a fixed β. We proved in [14] that $S_*(\beta)$ is bounded above by the line

$$C_*(\beta) = \{(\alpha, \gamma) : \ \alpha + \gamma = -\beta\},$$

and bounded below by the curve

$$C_0(\beta) = \{(\alpha, \gamma) : \ \alpha(\theta) = \beta + \frac{\theta \sin\theta}{1 - \cos\theta} \text{ and } \gamma(\theta) = \frac{1}{2}(\beta^2 - \alpha^2(\theta) - \theta^2), \ \theta \in (0, 2\pi)\}. \tag{2.3}$$

As an illustration, we drew the stability regions in the (α, γ)-plane for two different values of β, see Fig. 1.

2.2. **Runge-Kutta-Pouzet methods for delay integro-differential equations.** Let (A, b, c) denote a given Runge-Kutta method characterized by the $s \times s$ matrix $A = (a_{ij})$ and vectors $b = (b_1, \cdots, b_s)^T, c = (c_1, \cdots, c_s)^T$. In this paper we assume that $\sum_{j=1}^{s} b_j = 1$. Set $t_n = nh, n \in \mathbb{Z}$, with $h = 1/m$ for m a positive integer. Approximating both differential and integral parts in (2.1) with the Runge-Kutta method, we find the following scheme (cf. [15])

$$Y_i^{(n)} = y_n + h \sum_{j=1}^{s} a_{ij}(\alpha Y_j^{(n)} + \beta Y_j^{(n-m)} + \gamma G_j^{(n)}), \quad i = 1, \ldots, s, \tag{2.4}$$

$$G_i^{(n)} = h \sum_{j=1}^{s} a_{ij} Y_j^{(n)} + h \sum_{k=1}^{m} \sum_{j=1}^{s} b_j Y_j^{(n-k)} - h \sum_{j=1}^{s} a_{ij} Y_j^{(n-m)}, \quad i = 1, \ldots, s, \tag{2.5}$$

$$y_{n+1} = y_n + h \sum_{j=1}^{s} b_j(\alpha Y_j^{(n)} + \beta Y_j^{(n-m)} + \gamma G_j^{(n)}), \tag{2.6}$$

where y_n and $Y_j^{(n)}$ are approximations to $y(t_n)$ and $y(t_n + c_j h)$, respectively, and the argument $G_i^{(n)}$ denotes an approximation to the integral

$$\int_{-1}^{0} y(t_n + c_i h + \nu) d\nu = \int_{t_n}^{t_n + c_i h} y(\nu) d\nu + \int_{t_{n-m}}^{t_n} y(\nu) d\nu - \int_{t_{n-m}}^{t_{n-m} + c_i h} y(\nu) d\nu.$$

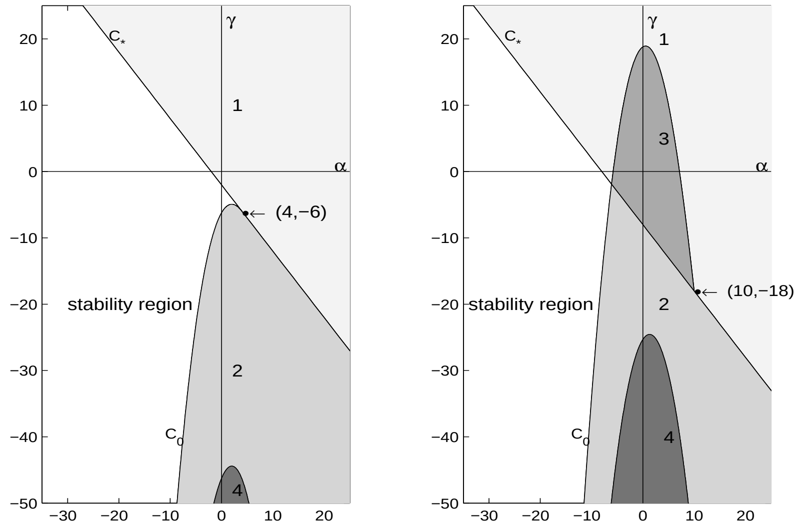

FIGURE 1. Stability region of (2.1), bounded by the line $C_*(\beta)$ and by the curve $C_0(\beta)$ in the $(\alpha,\gamma)-$plane (left: $\beta = 2$; right: $\beta = 8$) together with the curves separating regions with different numbers (as indicated) of roots in the right half-plane.

Koto [15] has pointed out that this scheme has the same order as the underlying Runge-Kutta method for ordinary differential equations. This follows from the Pouzet-type Runge-Kutta theory for Volterra integro-differential equations by Brunner, Hairer and Norsett [3] and Lubich [16]. Note that, here, we do not consider the computation of the needed initial values.

From the theory of difference equations, the characteristic equation of the scheme is given by

$$\det M(m,\xi) = 0, \tag{2.7}$$

where

$$M(m,\xi) = \begin{bmatrix} I - \frac{1}{m}(\alpha + \beta\xi^{-m})A - \frac{1}{m^2}\gamma(1-\xi^{-m})A^2 - \frac{1}{m^2}\gamma\sum_{k=1}^{m}\xi^{-k}Aeb^T & -e \\ -\frac{1}{m}(\alpha+\beta\xi^{-m})b^T - \frac{1}{m^2}\gamma(1-\xi^{-m})b^TA - \frac{1}{m^2}\gamma\sum_{k=1}^{m}\xi^{-k}b^T & \xi - 1 \end{bmatrix}, \tag{2.8}$$

where I stands for the $s \times s$ identity matrix and $e = (1,\ldots,1)^T \in \mathbb{R}^s$. The numerical stability region S_m of the scheme for any fixed m is given by

$$S_m = \{(\alpha,\beta,\gamma) : \text{ all roots of (2.7) satisfy } |\xi| < 1\}.$$

For any fixed β, we will use the notation $S_m(\beta)$ to denote the stability region in the (α, γ)-plane.

2.3. **Two examples.** We first consider Θ-methods, which can be represented by the Butcher tableau

$$\begin{array}{c|c} \Theta & \Theta \\ \hline & 1 \end{array}. \tag{2.9}$$

In this case, $M(m,\xi)$ will be denoted by $M_\Theta(m,\xi)$ and satisfies

$$M_\Theta(m,\xi) = \begin{bmatrix} 1 - \frac{1}{m}(\alpha+\beta\xi^{-m})\Theta - \frac{1}{m^2}\gamma(1-\xi^{-m})\Theta^2 - \frac{1}{m^2}\Theta\gamma\sum_{k=1}^{m}\xi^{-k} & -1 \\ -\frac{1}{m}(\alpha+\beta\xi^{-m}) - \frac{1}{m^2}\gamma(1-\xi^{-m})\Theta - \frac{1}{m^2}\gamma\sum_{k=1}^{m}\xi^{-k} & \xi - 1 \end{bmatrix}. \tag{2.10}$$

It is known that all $\Theta-$methods with $\Theta \in [1/2, 1]$ can preserve the stability of the analytical solution ($\tau(0)$-stability, cf. [8]) in the case of only fixed delay (i.e., $\gamma = 0$). Here we show that this does not necessarily hold when $\gamma \neq 0$.

Proposition 2.1. *If $\Theta > 5/6$, then $S_* \not\subseteq S_2$.*

Proof. We fix $\beta = 2$ and prove $S_*(2) \not\subseteq S_2(2)$. It is easy to see that $S_*(2)$ is bounded to the right by the line $C_*(2) : \alpha + \gamma = -2$ ($\alpha < 4$) and the curve $C_0(2)$, which intersect each other only at the point $(\alpha(0), \gamma(0)) = (4, -6)$ (see Fig. 1). On the other hand, it is easy to verify that

$$\det M_\Theta(m, -1) = 0$$

$$\Updownarrow$$

$$(2\Theta - 1)(\alpha + (-1)^m\beta + \frac{1}{2m}\gamma(1 - (-1)^m)(2\Theta - 1)) = 2m.$$

Hence, when $m = 2$, the line $\alpha = \frac{4}{2\Theta-1} - 2$ lies outside $S_2(2)$. From the assumption $\Theta > 5/6$ it follows that this line lies in the left half plane $\alpha < 4$, i.e. this line will cross the analytical stability region $S_*(2)$. Hence, $S_*(2) \not\subseteq S_2(2)$, which further gives $S_* \not\subseteq S_2$. This completes the proof. □

Remark 2.1. In fact, our numerical investigation shows that $S_* \subseteq \bigcap_{m=1}^{\infty} S_m$ possibly only for $\Theta = 1/2$ although $S_*(0) \subseteq \bigcap_{m=1}^{\infty} S_m(0)$ possibly for all $\Theta \in [1/2, 1]$ (see, for example, Fig. 2).

Next, we provide an example of an $A-$stable method which violates $S_*(0) \subseteq \bigcap_{m=1}^{\infty} S_m(0)$. Consider the 2-stage Lobatto IIIC method

$$\begin{array}{c|cc} 0 & 1/2 & -1/2 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}. \tag{2.11}$$

Using the boundary locus technique (cf. [1, 2]), which will be explained in detail in the following subsection, we draw a part of the boundary of the

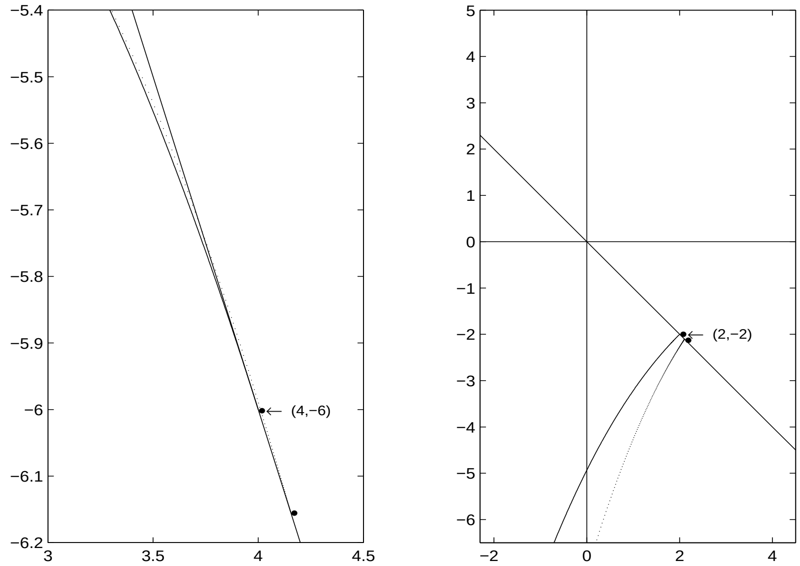

FIGURE 2. Local boundaries of analytical stability region and numerical stability region of the $\theta-$method with $\theta = 0.6$ for $m = 8$. Left picture: case $\beta = 2$; the analytical stability region is the area between the two solid lines, starting at $(4, -6)$. The numerical boundary locus (dotted line), starting near $(4.15, -6.15)$, crosses the analytical region. Right picture: case $\beta = 0$; the numerical boundary locus, starting near $(2.1, -2.1)$, lies below $C_0(0)$ starting at $(2, -2)$.

stability region in the (α, γ)-plane for $\beta = 0$ in Fig. 3 (left picture). One observes that the boundary of the stability region of the method intersects the boundary of the analytical stability region.

We remark that, for the case of fixed delay (i.e. $\gamma = 0$), it is known that the Lobatto IIIC methods can not completely preserve the asymptotic stability of the analytical solution (cf. [7, 10]).

2.4. **Gauss methods.** In this section, we derive a positive result on the stability of Runge-Kutta-Pouzet methods of Gauss type. Since the root ξ of (2.7) depends continuously on the parameters α, β and γ, we conclude that $S_* \subseteq \bigcap_{m=1}^{\infty} S_m$ if, for every $m \geq 1$, all the values (α, β, γ) satisfying (2.7) with $|\xi| = 1$ lie outside the analytical stability region S_*. This property is central to the so-called boundary locus or root locus technique (cf. [1, 2]).

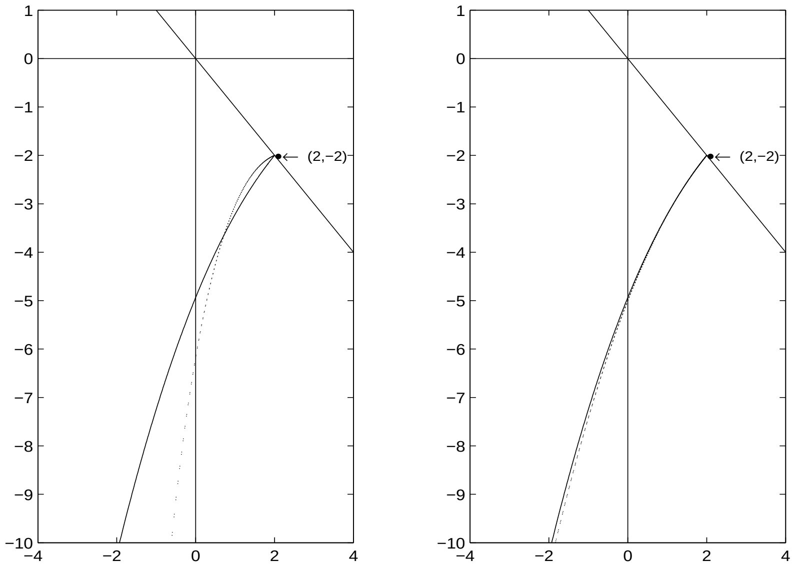

FIGURE 3. Local boundaries of analytical stability region (solid) and numerical stability region (dotted) for $\beta = 0$. Left: the 2-stage Lobatto IIIC method for $m = 2$, right: the 3-stage Gauss method for $m = 1$.

For our analysis, we introduce the notations

$$\begin{aligned} U_m &= \{(\alpha,\beta,\gamma) : \text{(2.7) has at least a root } \xi \text{ with } |\xi| = 1\}, \\ U_m^1 &= \{(\alpha,\beta,\gamma) : \text{(2.7) has at least a root } \xi \text{ with } \xi = 1\}, \\ U_m^2 &= \{(\alpha,\beta,\gamma) : \text{(2.7) has at least a root } \xi \text{ with } |\xi| = 1 \text{ but } \xi \neq 1\}. \end{aligned}$$

The notations $U_m(\beta), U_m^1(\beta)$ and $U_m^2(\beta)$ stand for the corresponding sets in the $(\alpha,\gamma)-$plane for any fixed β. Note that $\partial S_m \subseteq U_m = U_m^1 \bigcup U_m^2$. We remark that it is possible that $U_m \setminus U_m^1 \neq U_m^2$.

Lemma 2.2. *For any Runge-Kutta method* (2.4)-(2.5)-(2.6) *and* $m \geq 1$, $C_* \subseteq U_m^1 \subseteq U_m$.

Proof. A straightforward computation shows

$$M(m,1) = \begin{bmatrix} I - \frac{1}{m}(\alpha+\beta)A - \frac{1}{m}\gamma Aeb^T & -e \\ -\frac{1}{m}(\alpha+\beta+\gamma)b^T & 0 \end{bmatrix}. \tag{2.12}$$

Therefore, $\det M(m,1) = 0$ if $\alpha + \beta + \gamma = 0$. This completes the proof. □

In general, we do not necessarily have $C_* = U_m^1$. For example, U_m^1 also contains the line $\alpha + \beta = 2m$ for the 2-stage Lobatto IIIC method (2.11). By the way, we obtain $S_*(2) \nsubseteq S_1(2)$ for this method.

Lemma 2.3. *For any Gauss method* (2.4)-(2.5)-(2.6) *and* $m \geq 1$, $C_* = U_m^1$.

Proof. Our proof is based on the so-called W-transformation. According to Theorem 5.6 in [12, Chapter IV], there exists for each Gauss method a matrix W such that

$$W^{-1}AW = \begin{bmatrix} 1/2 & -\xi_1 & & & \\ \xi_1 & 0 & -\xi_2 & & \\ & \xi_2 & \ddots & \ddots & \\ & & \ddots & 0 & -\xi_{s-1} \\ & & & \xi_{s-1} & 0 \end{bmatrix} := X_G,$$

with $\xi_k = 1/(2\sqrt{4k^2-1})$. This matrix also satisfies $W^TB = W^{-1}$ and $W^Tb = e_1$, where $B = \mathrm{diag}(b_1, \dots, b_s), e_1 = (1, 0, \dots, 0)^T$. Hence,

$$\begin{bmatrix} W^{-1} & 0 \\ 0 & 1 \end{bmatrix} M(m,1) \begin{bmatrix} W & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I - \frac{\alpha+\beta}{m}X_G - \frac{\gamma}{m}X_G e_1 e_1^T & -e_1 \\ -\frac{\alpha+\beta+\gamma}{m}e_1^T & 0 \end{bmatrix}.$$

Expanding the determinant of the right-hand side matrix along the last column, we find

$$\det M(m,1) = -\frac{\alpha+\beta+\gamma}{m} \det \begin{bmatrix} 1 & \frac{\xi_2(\alpha+\beta)}{m} & & \\ -\frac{\xi_2(\alpha+\beta)}{m} & \ddots & \ddots & \\ & \ddots & 1 & \frac{\xi_{s-1}(\alpha+\beta)}{m} \\ & & -\frac{\xi_{s-1}(\alpha+\beta)}{m} & 1 \end{bmatrix}.$$

Therefore, $\det M(m,1) = 0 \Longleftrightarrow \alpha+\beta+\gamma = 0$, which gives $U_m^1 = C_*$. □

In order to determine U_m^2, we present some preliminary results first.

Lemma 2.4. *For $\xi \neq 1$, the characteristic equation* (2.7) *is equivalent to*
(2.13)
$$\det[I - \frac{1}{m}(\alpha+\beta\xi^{-m})(A + \frac{1}{\xi-1}eb^T) - \frac{1}{m^2}\gamma(1-\xi^{-m})(A + \frac{1}{\xi-1}eb^T)^2] = 0.$$

Proof. The conclusion follows from the relation

$$\begin{bmatrix} I & \frac{1}{\xi-1}e \\ 0 & 1 \end{bmatrix} M(m,\xi) =$$

$$\begin{bmatrix} I - \frac{1}{m}(\alpha+\beta\xi^{-m})(A+\frac{1}{\xi-1}eb^T) - \frac{1}{m^2}\gamma(1-\xi^{-m})(A+\frac{1}{\xi-1}eb^T)^2 & 0 \\ -\frac{1}{m}(\alpha+\beta\xi^{-m})b^T - \frac{1}{m^2}\gamma(1-\xi^{-m})b^T(A+\frac{1}{\xi-1}eb^T) & \xi - 1 \end{bmatrix}.$$

□

Remark 2.2. The proof actually implies

$$\det M(m,\xi) =$$

$$(\xi-1)\det[I - \frac{1}{m}(\alpha+\beta\xi^{-m})(A+\frac{1}{\xi-1}eb^T) - \frac{1}{m^2}\gamma(1-\xi^{-m})(A+\frac{1}{\xi-1}eb^T)^2].$$

$\xi = 1$ is a removable singularity of the right-hand side because the matrix entries of $\det M(m, \xi)$ are continuous function of ξ, hence also $\det M(m, \xi)$ is continuous w.r.t. ξ.

Let $R(z) = 1 + b^T z(I - Az)^{-1}e$ be the stability function of a Runge-Kutta method (cf. [12]). Considering

$$R(z) = \frac{\det(I - zA + zeb^T)}{\det(I - zA)}, \tag{2.14}$$

we have the following result.

Lemma 2.5. *The following equality holds:*

$$(\xi - 1)\det(I - z(A + \frac{1}{\xi - 1}eb^T)) = \xi \det(I - zA) - \det(I - zA + zeb^T), \tag{2.15}$$

where the left-hand side is interpreted as the limit for $\xi \to 1$ *when* $\xi = 1$. *Furthermore, if and only if the polynomials* $\det(I - zA)$ *and* $\det(I - zA + zeb^T)$ *have no common root, the following two equations are equivalent to each other:*

$$(\xi - 1)\det[I - z(A + \frac{1}{\xi - 1}eb^T)] = 0, \tag{2.16}$$

$$\xi = R(z). \tag{2.17}$$

Proof. The conclusion follows from (2.14) and the equality

$$\begin{bmatrix} I & -\frac{1}{\xi-1}e \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I - z(A + \frac{1}{\xi-1}eb^T) & 0 \\ -zb^T & \xi - 1 \end{bmatrix} = \begin{bmatrix} I - zA & 0 \\ -zb^T & \xi - R(z) \end{bmatrix} \begin{bmatrix} I & -(I - zA)^{-1}e \\ 0 & 1 \end{bmatrix}.$$

□

Remark 2.3. Lemma 2.5 shows a relation between the eigenvalues of matrix $A + \frac{1}{\xi-1}eb^T$ and the boundary of the stability region of the method for ODEs if we set $|\xi| = 1$. For reducible methods, however, this equivalence no longer holds. For example, consider the following method

$$\begin{array}{c|cc} \frac{1}{2} & \frac{1}{2} & 0 \\ c_* & 0 & c_* \\ \hline & 1 & 0 \end{array},$$

where c_* is a pending constant. It is easy to verify that $c_* \in \sigma(A + \frac{1}{\xi-1}eb^T)$ for all $\xi \neq 1$. On the other hand, the method can reduce to the implicit midpoint rule, which has the whole imaginary axis as its boundary of stability region. So (2.16) is not equivalent to (2.17) if we set c_* as a non-zero constant.

Using order stars, Guglielmi and Hairer [10] proved the following result which will play an important role in our stability analysis.

Lemma 2.6. *Let $R(z)$ be symmetric and assume the order star A has the whole imaginary axis as boundary with A lying to the left. Let the function $\varphi(y)$ be the argument of $R(iy)$,*

$$R(iy) = \exp(i\varphi(y)), \tag{2.18}$$

in such a way that $\varphi(0) = 0$ and $\varphi(y)$ is continuous. Then, the function $\varphi(y)$ is strictly monotonically increasing and it satisfies the following properties:

$$\varphi(-y) = -\varphi(y), \ \ \varphi(y) < y \text{ for } y > 0, \text{ and } \lim_{y\to\infty} \varphi(y) = s\pi,$$

where s is the number of poles of $R(z)$.

Let $\xi = \exp(i\varphi), \xi \neq 1$. From Lemma 2.5, we conclude that, for the Gauss methods, all the eigenvalues of $A + \frac{1}{\xi-1}eb^T$ lie on the imaginary axis. We denote them as $1/iy$, and using also (2.18), we find that the characteristic equation (2.13) is equivalent to

$$1 + \frac{i}{my}(\alpha + \beta\exp(-im\varphi)) + \frac{\gamma}{m^2y^2}(1 - \exp(-im\varphi)) = 0.$$

Separating real and imaginary parts yields

$$\begin{cases} 1 + \frac{1}{my}\beta\sin m\varphi + \frac{\gamma}{m^2y^2}(1 - \cos m\varphi) = 0, \\ \\ \frac{1}{my}(\alpha + \beta\cos m\varphi) + \frac{\gamma}{m^2y^2}\sin m\varphi = 0. \end{cases}$$

These two equalities determine the parameter sets (α, β, γ) for which the characteristic equation has roots of modulus one. For a given β, this leads to a set of curves in the (α, γ) plane, parameterized by φ for $\varphi \in [0, s\pi]$. We will denote the curves by $\bar{\alpha}(\varphi)$ and $\bar{\gamma}(\varphi)$, in order to avoid confusion with the curves $\alpha(\theta)$ and $\gamma(\theta)$ defined in (2.3). The set $U_m^2(\beta)$ is given by

$$U_m^2(\beta) = \bigcup_{k=0}^{\lfloor\frac{ms-1}{2}\rfloor} \bar{C}_k(\beta),$$

where $\lfloor x \rfloor$ stands for the integer part of x, and

$$\bar{C}_k(\beta) = \{(\bar{\alpha}(\varphi), \bar{\gamma}(\varphi)) : \ \varphi \in (\frac{2k\pi}{m}, \frac{2(k+1)\pi}{m})\},$$

$$\bar{\alpha}(\varphi) = \beta + \frac{my\sin m\varphi}{1 - \cos m\varphi}, \quad \bar{\gamma}(\varphi) = \frac{1}{2}(\beta^2 - \bar{\alpha}^2(\varphi) - m^2y^2),$$

where y and φ satisfy (2.18), and, if ms is odd, we specifically define

$$\bar{C}_{\lfloor\frac{ms-1}{2}\rfloor}(\beta) = \{(\bar{\alpha}(\varphi), \bar{\gamma}(\varphi)) : \ \varphi \in (\frac{(ms-1)\pi}{m}, s\pi)\}.$$

All the curves $\bar{C}_k(\beta)$ lie in the area $\gamma \leq (\beta^2 - \alpha^2)/2$. $\bar{C}_0(\beta)$ starts at $(\beta+2, -2(\beta+1))$ and approximates $C_0(\beta)$. The other curves $\bar{C}_k(\beta)$ start at ∞ and end at ∞ if ms is even. More precisely, we have the following result.

Lemma 2.7.

$$\lim_{m\varphi\to 2k\pi+0}\alpha(\theta)\to+\infty,\ \lim_{m\varphi\to 2k\pi-0}\alpha(\theta)\to-\infty,\ \lim_{m\varphi\to 2k\pi}\gamma(\theta)\to-\infty\ ,\ k=1,2,\ldots$$

These curves are very similar to the ones of the continuous case in Fig. 1. So we do not plot them here any more. The only case left, that of $\varphi\to s\pi$ and ms is odd, is more complicated because $\lim\limits_{\varphi\to s\pi}(1-\cos m\varphi)=2$ and we have to determine the limit $\lim\limits_{y\to\infty} y\sin m\varphi$.

It is well known that, for the s-stage Gauss method, the stability function $R(z)$ is the s-stage diagonal Padé-approximation (see [12]), i.e. $R_{ss}(z)=P_{ss}(z)/Q_{ss}(z)$, where

$$P_{ss}(z)=1+\frac{s}{2s}z+\frac{s(s-1)}{2s(2s-1)}\frac{z^2}{2!}+\cdots+\frac{s(s-1)\ldots 1}{2s\ldots(s+1)}\frac{z^s}{s!},$$

and $Q_{ss}(z)=P_{ss}(-z)$. We have the following result.

Lemma 2.8. *For the s-stage diagonal Padé-approximation, it holds that*

$$\lim_{y\to\infty} y\sin\varphi(y)=(-1)^{s-1}2s(s+1),$$

where y,φ are defined by (2.18).

Proof. A straightforward but technical computation shows that

$$\lim_{y\to\infty} y^2R'_{ss}(iy)/R_{ss}(iy)=2s(s+1),$$

which, combined with $\lim\limits_{y\to\infty}\varphi(y)=s\pi$ (Lemma 2.6), gives

$$\lim_{y\to\infty}\frac{\sin\varphi(y)}{y^{-1}}=\lim_{y\to\infty}-\frac{\varphi'(y)\cos\varphi(y)}{y^{-2}}=(-1)^{s-1}2s(s+1).$$

This completes the proof. □

Using the above lemma, we get the following limits for ms odd:

$$\lim_{\varphi\to s\pi}\bar\alpha(\varphi)=\beta+\lim_{\varphi\to s\pi}(y\sin\varphi)\frac{m\sin m\varphi}{\sin\varphi(1-\cos m\varphi)}=\beta+s(s+1)m^2, \tag{2.19}$$

and

$$\lim_{\varphi\to s\pi}\bar\gamma(\varphi)=-\infty. \tag{2.20}$$

Now we give some further properties of the curves $\bar C_k(\beta)$.

Lemma 2.9. *The curves $\bar C_k(\beta)$ do not intersect.*

Proof. Suppose that there exist $\varphi_1,\varphi_2\in(0,s\pi)$ and corresponding y_1,y_2 defined by (2.18), such that $\bar\alpha(\varphi_1)=\bar\alpha(\varphi_2)$ and $\bar\gamma(\varphi_1)=\bar\gamma(\varphi_2)$. These equalities immediately lead to $y_1=y_2$, which, combined with Lemma 2.6, gives $\varphi_1=\varphi_2$. This completes the proof. □

Lemma 2.10. *The curves $\bar C_k(\beta)$ are ordered according to k.*

Proof. We consider the intersection of the curves $\bar{C}_k(\beta)$ and the line $\alpha = \beta$. The equality $\bar{\alpha}(\varphi_k) = \beta, m\varphi_k \in (2k\pi, 2(k+1)\pi), k = 0, \ldots, \lfloor \frac{ms-1}{2} \rfloor$, implies

$$\varphi_k = \frac{(2k+1)\pi}{m} \text{ and } \gamma(\varphi_k) = \frac{-m^2 y^2(\varphi_k)}{2},$$

where $y(\varphi_k)$ is determined by (2.18) corresponding to φ_k. From the monotonicity of $y(\varphi)$ by Lemma 2.6, we see that the curves $\bar{C}_k, k = 0, \ldots, \lfloor \frac{ms-1}{2} \rfloor$, are ordered in a natural way if ms is even. When ms is odd, considering Lemma 2.9, (2.19) and (2.20), we conclude that $\bar{C}_{\lfloor \frac{ms-1}{2} \rfloor}$ is the lowest curve. This completes the proof. □

Theorem 2.11. *For all Gauss methods, we have* $S_* \subset \bigcap_{m=1}^{\infty} S_m$.

Proof. For any fixed β, we consider the reference point $(\alpha, \gamma) = (-|\beta| - 1, 0)$. It is easy to verify by Lemma 2.5 that, for the reference point, no ξ with $|\xi| \geq 1$ is a solution to (2.13). By continuity arguments, the whole connected region containing the reference point *belongs to* the stability region of the method (In the next theorem we will show that the connected region actually *is* the stability region).

First note that both regions are bounded above by $C_*(\beta) = U_m^1(\beta)$. We prove that the curve $\bar{C}_0(\beta)$ lies below or to the right of $C_0(\beta)$ for any β and m. To this end, we first prove $C_0(\beta)$ intersects $\bar{C}_0(\beta)$ only at $(\alpha(0), \gamma(0))$. Suppose there exist $\theta, m\varphi \in (0, 2\pi)$ such that $\alpha(\theta) = \bar{\alpha}(\varphi)$ and $\gamma(\theta) = \bar{\gamma}(\varphi)$. Then we have $\theta = my$, which gives $my \in (0, 2\pi)$. From the monotonicity of function $\frac{\sin x}{1-\cos x}$ and the fact that $\varphi(y) < y$ (Lemma 2.6), it follows that

$$\alpha(\theta) = \beta + \frac{my \sin my}{1 - \cos my} < \beta + \frac{my \sin m\varphi}{1 - \cos m\varphi} = \bar{\alpha}(\varphi),$$

which contradicts the earlier assumption $\alpha(\theta) = \bar{\alpha}(\varphi)$.

Next we prove that $\bar{C}_0(\beta)$ lies below $C_0(\beta)$ (in the case $ms > 1$) or to the right of $C_0(\beta)$ (in the case $ms = 1$). In the case of $ms > 1$, it is easy to verify that $\alpha(\pi) = \bar{\alpha}(\pi/m) = \beta$ and $\gamma(\pi) = -\pi^2/2 > \bar{\gamma}(\pi/m)$. In the case of $ms = 1$, for any $\gamma(\theta) = \bar{\gamma}(\varphi)$, one has $\alpha(\theta) < \beta + 2 = \bar{\alpha}(\varphi)$ because $\bar{C}_0(\beta)$ is the line $\{(\beta + 2, \gamma); \gamma \in (-\infty, -2(\beta + 1))\}$ (cf. [14]). This completes the proof. □

Theorem 2.12. *For all Gauss methods, the stability region $S_m(\beta)$ is bounded above by the line C_* and below by the curve $\bar{C}_0(\beta)$.*

Proof. At each point of a curve $\bar{C}_k(\beta)$, there exists a pair of critical roots $\exp(\pm i\varphi)$ exactly on the unit disk. Now we study to which side of the curves the critical roots move outside the unit disk. Let $\xi = \exp(\mu + i\varphi)$ and let $1/(x + iy)$ be an eigenvalue of matrix $A + \frac{1}{\xi - 1} e b^T$. Then the characteristic equation (2.13) is equivalent to

$$1 - \frac{\alpha + \beta \exp(-m(\mu + i\varphi))}{m(x + iy)} - \frac{\gamma(1 - \exp(-m(\mu + i\varphi)))}{m(x + iy)^2} = 0.$$

We denote the left-hand side by $F(\alpha, \gamma, \mu + i\varphi)$. Define

$$\begin{cases} F_1(\alpha, \gamma, \mu, \varphi) = \mathrm{Re} F(\alpha, \gamma, \mu + i\varphi), \\ F_2(\alpha, \gamma, \mu, \varphi) = \mathrm{Im} F(\alpha, \gamma, \mu + i\varphi), \end{cases}$$

Calculating the determinant $\det \mathcal{M}$, where

$$\mathcal{M} = \begin{bmatrix} \frac{\partial F_1}{\partial \alpha} & \frac{\partial F_1}{\partial \gamma} \\ \frac{\partial F_2}{\partial \alpha} & \frac{\partial F_2}{\partial \gamma} \end{bmatrix}_{\mu=0},$$

and noting $\mu = 0$ implies $x = 0$, we find

$$\det \mathcal{M} = \frac{1 - \cos m\varphi}{-m^3 y^3},$$

where y and φ satisfy (2.18). For $y > 0$, we have that $\det M < 0$, and for $y < 0$, we have $\det M > 0$. We now apply Proposition 2.13 in Diekmann et al. [6, Chapter XI]. If we cross a curve $\bar{C}_k(\beta)$ to the left with "left" defined w.r.t. the direction of increasing φ-coordinate for $\varphi > 0$, the critical roots $\pm i\varphi$ enter the right half plane, i.e. the number of roots ξ with $|\xi| > 1$ increases by 2. On the other hand, from a result in [10], one has that the scheme is not asymptotically stable for any $\alpha \geq -\beta$, $\gamma = 0$. By continuity arguments, we may conclude that the area $\alpha + \beta + \gamma \geq 0$ does not belong to the stability region. This completes the proof. □

Remark 2.4. Specializing Theorems 2.11 and 2.12 to the case $\gamma = 0$, one regains the stability result for Gauss methods as obtained by Guglielmi and Hairer in [10].

Remark 2.5. Set $z(t) = \int_{t-\tau}^{t} y(\nu) d\nu$. Then, (2.1) can be transformed into a system of delay differential equations

$$\begin{bmatrix} y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} \alpha & \gamma \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} \beta & 0 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} y(t-1) \\ z(t-1) \end{bmatrix}. \tag{2.21}$$

This equation is unstable in our sense because the characteristic equation given by

$$\det \begin{bmatrix} \lambda - \alpha - \beta \exp(-\lambda) & -\gamma \\ -(1 - \exp(-\lambda)) & \lambda \end{bmatrix} = 0,$$

always has a root $\lambda = 0$. In addition, we are not aware of any positive results on *delay dependent* stability of numerical methods for general multi-dimensional systems of delay equations that would cover equation (2.21). Some negative results, i.e., showing instability, have been obtained (cf. [9, 19]).

3. Delay integral equations

In this section, we study the stability of Runge-Kutta methods for Volterra delay integral equations of the second kind

$$y(t) = (\alpha + i\beta)\int_{t-1}^{t} y(\nu)d\nu,\ t > 0, \tag{3.1}$$

where $\alpha, \beta \in \mathbb{R}$ and $y(t) = \psi(t), t \in [-1, 0]$. The characteristic equation is given by

$$1 - (\alpha + i\beta)\int_{-1}^{0} \exp(\lambda\nu)d\nu = 0.$$

Baker and Ford [1] have studied the analytical stability region S_I of this integral equation:

$$S_I = \{(\alpha, \beta) : \alpha < \beta \cot \beta, \beta \in (-\pi, \pi)\},$$

Hence,

$$\partial S_I = \{(\alpha, \beta) : \alpha = \beta \cot \beta, \beta \in (-\pi, \pi)\}.$$

In Fig. 4 we have drawn the boundary ∂S_I together with the curves separating regions with different numbers of roots in the right half plane.

Applying a Runge-Kutta method (A, b, c) of Pouzet type with constant stepsize $h = 1/m$ to (3.1), we find

$$Y_i^{(n)} = h(\alpha + i\beta)[\sum_{j=1}^{s} a_{ij}Y_j^{(n)} + \sum_{k=1}^{m}\sum_{j=1}^{s} b_jY_j^{(n-k)} - \sum_{j=1}^{s} a_{ij}Y_j^{(n-m)}], \quad i = 1, \ldots, s, \tag{3.2}$$

$$y_{n+1} = h(\alpha + i\beta)\sum_{k=0}^{m-1}\sum_{j=1}^{s} b_jY_j^{(n-k)}, \tag{3.3}$$

where $t_n = nh, y_n$ and $Y_j^{(n)}$ are approximations to $y(t_n)$ and $y(t_n + c_jh)$, respectively. Since (3.2) is independent of y_{n+1}, the characteristic equation is given by

$$\det(I - \frac{\alpha + i\beta}{m}(A(1 - \xi^{-m}) + eb^T\sum_{k=1}^{m}\xi^{-k})) = 0. \tag{3.4}$$

Hence, the stability region S_m^I of (3.2)-(3.3) for any fixed m is given by

$$S_m^I = \{(\alpha, \beta) :\ \text{all roots of (3.4) satisfy } |\xi| < 1\}.$$

In order to determine S_m^I, we use the boundary locus technique again. We introduce the notation

$$U_m^I = \{(\alpha, \beta) :\ \text{(3.4) has at least a root } \xi \text{ with } |\xi| = 1\ \}.$$

When $\xi = 1$, (3.4) implies $\alpha = 1$ and $\beta = 0$. In the case $\xi \neq 1$, (3.4) is equivalent to

$$\det(I - \frac{\alpha + i\beta}{m}(1 - \xi^{-m})(A + \frac{1}{\xi - 1}eb^T)) = 0. \tag{3.5}$$

Let $\xi = \exp(i\varphi)$ and let $1/(x+iy)$ be an eigenvalue of matrix $A + \frac{1}{\xi-1}eb^T$. Then

$$(\alpha + i\beta)(1 - \cos m\varphi + i \sin m\varphi) = m(x + iy).$$

Separating real and imaginary parts, we find two real equations

$$\begin{cases} \alpha(1-\cos m\varphi) - \beta \sin m\varphi = mx, \\ \alpha \sin m\varphi + \beta(1-\cos m\varphi) = my. \end{cases}$$

Solving them for α, β, we have

$$\bar{\alpha}(\varphi) = \frac{my \sin m\varphi}{2(1-\cos m\varphi)} + \frac{1}{2}mx, \tag{3.6}$$

$$\bar{\beta}(\varphi) = \frac{1}{2}my - \frac{mx \sin m\varphi}{2(1-\cos m\varphi)}. \tag{3.7}$$

Here, we also use $(\bar{\alpha}, \bar{\beta})$ to denote the numerical boundary locus. We remark that $\bar{\alpha}(\varphi)$ is even and $\bar{\beta}(\varphi)$ is odd.

Baker and Ford [1] have proved that the implicit Euler rule and the trapezium rule preserve the asymptotic stability of (3.1). It is natural to wonder whether all A-stable methods possess this character. Similarly to the ideas in [10], we study the necessary condition for $S_I \subset S_m^I$ close to the point $(\alpha, \beta) = (1, 0)$, which corresponds to $(x, y) = (0, 0)$ and $\varphi = 0$. From Lemma 2.5, it follows that

$$\exp(i\varphi) = R(x + iy). \tag{3.8}$$

Assume that $R(z)$ satisfies

$$R(z) = \exp(z)(1 - Cz^{p+1} - Dz^{p+2} - \dots),$$

where $C \neq 0$ is the error constant of the approximation. Guglielmi and Hairer [10] parametrized the local trajectory defined by (3.8) close to $(x, y) = (0, 0)$ and $\varphi = 0$ as a function of y (assume $p > 1$). We recall this result:

$$x(y) = \begin{cases} (-1)^{k+1}Dy^{2k+2} + \dots & \text{if } p = 2k \text{ is even} \\ (-1)^k Cy^{2k} + \dots & \text{if } p = 2k-1 \text{ is odd,} \end{cases} \tag{3.9}$$

$$\varphi(y) - y = \begin{cases} (-1)^{k+1}Cy^{2k+1} + \dots & \text{if } p = 2k \text{ is even} \\ (-1)^{k+1}Dy^{2k+1} + \dots & \text{if } p = 2k-1 \text{ is odd.} \end{cases} \tag{3.10}$$

Theorem 3.1. *Assume $p > 1$. If, close to $(\alpha, \beta) = (1, 0)$, we have $S_I \subset S_m^I$ for every $m \geq 1$, then it holds*

$(-1)^k C > 0$ *if* $p = 2k$

$(-1)^k C > 0$ *and* $(-1)^{k+1} D < (-1)^k C/6$ *if* $p = 2k - 1$

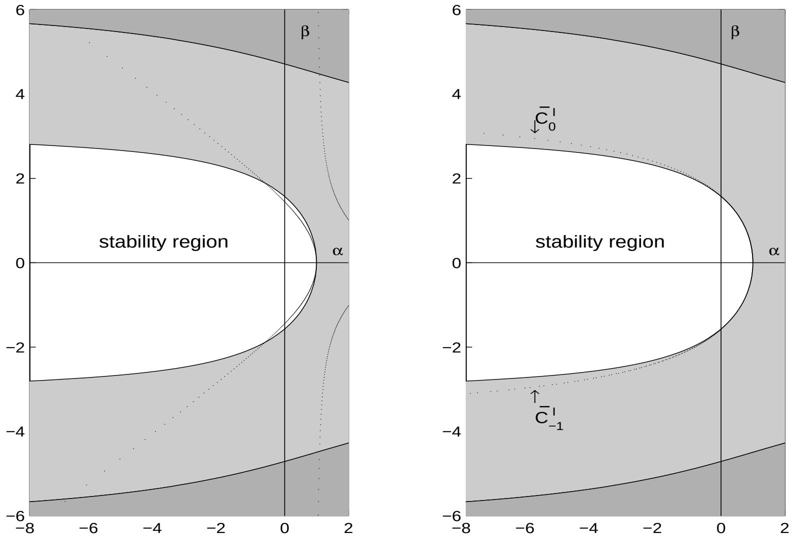

FIGURE 4. Analytical stability region (solid) vs numerical stability region (dotted). Left: the 2-stage Lobatto IIIC method for $m = 2$, right: the 3-stage Gauss method for $m = 1$. The analytical stability region is the white area. The gray regions are regions with different numbers of roots in the right half plane. The numerical stability region is bounded to the right by the dotted line.

Proof. Using a Taylor expansion, it follows from (3.6) and (3.7) that

$$\begin{aligned}\bar{\alpha} &= \bar{\beta}\cot\bar{\beta} + \frac{1}{2}m(y-\varphi)\cot\frac{1}{2}m\varphi + [\frac{1}{2}m\varphi\cot\frac{1}{2}m\varphi - \bar{\beta}\cot\bar{\beta}] + \frac{1}{2}mx \\ &= \bar{\beta}\cot\bar{\beta} + (y-\varphi)\varphi^{-1}(1+O(\varphi^2)) + \frac{1}{2}mx \\ &\quad -\frac{\sin\frac{1}{2}m\varphi\cos\frac{1}{2}m\varphi - \frac{1}{2}m\varphi}{\sin^2\frac{1}{2}m\varphi}(\bar{\beta} - \frac{1}{2}m\varphi) + O(\varphi^{2k+2} + \varphi^{-2}x^2) \\ &= \bar{\beta}\cot\bar{\beta} + (y-\varphi)\varphi^{-1} + \frac{1}{6}mx + O(\varphi^{2k+2} + \varphi^{-2}x^2),\end{aligned}$$

which, combined with (3.9) and (3.10), gives the conclusion of the theorem. □

Using the knowledge of Padé-approximations, one can see that the Gauss and Radau methods satisfy the necessary condition given by Theorem 3.1, but the Lobatto IIIC methods do not (cf. [10]). To illustrate the latter, we plot the boundary of the stability region of the 2-stage Lobatto IIIC method in Fig.4 (left picture). One can see that the analytical stability is not a subset of the numerical stability region.

In the following, we consider the Gauss methods. Because all the eigenvalues of matrix $A+\frac{1}{\xi-1}eb^T$ lie on the imaginary axis, we have $x=0$. For the numerical stability region, we find the following expression:

$$U_m^I=\bigcup_{k=-\lfloor\frac{ms+1}{2}\rfloor}^{\lfloor\frac{ms-1}{2}\rfloor}\bar{C}_k^I\bigcup\{(1,0)\},$$

where

$$\bar{C}_k^I=\{(\bar{\alpha}(\varphi),\bar{\beta}(\varphi)):\ \varphi\in(\frac{2k\pi}{m},\frac{2(k+1)\pi}{m})\},$$
$$\bar{\alpha}(\varphi)=\frac{my\sin m\varphi}{2(1-\cos m\varphi)},\quad \bar{\beta}(\varphi)=\frac{my}{2},$$

y,φ satisfy (2.18), and if ms is odd, we define

$$\bar{C}^I_{\lfloor\frac{ms-1}{2}\rfloor}=\{(\bar{\alpha}(\varphi),\bar{\beta}(\varphi));\ \varphi\in(\frac{(ms-1)\pi}{m},s\pi)\},$$
$$\bar{C}^I_{-\lfloor\frac{ms+1}{2}\rfloor}=\{(\bar{\alpha}(\varphi),\bar{\beta}(\varphi));\ \varphi\in(-s\pi,-\frac{(ms-1)\pi}{m})\}.$$

$\bar{C}_0^I$ and $\bar{C}_{-1}^I$ start at $(1,0)$ and approximate ∂S_I. The other curves separate regions with different numbers of roots outside the unit disk. As an illustration, we draw the boundary of the stability region of the 3-stage Gauss method in Fig.4 (right picture). When $m>1$, the other curves are very similar to the ones of the continuous case in Fig. 4.

Theorem 3.2. *For all Gauss methods, the stability region S_m^I is bounded above by the curve $\bar{C}_0^I$ and below by the curve $\bar{C}_{-1}^I$.*

Proof. By Lemma 2.6, $\varphi(y)$ is strictly monotonically increasing. Hence, the curves $\bar{C}_k^I$ do not intersect. $\bar{C}_0^I$ and $\bar{C}_{-1}^I$ are joined at the following limit point:

$$\lim_{\varphi\to 0}(\bar{\alpha}(\varphi),\bar{\beta}(\varphi))=(1,0).$$

Also, the curves $\bar{C}_k^I$ are ordered according to k. By continuity arguments, the number of roots outside the unit disk is constant in each region separated by the curves. Next we prove that the critical root lying on the unit disk will move outside the unit disk if we cross a curve $\bar{C}_k^I$ to the right in the parameter space. To this end, we consider the characteristic equation (3.5) again. Let $\xi=\exp(\mu+i\varphi)$ and let $1/(x+iy)$ be an eigenvalue of matrix $A+\frac{1}{\xi-1}eb^T$. This yields

$$1-\frac{\alpha+i\beta}{m(x+iy)}(1-\exp(-m(\mu+i\varphi)))=0.$$

We denote the left-hand side by $F^I(\alpha,\beta,\mu+i\varphi)$. Define

$$\begin{cases}F_1^I(\alpha,\beta,\mu,\varphi)=\mathrm{Re}F^I(\alpha,\beta,\mu+i\varphi),\\ F_2^I(\alpha,\beta,\mu,\varphi)=\mathrm{Im}F^I(\alpha,\beta,\mu+i\varphi).\end{cases}$$

Calculating the determinant $\det \mathcal{M}_I$, where

$$\mathcal{M}_I = \begin{bmatrix} \frac{\partial F_1^I}{\partial \alpha} & \frac{\partial F_1^I}{\partial \beta} \\ \frac{\partial F_2^I}{\partial \alpha} & \frac{\partial F_2^I}{\partial \beta} \end{bmatrix}_{\mu=0},$$

and noting $\mu = 0$ implies $x = 0$, we have

$$\det \mathcal{M}_I = \frac{2(1 - \cos m\varphi)}{m^2 y^2} > 0,$$

where y and φ satisfy (2.18). Applying again Proposition 2.13 in Diekmann et al. [6, Chapter XI], we conclude that the critical root moves outside the unit disk for parameter sets in the (α, β) parameter region to the right of the curve $\bar{C}_k^I$ when we follow this curve in the direction of increasing φ. On the other hand, consider the reference point $(\alpha, \beta) = (-1, 0)$. It is easily verified by Lemma 2.5 that, for this reference point, any $\xi \in \mathbb{C}$ with $|\xi| \geq 1$ is not a solution to (3.5). This completes the proof. □

Theorem 3.3. *All Gauss methods satisfy* $S_I \subset \bigcap_{m=1}^{\infty} S_m^I$.

Proof. By the symmetry of $\bar{C}_0^I$ vs $\bar{C}_{-1}^I$, it is sufficient to prove that the curve $\bar{C}_0^I$ lies above ∂S_I. In fact, for any $\beta = \bar{\beta}(\varphi)$, we have $\beta = my/2$, which gives

$$\alpha(\beta) = \frac{my \sin my}{2(1 - \cos my)} < \frac{my \sin m\varphi}{2(1 - \cos m\varphi)} = \bar{\alpha}(\varphi),$$

where we have also used the monotonicity of function $\frac{\sin x}{1-\cos x}$ and the fact that $\varphi(y) < y$ (Lemma 2.6). This completes the proof. □

Remark 3.1. By differentiation, (3.1) reduces to a delay differential equation with a discrete delay

$$y'(t) = (\alpha + i\beta)(y(t) - y(t-1)). \tag{3.11}$$

This equation is unstable in our sense because the characteristic equation given by

$$\lambda - (\alpha + i\beta)(1 - \exp(-\lambda)) = 0,$$

always has a root $\lambda = 0$. In addition, it is known that any symmetric method (including the Gauss methods) is not τ-stable, i.e. the method can not completely retain the asymptotic stability of delay differential equations with complex coefficients (cf. [11]). Hence, Theorem 3.3 is not covered by the known results.

Remark 3.2. For the case of a multi-dimensional system of the form

$$y(t) = A_0 \int_{t-1}^{t} y(\nu) d\nu,$$

where $A_0 \in \mathbb{C}^{d \times d}$, the sufficient and necessary condition on asymptotic stability of the equation is given by $\sigma(A_0) \subset S_I$. Under this condition, the corresponding numerical solution based on the Gauss methods is asymptotically stable too.

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