On the stability of normalized Powell–Sabin B–splines

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Report TW 359, May 2003

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Abstract

In this paper we show that the normalized Powell–Sabin B–splines form a stable basis for the max norm. The approximation constants depend only on the smallest angle in the underlying triangulation. Because the B–splines refer to the size of the Powell–Sabin triangles, we have that small Powell–Sabin triangles correspond to better approximation constants than big Powell–Sabin triangles. Next, in addition to the max norm, we treat the $L_p$ norm. Here the approximation constants depend also on a fraction proper to the triangulation. Finally, as a special case, we consider the B–spline bases obtained from Powell–Sabin triangles with minimal area and pay extra attention to the approximation constants for the max norm.

Keywords: Powell–Sabin splines, stable bases, approximation
AMS(MOS) Classification: 41A15, 65D07
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Abstract

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1 Introduction

Let $\Delta$ be a triangulation of a subset $\Omega \in \mathbb{R}^2$ with polygonal boundary $\partial \Omega$. The polynomial spline space $S^r_d(\Delta)$ is defined as

$$S^r_d(\Delta) := \{ s \in C^r(\Omega) : s|_T \in \Pi_d \text{ for all } T \in \Delta \}$$

where $d > r \geq 0$ are given integers and $\Pi_d$ is the linear space of bivariate polynomials of degree $\leq d$. A basis $\{B_i\}_{i=1}^n$ for a spline space $S^r_d$ which satisfies

$$k_1||c||_\infty \leq \left| \sum_{i=1}^n c_i B_i \right|_\infty \leq k_2||c||_\infty$$

for all choices of the coefficient vector $c$, is called a stable basis. Here $k_1$ and $k_2$ are constants which depend only on the smallest angle in $\Delta$.

Finding stable bases for spline spaces $S^r_d(\Delta)$ is a non trivial task for $r > 0$, and can only be done for general triangulations $\Delta$ when $d \geq 3r + 2$ [4]. In this paper we study $C^1$ continuous piecewise quadratic splines, with $r = 1$ and $d = 2$. Because there exists no solution for general triangulations, we restrict ourselves to Powell–Sabin (PS) refinements $\Delta^*$ of $\Delta$. The corresponding splines are called Powell–Sabin splines. They appear to be very valuable for CAGD applications [10]. Dierckx [1] proposed a stable algorithm to construct a normalized B–spline representation for such a spline space $S^1_2(\Delta^*)$.

In this paper we prove that the normalized B–spline basis for Powell–Sabin splines is a stable basis. We follow a similar approach as in [5], where it is proven that the Bernstein polynomials of degree $d$ on a triangle $T$ form a stable basis for $\Pi_d$. Related work has been done for a Hermite basis for quadratic splines. Upper bounds were derived for the Hermite basis functions and for their first derivatives [8].
2 Powell–Sabin splines

2.1 Polynomials on triangles

Consider a non-degenerated triangle $T(V_1, V_2, V_3)$ in a plane, having vertices $V_i$ with Cartesian coordinates $(x_i, y_i)$, $i = 1, 2, 3$. This triangle will be denoted as the domain triangle. We define the barycentric coordinates $\tau = (\tau_1, \tau_2, \tau_3)$ of an arbitrary point $(x, y) \in \mathbb{R}^2$ with respect to $T$ as the unique solution to the system

$$
\begin{bmatrix}
x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3 \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
\tau_1 \\
\tau_2 \\
\tau_3
\end{bmatrix}
=
\begin{bmatrix}
x \\
y \\
1
\end{bmatrix}.
$$

(3)

Each polynomial $P_d(x, y) \in \Pi_d$ on $T$ has a unique representation

$$
P_d(x, y) := b^d_T(\tau) = \sum_{|\lambda|=d} b_\lambda B_\lambda^d(\tau),
$$

(4)

with $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, $\lambda_i \geq 0$ a multi-index of length $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3 = d$, and

$$
B_\lambda^d(\tau) = \frac{d!}{\lambda_1! \lambda_2! \lambda_3!} \tau_1^{\lambda_1} \tau_2^{\lambda_2} \tau_3^{\lambda_3}
$$

(5)

the Bernstein–Bézier polynomials on the triangle [2].

![Positions of the Bézier ordinates for $d = 2$.](image)

The coefficients $b_\lambda$ are called the Bézier ordinates. By associating each ordinate $b_\lambda$ with the Bézier domain point $(\frac{\lambda_1}{d}, \frac{\lambda_2}{d}, \frac{\lambda_3}{d})$ in the triangle $T$ we can display this Bernstein–Bézier representation schematically, as in figure 1.

2.2 The linear space $S^1_2(\Delta^*)$

Consider a simply connected subset $\Omega \subset \mathbb{R}^2$ with polygonal boundary $\delta \Omega$. Suppose we have a conforming triangulation $\Delta$ of $\Omega$, being constituted of triangles $T_j$, $j = 1, \ldots, t$, and having
vertices $V_i$ with Cartesian coordinates $(x_i, y_i)$, $i = 1, \ldots, n$. The Powell-Sabin refinement $\Delta^*$ of $\Delta$ divides each triangle $T_j$ into six smaller triangles with a common vertex. It can be constructed as follows (see figure 2):

![Figure 2: A PS-refinement $\Delta^*$.](image)

1. Choose an interior point $Z_j$ for each triangle $T_j$, so that if two triangles $T_i$ and $T_j$ have a common edge, the line joining $Z_i$ and $Z_j$ intersects this common edge at a point $R_{ij}$ between its vertices. We will choose $Z_j$ as the incenter of triangle $T_j$.

2. Join the points $Z_j$ to the vertices of $T_j$.

3. For each edge of $T_j$
   - which belongs to the boundary $\delta \Omega$, join $Z_j$ to any point on this edge.
   - which is common to a triangle $T_i$, join $Z_j$ to $R_{ij}$.

Now we consider the space of piecewise quadratic $C^1$ continuous polynomials on $\Omega$, the Powell-Sabin splines. This space is denoted by $S_2^1(\Delta^*)$. Each of the 6$t$ triangles resulting from the PS-refinement becomes the domain triangle of a quadratic Bernstein-Bézier polynomial, i.e. we choose $d = 2$ in equation (4) and (5), as indicated for one subtriangle in figure 2. Powell and Sabin [7] showed that the following interpolation problem:

$$s(V_k) = f_k, \quad \frac{\partial s}{\partial x}(V_k) = f_{xk}, \quad \frac{\partial s}{\partial y}(V_k) = f_{yk}, \quad k = 1, \ldots, n$$

has a unique solution $s(x, y)$ in $S_2^1(\Delta^*)$. Hence, the dimension of the space $S_2^1(\Delta^*)$ equals $3n$.

### 2.3 A normalized B–spline representation

Dierckx [1] presented a normalized B–spline representation for Powell–Sabin splines

$$s(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{3} c_{ij} B_i^j(x, y), \quad (x, y) \in \Omega$$

where the B–splines form a convex partition of unity on $\Omega$, i.e.

$$B_i^j(x, y) \geq 0 \text{ for all } x, y \in \Omega,$$
\[
\sum_{i=1}^{n} \sum_{j=1}^{3} B_{ij}^3(x, y) = 1 \text{ for all } x, y \in \Omega.
\]  

Furthermore these basis functions have local support: \( B_{ij}^3(x, y) \) vanishes outside the so-called molecule \( M_i \) of vertex \( V_i \), which is the union of all triangles \( T_k \) containing \( V_i \). The molecule number \( m_i \) is defined as the number of triangles in the molecule \( M_i \).

The basis functions \( B_{ij}^3(x, y) \) can be obtained as follows: find three linearly independent sets \( (\alpha_{ij}, \beta_{ij}, \gamma_{ij}) \), \( j = 1, 2, 3 \) for each vertex \( V_i \). \( B_{ij}^3(x, y) \) is the unique solution of the interpolation problem (6) with \( (f_k, f_{ek}, f_{yk}) = (\delta_{ki}\alpha_{ij}, \delta_{ki}\beta_{ij}, \delta_{ki}\gamma_{ij}) \), where \( \delta_{ki} \) is the Kronecker delta.

The sets \( (\alpha_{ij}, \beta_{ij}, \gamma_{ij}) \), \( j = 1, 2, 3 \) must be determined in such a way that equations (8) and (9) are satisfied. To find appropriate sets \( (\alpha_{ij}, \beta_{ij}, \gamma_{ij}) \), \( j = 1, 2, 3 \) we use the algorithm from [1].

1. For each vertex \( V_i \in \Delta \), find its PS-points. This is a number of particular surrounding Bézier domain points and the vertex \( V_i \) itself. Figure 3 shows the PS-points \( S, \tilde{S}, S' \) and \( V_i \) for the vertex \( V_i \) in the triangle \( T(V_1, V_2, V_3) \).

2. For each vertex \( V_i \), find a triangle \( t_i(Q_{i1}, Q_{i2}, Q_{i3}) \) which contains all the PS-points of \( V_i \) from all the triangles \( T_k \) in the molecule \( M_i \). These triangles \( t_i \), \( i = 1, \ldots, n \) are called PS-triangles and we denote their vertices with \( Q_{ij} \) \( (X_{ij}, Y_{ij}) \). Figure 3 also shows such a PS-triangle \( t_i \).

3. Three linearly independent triplets of real numbers \( (\alpha_{ij}, \beta_{ij}, \gamma_{ij}) \), \( j = 1, 2, 3 \) can be derived from the PS-triangle \( t_i \) of a vertex \( V_i \) as follows:

\[
\alpha_i = (\alpha_{i1}, \alpha_{i2}, \alpha_{i3}) \text{ are the barycentric coordinates of } V_i \text{ with respect to } t_i,
\]

\[
\beta_i = (\beta_{i1}, \beta_{i2}, \beta_{i3}) = \left( \frac{Y_{i2} - Y_{i3}}{f}, \frac{Y_{i3} - Y_{i1}}{f}, \frac{Y_{i1} - Y_{i2}}{f} \right),
\]

\[
\gamma_i = (\gamma_{i1}, \gamma_{i2}, \gamma_{i3}) = \left( \frac{X_{i3} - X_{i2}}{f}, \frac{X_{i1} - X_{i3}}{f}, \frac{X_{i2} - X_{i1}}{f} \right),
\]

where

\[
f = \begin{vmatrix}
X_0 & Y_0 & 1 \\
X_2 & Y_2 & 1 \\
X_3 & Y_3 & 1
\end{vmatrix}.
\]

We have \( |\alpha_i| = 1 \) and \( |\beta_i| = |\gamma_i| = 0 \).

A useful consequence is the notion of control triangles. First, we define the PS-control points as

\[
C_{ij}(X_{ij}, Y_{ij}, c_i, c_j).
\]

For fixed \( i \), they constitute a triangle \( T_i(C_{i1}, C_{i2}, C_{i3}) \) that is tangent to the surface at \( (V_i, s(V_i)) \). The projection of the control triangles \( T_i \) in the \( (x, y) \) plane are the PS-triangles \( t_i \). The area of a PS-triangle \( t_i \) equals

\[
A_{t_i(Q_{i1}, Q_{i2}, Q_{i3})} = \frac{1}{2|\beta_{i1}\gamma_{i2} - \gamma_{i1}\beta_{i2}|} = \frac{|f|}{2}
\]

(11)
3 Properties of triangulations and Powell–Sabin refinements

In this section we introduce some useful notation which will be used throughout the remainder of this text, and we collect several properties needed later. Suppose $\mathcal{T}$ is a triangle, then

$$\begin{align*}
|\mathcal{T}| & := \text{the diameter of the smallest disk containing } \mathcal{T}, \\
\rho_T & := \text{the radius of the largest disk contained in } \mathcal{T}, \\
\theta_T & := \text{the smallest angle in the triangle } \mathcal{T}, \\
A_T & := \text{the area of the triangle } \mathcal{T}.
\end{align*}$$

Consider a triangulation $\Delta$ of a subset $\Omega \in \mathbb{R}^2$ and its PS-refinement $\Delta^\ast$. Denote the PS-refinement of triangle $\mathcal{T} \in \Delta$ as $\mathcal{T}^\ast$. We define

$$\begin{align*}
|\mathcal{T}^\ast| & := \min_{T_{PS} \in \mathcal{T}^\ast} |T_{PS}|, \\
\rho_{T^\ast} & := \min_{T_{PS} \in \mathcal{T}^\ast} \rho_{T_{PS}}, \\
\theta_{T^\ast} & := \min_{T_{PS} \in \mathcal{T}^\ast} \theta_{T_{PS}}, \\
\theta_\Delta & := \text{the smallest angle in the triangulation } \Delta, \\
\theta_{\Delta^\ast} & := \text{the smallest angle in the PS-refinement } \Delta^\ast, \\
A_\Omega & := \text{the area of } \Omega.
\end{align*}$$

The following lemmas give estimates of the above quantities.

**Lemma 3.1** Consider a triangle $\mathcal{T}$. Then

$$\frac{|\mathcal{T}|}{\rho_T} \leq \frac{4}{\tan(\theta_T/2)}. $$

**Proof** It is well-known that

$$\rho_T = \tan(\theta_T/2) \cdot \frac{a + b - c}{2},$$

with $a, b$ and $c$ the side lengths of the triangle. Side length $c$ corresponds to the side opposite to the angle $\theta_T$, and thus has the smallest value. Denote the longest edge of $\mathcal{T}$ with $e_{\text{max}}$, then the following inequalities hold:

$$\frac{2}{\tan(\theta_T/2)} = \frac{a + b - c}{\rho_T} \geq \frac{|e_{\text{max}}|}{\rho_T} \geq \frac{|\mathcal{T}|/2}{\rho_T}.$$
The following lemma is due to Lai and Schumaker [6].

**Lemma 3.2** Suppose $\Delta^*$ is the Powell–Sabin refinement of a given triangulation $\Delta$. Then $\theta_{\Delta^*} \geq \theta_{\Delta} \sin(\theta_{\Delta})/4$.

**Proof** See [6]. □

**Lemma 3.3** Suppose $T$ is a triangle in $\Delta$ with PS-refinement $T^*$. Denote the longest edge in the PS-refinement $T^*$ of the triangle $T$ as $e_{\text{max}}$. Then

$$\frac{1}{\rho_{T^*}} \leq \frac{4}{\sin(\theta_{T^*}) \tan(\theta_{T^*}/2)|e_{\text{max}}|}.$$  

**Proof** Let $e$ and $\bar{e}$ be two edges of the same triangle $T_{PS} \in T^*$. Then

$$\sin(\theta_{T^*})|e| \leq |\bar{e}|.$$  \hfill (12)  

Suppose we want to compare two arbitrary edges $e_1$ and $e_2$ in $T^*$. Then there always exists a series of edges in $T^*$ such that

$$|e_1| \leq \left(\frac{1}{\sin(\theta_{T^*})}\right)^4 |e_2|.$$  \hfill (13)  

Evidently this equation also holds for the maximum and minimum edge. By Lemma 3.1 the following holds:

$$\frac{|e_{\text{min}}|}{\rho_{T^*}} \leq \frac{|T^*|}{\rho_{T^*}} \leq \frac{4}{\tan(\theta_{T^*}/2)}.\hfill (14)$$

Substitute (13) in (14) to prove the lemma. □

**Lemma 3.4** Consider two triangles $T_1$ and $T_2$ in $\Delta$ with a common edge. Denote the longest edge in the PS-refinement $T_1^*$ of the triangle $T_1$ as $e_{\text{max}}(T_1^*)$ and the longest edge in the PS-refinement $T_2^*$ of the triangle $T_2$ as $e_{\text{max}}(T_2^*)$. Then

$$\sin(\theta_{\Delta^*})^4 \leq \frac{|e_{\text{max}}(T_2^*)|}{|e_{\text{max}}(T_1^*)|} \leq \left(\frac{1}{\sin(\theta_{\Delta^*})}\right)^4.$$  

**Proof** From equation (13) we find that

$$|e_{\text{max}}(T_1^*)| \leq \left(\frac{1}{\sin(\theta_{\Delta^*})}\right)^4 |\bar{e}|$$  

$$\leq \left(\frac{1}{\sin(\theta_{\Delta^*})}\right)^4 |e_{\text{max}}(T_2^*)|$$  \hfill (15)  

and likewise

$$|e_{\text{max}}(T_2^*)| \leq \left(\frac{1}{\sin(\theta_{\Delta^*})}\right)^4 |e_{\text{max}}(T_1^*)|$$  \hfill (16)  

with $\bar{e}$ a common edge of $T_1^*$ and $T_2^*$. Combining equation (15) and (16) yields the result. □
4 Stability for the max norm

We will now prove that the basis functions \( B_j^i(x, y) \) for \( S_3^1(\Delta^*) \), introduced in section 2.3, form a stable basis for \( S_3^1(\Delta^*) \), i.e. that there exist constants \( k_1 \) and \( k_2 \) such that for all choices of the coefficient vector \( c \)

\[
k_1 \|c\|_\infty \leq \left\| \sum_{i=1}^{n} \sum_{j=1}^{3} c_{i,j} B_j^i(x, y) \right\|_\infty \leq k_2 \|c\|_\infty
\]

with \( \|c\|_\infty := \max_{i,j} |c_{i,j}| \) and \( \|s\|_\infty := \max_{x,y} |s(x, y)| \).

Before we prove the main theorem, we introduce two lemmas. The first lemma (Lemma 4.1) gives an upper bound for \( \|D z s(x, y)\|_{\infty, TPS} \) and \( \|D y s(x, y)\|_{\infty, TPS} \), where \( \|D z s\|_{\infty, TPS} := \max_{TPS} |D z s(x, y)| \) and \( TPS \) is a triangle in the PS-refinement \( \Delta^* \). This upper bound will be useful in the proof of Theorem 4.3.

**Lemma 4.1** Suppose \( s(x, y) \in S_3^1(\Delta^*) \). Consider a triangle \( TPS \) of the PS-refinement \( \Delta^* \) of \( \Delta \). Then

\[
\|D z s(x, y)\|_{\infty, TPS} \leq \frac{12}{\rho_{TPS}} \|s(x, y)\|_{\infty, TPS},
\]

and

\[
\|D y s(x, y)\|_{\infty, TPS} \leq \frac{12}{\rho_{TPS}} \|s(x, y)\|_{\infty, TPS}.
\]

**Proof** We can write \( s(x, y)|_{TPS} \) in its unique Bézier representation:

\[
s(x, y)|_{TPS} := s(\tau) = \sum_{|\lambda|=2} b_\lambda B^2_\lambda(\tau).
\]

Denote the vertices of \( TPS \) as \( V_i(x_i, y_i), i = 1, 2, 3 \). Let \( u = V_2 - V_1 = (x_2 - x_1, y_2 - y_1) \) and \( v = V_3 - V_1 = (x_3 - x_1, y_3 - y_1) \) define two vectors. Then the derivatives of \( s(x, y)|_{TPS} \) with respect to \( u \) respectively \( v \) are given by

\[
D_u s(\tau) = (x_2 - x_1) D z s(\tau) + (y_2 - y_1) D y s(\tau),
\]

\[
D_v s(\tau) = (x_3 - x_1) D z s(\tau) + (y_3 - y_1) D y s(\tau).
\]

Solving for \( D z s(\tau) \) and \( D y s(\tau) \) gives

\[
D z s(\tau) = \frac{(y_3 - y_1) D_u s(\tau) - (y_2 - y_1) D_v s(\tau)}{f},
\]

\[
D y s(\tau) = \frac{(x_2 - x_1) D_u s(\tau) - (x_3 - x_1) D_v s(\tau)}{f},
\]

from which we find that

\[
\|D z s(\tau)\|_\infty \leq \frac{|y_3 - y_1|}{2A_{TPS}} \|D u s(\tau)\|_\infty + \frac{|y_2 - y_1|}{2A_{TPS}} \|D v s(\tau)\|_\infty.
\]

The area \( A_{TPS} \) is bounded below by

\[
\rho_{TPS} |y_3 - y_1| \leq A_{TPS}, \quad \rho_{TPS} |y_2 - y_1| \leq A_{TPS}.
\]

Substituting in the previous equation gives

\[
\|D z s(\tau)\|_\infty \leq \frac{1}{2\rho_{TPS}} (\|D u s(\tau)\|_\infty + \|D v s(\tau)\|_\infty).
\]
The estimate for $\|D_y s(\tau)\|_\infty$ can be established in the same way. Vector $u$ has barycentric coordinates (-1,1,0). The derivative of $s(\tau)$ with respect to $u$ is given by [3]

$$D_u s(\tau) = 2 \sum_{|\lambda| = 1} (-b_{\lambda + \epsilon_1} + b_{\lambda + \epsilon_2}) B_\lambda^1(\tau)$$

with $\epsilon_1 = (1,0,0)$ and $\epsilon_2 = (0,1,0)$. We now have

$$\|D_u s(x,y)\|_{\infty,T_{ps}} \leq 2 \sum_{|\lambda| = 1} (2\|b\|_\infty) B_\lambda^1(\tau) = 4\|b\|_\infty.$$  

The same reasoning gives an analogous estimate for $\|D_x s(x,y)\|_{\infty,T_{ps}}$. Combining these two estimates yields

$$\|D_x s(x,y)\|_{\infty,T_{ps}} \leq \frac{4}{\rho_{_{T_{ps}}}} \|b\|_\infty$$

and

$$\|D_y s(x,y)\|_{\infty,T_{ps}} \leq \frac{4}{\rho_{_{T_{ps}}}} \|b\|_\infty.$$  

It suffices to prove that

$$\|b\|_\infty \leq 3\|s(x,y)\|_{\infty,T_{ps}}.$$  

Define

$$\xi := \{ \left( \frac{\lambda_1}{2}, \frac{\lambda_2}{2}, \frac{\lambda_3}{2} \right) | \lambda_1 + \lambda_2 + \lambda_3 = 2, \lambda_i \geq 0 \}$$

as the set of Bézier domain points. Then

$$[s(\xi)]_{6 \times 1} = [B_\lambda^3(\xi)]_{6 \times 6} \cdot [b_\lambda]_{6 \times 1}.$$  

Since interpolation at the Bézier domain points $\xi$ by polynomials in $P_2$ is unique, $[B_\lambda^3(\xi)]_{6 \times 6}$ is invertible, and we find

$$\|b\|_\infty \leq \|[B_\lambda^3(\xi)]_{6 \times 6}^{-1}\| \cdot \|[s(\xi)]_{6 \times 1}\| \leq \|[B_\lambda^3(\xi)]_{6 \times 6}^{-1}\| \cdot \|[s(x,y)]_{\infty,T_{ps}}\|.$$  

It can easily be verified that $\|[B_\lambda^3(\xi)]_{6 \times 6}^{-1}\|_\infty = 3$. □

The second lemma (Lemma 4.2) deals with the choice of the PS-triangles [9]. Recall figure 3. It is clear that there are infinity many choices for the PS-triangle $t_i(Q_{11}, Q_{12}, Q_{13})$, because the only condition for $t_i$ to be a valid PS-triangle is that $t_i$ contains the PS-points $V_i, S, S$ and $S'$. Also in a general situation there are infinity many triangles that form a valid PS-triangle for a vertex $V_i$. The actual choice of a PS-triangle is important, because, as explained in section 2.3, the B–spline basis functions depend on these PS-triangles. As a logic consequence the approximation constants in (2) will be different for another choice of basis functions or another choice of PS-triangles.

Now, suppose we are given a vertex $V_i$ and its surrounding PS-points. Let $C_i$ be the smallest circle with center $V_i$ that contains all the PS-points and denote its radius as $r_i$. It is clear that an equilateral triangle with barycenter $V_i$ and inradius $k r_i$ with $k \geq 1$ is a valid PS-triangle for $V_i$. It is also clear that for every vertex $V_i$ there exists a constant $K_i \geq 1$ such that the actual PS-triangle $t_i$ is contained in such an equilateral triangle with barycenter $V_i$ and inradius $K_i r_i$.

Let $K = \max_i K_i$, then Lemma 4.2 is used in Theorem 4.3 to reduce the dependence of the approximation constants in (2) on the PS-triangles $t_i$ to dependence on the constant $K$. We mention that a scaling operation on the domain does not change the value of $K$.  

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Lemma 4.2 Let an arbitrary vertex $V_i \in \Delta$ be given. Consider the surrounding PS-points of vertex $V_i$ and denote the PS-point with the longest distance to vertex $V_i$ as $S$. Define $C_i$ as the circle with center $V_i$ and radius $K|SV_i|$ where $K \in \mathbb{R}$ is a constant and $K \geq 1$. It is clear that $C_i$ contains all the PS-points. Define $T_S \in \Delta^*$ as either one of the two triangles that contains the PS-point $S$ and define $T_{C_i}$ as the set of equilateral triangles that have $C_i$ as its inscribed circle. Suppose $t_i$ is a valid PS-triangle and suppose $t_i$ is contained in a triangle $T'_{C_i} \in T_{C_i}$. Then

$$\frac{|e_{\text{max}}(t_i)|}{|e_{\text{max}}(T_S)|} \leq \sqrt{3}K. \tag{See figure 4}$$

![Figure 4: Circle $C_i$ and a triangle $T'_{C_i} \in T_{C_i}$ for $K = 1$.](image)

Proof Because $t_i$ is contained in $T'_{C_i}$, it is sufficient to prove that

$$\frac{|e_{\text{max}}(T_{C_i})|}{|e_{\text{max}}(T_S)|} \leq \sqrt{3}K.$$

Clearly,

$$|e_{\text{max}}(T_{C_i})| = 2\sqrt{3}K|SV_i|. \tag{17}$$

We also know that

$$|e_{\text{max}}(T_S)| \geq 2|SV_i|. \tag{18}$$

Combining (17) and (18) proves the lemma. \qed

Now we come to the main theorem of this paper which states that the normalized B-spline basis functions form a stable basis.

Theorem 4.3 Consider a triangulation $\Delta$ of a subset $\Omega \subset \mathbb{R}^2$ with polygonal boundary $\partial \Omega$. Suppose $\Delta$ is constituted of triangles $T_j$, $j = 1, \ldots, t$ which have vertices $V_i$, $i = 1, \ldots, n$. Define $\Delta^*$ as the PS-refinement of $\Delta$. Suppose that there exists a constant $K$ such that every PS-triangle $t_i$ is contained in a triangle $T'_{C_i} \in T_{C_i}$ where $T_{C_i}$ is defined as in Lemma 4.2. Then there exists a constant $K_1$ depending only on $K$ and on the smallest angle in the underlying triangulation such that for all Powell–Sabin splines $s(x, y) \in S^1_4(\Delta^*)$ in their normalized B–spline representation (7),

$$\|s(x, y)\|_\infty \leq \|e\|_\infty \leq K_1\|s(x, y)\|_\infty. \tag{19}$$
Proof The left inequality immediately follows from equation (9). We now establish the right inequality. From (7) and the construction of the B-spline basis functions we have

\[
\begin{align*}
    s(V_i) &= \alpha_1 c_1 + \alpha_2 c_2 + \alpha_3 c_3, \\
    D_x s(V_i) &= \beta_1 c_1 + \beta_2 c_2 + \beta_3 c_3, \\
    D_y s(V_i) &= \gamma_1 c_1 + \gamma_2 c_2 + \gamma_3 c_3,
\end{align*}
\]

or

\[
\begin{bmatrix}
    s(V_i) \\
    D_x s(V_i) \\
    D_y s(V_i)
\end{bmatrix} =
\begin{bmatrix}
    \alpha_1 & \alpha_2 & \alpha_3 \\
    \beta_1 & \beta_2 & \beta_3 \\
    \gamma_1 & \gamma_2 & \gamma_3
\end{bmatrix}
\begin{bmatrix}
    c_1 \\
    c_2 \\
    c_3
\end{bmatrix}.
\]

If we take into account that \(\alpha_3 = 1 - \alpha_1 - \alpha_2\), \(\beta_3 = -\beta_1 - \beta_2\) and \(\gamma_3 = -\gamma_1 - \gamma_2\), then we find that the inverse of \(A\) is equal to

\[
A^{-1} = \begin{bmatrix}
    1 & \eta_{12} & \eta_{13} \\
    \eta_{12} & 1 & \eta_{13} \\
    \eta_{13} & \eta_{13} & 1
\end{bmatrix},
\]

where

\[
\begin{align*}
\eta_{12} &= \frac{\alpha_2 \gamma_1 - \alpha_1 \gamma_2 + \delta_1 \gamma_2 - \delta_2 \gamma_1}{\beta_1 \gamma_2 - \beta_2 \gamma_1}, \\
\eta_{13} &= \frac{\alpha_3 \beta_1 - \alpha_1 \beta_2 - \delta_1 \beta_2 + \delta_2 \beta_1}{\beta_1 \gamma_2 - \beta_2 \gamma_1}.
\end{align*}
\]

Suppose that \(\|c\|_\infty = |c_{ij}|\). Then

\[
\|c\|_\infty = |s(V_i) + \eta_{|ij|} D_x s(V_i) + \xi_{|ij|} D_y s(V_i)|.
\]

Define \(T_S\) as in Lemma 4.2. Lemma 4.1 applied to triangle \(T_S\), together with equation (11), yields

\[
\|c\|_\infty \leq \|s(x, y)\|_{\infty, T_S} \left(1 + |\alpha_2 \gamma_1 - \alpha_1 \gamma_2 + \delta_1 \gamma_2 - \delta_2 \gamma_1| \cdot 2A_t \cdot \frac{12}{\rho T_S} + |\alpha_3 \beta_1 - \alpha_2 \beta_2 - \delta_1 \beta_2 + \delta_2 \beta_1| \cdot 2A_t \cdot \frac{12}{\rho T_S}\right).
\]

If we use the explicit formulas for \(\beta_{ij}\) and \(\gamma_{ij}\), we get

\[
\|c\|_\infty \leq \|s(x, y)\|_{\infty, T_S} \left[1 + \frac{24A_t}{\rho T_S} \left(2|X_{ij} - X_{ij}| + 2|X_{ij} - X_{ij}| + 2|Y_{ij} - Y_{ij}| + 2|Y_{ij} - Y_{ij}| \right)\right].
\]

Here we have also used the fact that \(|\alpha_{ij}| \leq 1\) and \(|\beta_{ij}| \leq 1\). We apply Lemma 3.3 to the equation and find that

\[
\|c\|_\infty \leq \|s(x, y)\|_{\infty, T_S} \left[1 + \frac{48 \cdot 8}{\sin(\theta_\Delta)^4 \tan(\theta_\Delta)} [\max(\tau_{ij})]\right].
\]

By Lemma 4.2 and Lemma 3.2 it follows that

\[
\|s(x, y)\|_{\infty} \leq \|c\|_{\infty} \leq K_1 \|s(x, y)\|_{\infty}
\]

with

\[
K_1 = \left[1 + \frac{384 \sqrt{3} K}{\sin(\theta_\Delta) \sin(\theta_\Delta)} \right].
\]

\(\square\)

The assumption that there exists a constant \(K\) such that every PS-triangle \(t_i\) is contained in a triangle \(T\) is equivalent with the statement that the area of the PS-triangles has to be bounded. As a consequence we remark that the smaller the PS-triangles the better the approximation constants.
5 Stability for the $L_p$ norm

Theorem 5.1 extends Theorem 4.3 to the $L_p$ norm. As before, $\Delta$ is a given triangulation of a subset $\Omega \in \mathbb{R}^2$ with polygonal boundary $\partial \Omega$. A basis $\{B_i\}_{i=1}^n$ for a spline space $S^p_\Omega(\Delta)$ (1) satisfies

$$k_1 ||c||_p \leq \frac{1}{A_\Omega^{1/p}} \sum_{i=1}^n |c_i B_i| \leq k_2 ||c||_p \tag{20}$$

for all choices of the coefficient vector $c$, is called a stable basis for the $L_p$ norm. Here $||c||_p := (\sum_{i=1}^n |c_i|^p)^{1/p}$ and $||s||_p := (\int_{\Omega} |s(x,y)|^p dxdy)^{1/p}$. We are interested in constants $k_1$ and $k_2$ which depend only on the smallest angle in the triangulation $\Delta$.

**Theorem 5.1** Consider a triangulation $\Delta$ of a subset $\Omega \in \mathbb{R}^2$ with polygonal boundary $\partial \Omega$. Suppose $\Delta$ is constituted of triangles $T_j$, $j = 1, \ldots, t$ which have vertices $V_i$, $i = 1, \ldots, n$. Define $\Delta^*$ as the PS-refinement of $\Delta$. Suppose that there exists a constant $K$ such that every PS-triangle $t_i$ is contained in a triangle $T_{c_i} \subset T_C$, where $T_C$ is defined as in Lemma 4.2. Then there exists a constant $K_2$ depending only on $K$ and on the smallest angle in the underlying triangulation such that for all Powell–Sabin splines $s(x,y) \in S_2^p(\Delta^*)$ in their normalized $B$-spline representation (7)

$$\frac{\min_{T \in \Delta} A_T^{1/p}}{A_\Omega^{1/p}} \frac{1}{K_2} ||c||_p \leq \frac{||s(x,y)||_p}{A_\Omega^{1/p}} \leq ||c||_p \tag{21}$$

for $1 \leq p < \infty$.

**Proof** We have

$$||s(x,y)||_p^p = \frac{1}{\Omega} \left| \sum_{i=1}^n \sum_{j=1}^3 c_{ij} B_i^j(x,y) \right|^p dx dy.$$

Let $1/p + 1/q = 1$, then by Hölder's inequality:

$$||s(x,y)||_p^p \leq \frac{1}{\Omega} \left| \sum_{i=1}^n \sum_{j=1}^3 c_{ij} B_i^j(x,y) \right|^p dx dy \leq \frac{1}{\Omega} \left| \sum_{i=1}^n \sum_{j=1}^3 B_i^j(x,y) \right|^p dx dy = ||c||_p^p : A_\Omega.$$

This proves the right inequality.

To prove the left-hand side of (21), we use the fact that all norms on a finite dimensional vector space are equivalent. Consider a triangle $T \in \Delta$. By mapping $T$ to the standard simplex $T_e = \{(x,y) | 0 \leq x, y \leq 1, x + y \leq 1\}$, we get that

$$||s(x,y)||_{\infty,T_e} \leq K_3 ||s(x,y)||_{B,T_e}.$$

This implies that

$$||s(x,y)||_{\infty,T} \leq \frac{K_3}{A_\Omega^{1/p}} ||s(x,y)||_{B,T}.$$

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So,
\[ ||e||_p^p \leq \sum_{T \in \Delta} \left( \sum_{i \in v, \tau} \sum_{j=1}^{3} |c_{ij}|^p \right) \]
\[ \leq \sum_{T \in \Delta} \left( 9||e||_{\infty, T}^p \right) \]
\[ \leq \sum_{T \in \Delta} \left( 9K_1^p ||s(x, y)||_{\infty, T}^p \right) \]
\[ \leq \sum_{T \in \Delta} \left( 9K_1^p \frac{K_3}{A_T} ||s(x, y)||_{p, T}^p \right) \]
\[ \leq 9K_1^p \frac{K_3}{\min_{T \in \Delta} A_T^{1/p}} \sum_{T \in \Delta} ||s(x, y)||_{p, T}^p \]

Because \( \sum_{T \in \Delta} ||s(x, y)||_{p, T}^p = ||s(x, y)||_{p}^p \), we have proven that there exists a constant \( K_2 = 9^{1/s}K_1K_3 \) such that
\[ ||e||_p \leq \frac{K_2}{\min_{T \in \Delta} A_T^{1/p}} ||s(x, y)||_p. \]

\[ \square \]

Theorem 5.1 shows that the basis functions form a stable basis for the \( L_p \) norm, but the constant \( k_1 \) from (20) contains a factor \( \frac{\min_{T \in \Delta} A_T^{1/p}}{A_{v_i}^{1/p}} \). Nevertheless our approximation constant is satisfactory because its value does not change by a scaling operation on the domain.

6 Minimal PS–triangles

By using the PS–control points (10) we can interactively change the shape of a PS–spline surface. In order to have a good local control over the spline surface we want the PS–triangles \( t_i \) (which are the projection of the control triangles in the \((x, y)\) plane) to be as small as possible. Therefore we are interested in the PS–triangle \( t_i \) with the smallest area. To determine these minimal PS–triangles we have to solve a quadratic programming problem as mentioned in [1].

In section 4 we explained that there are infinity many choices for the PS–triangles. In this section we will assume that only the PS–triangles with minimal area are used. For this special case we derive an estimate for the constant \( K \), introduced in section 4, which only depends on the smallest angle in the triangulation \( \Delta \).

**Theorem 6.1** Suppose \( V_i \) is a vertex in the triangulation \( \Delta \) with molecule \( M_i \) and molecule number \( m_i \) and suppose that \( V_i \notin \delta \Omega \). Let \( t_i \) be the PS–triangle with minimal area that contains all the PS–points of vertex \( V_i \). Denote the PS–point with the largest distance to vertex \( V_i \) as \( S \) and define \( C_i \) as the circle with center \( V_i \) and radius \( K|SV_i| \). If
\[ K \geq \frac{3}{2 \sin(\theta_{\Delta}) \sin(\theta_{\Delta}) / 2 \sin(\theta_{\Delta}) \sin(\theta_{\Delta}) / 4)^{4/\theta_{\Delta}} + 4} \]
then there exists an equilateral triangle \( T_{C_i} \) with \( C_i \) as its inscribed circle that contains \( t_i \) (see figure 4).

**Proof** The PS–points surrounding vertex \( V_i \) form a polygon \( P \) with \( m_i \) corners. It is clear that the PS–triangle \( t_i \) contains this \( m_i \)-gon.

Because \( V_i \notin \delta \Omega \) we know that \( V_i \) lies inside this polygon \( P \). The distance from an edge of the
polygon $P$ to vertex $V_i$ can be bounded below. If we use the same reasoning as in equation (12), then we get for an arbitrary edge $e_P$ of the polygon $P$ that

$$|e_P| \geq \frac{1}{2} \min_{e \in M_i^*} |e| \sin(\theta_{\Delta^*}/2).$$

Here, $M_i^*$ is defined as the molecule of vertex $V_i$ in the PS-refinement $\Delta^*$ of $\Delta$.

From this inequality we can conclude that the height $h_{t_i}$ of PS-triangle $t_i$ can be bounded below by

$$h_{t_i} \geq \min_{e \in M_i^*} |e| \sin(\theta_{\Delta^*}/2). \quad (23)$$

As in Lemma 4.2, we know that there exists a triangle $T_S \in M_i^*$ such that (18) is satisfied. Define triangle $T_t_i \in \Delta$ as the triangle that contains triangle $T_S \in \Delta^*$ and define triangle $T_j \in \Delta$ as the triangle that contains edge $e \in M_i^*$ for which $|e| = \min_{e \in M_i^*} |e|$. Then $T_t$ and $T_j$ belong to the same molecule $M_i$ and by Lemma 3.4

$$|e_{max}(T_t^*)| \leq \left( \frac{1}{\sin(\theta_{\Delta^*})} \right)^{4 \cdot \frac{m_i}{m_i}} |e_{max}(T_j^*)|,$$

and by (18) and (13)

$$2|SV_i| \leq \left( \frac{1}{\sin(\theta_{\Delta^*})} \right)^{2m_i+4} |e_{min}(T_j^*)|,$$

where $T_t^*$ and $T_j^*$ are the corresponding PS-refinements. Combining this with (23) gives

$$|e_{max}(T_t)| \geq 2\sin(\theta_{\Delta^*}/2) \sin(\theta_{\Delta^*})^{2m_i+4} |SV_i|.$$  \hspace{1cm} (24)

Now, assume that for an arbitrary constant $K \geq 1$ we have that

$$|e_{max}(t_i)| > 2\sqrt{K} |SV_i|. \quad (25)$$

This means that the PS-triangle $t_i$ is not contained in a triangle $T^*_{C_i}$. The area of PS-triangle $t_i$ can be bounded below by combining equation (24) and (25). We get

$$A_{t_i} > \frac{1}{2} \left( 2\sin(\theta_{\Delta^*}/2) \sin(\theta_{\Delta^*})^{2m_i+4} |SV_i| \right) \left( 2\sqrt{K} |SV_i| \right). \quad (26)$$

The area of a triangle $T^*_{C_i}$ with $K = 1$ is equal to $3\sqrt{3} |SV_i|^2$. If this area is smaller than the right-hand side of (26) we have a contradiction, because $t_i$ is supposed to be the PS-triangle with minimal area. So, we have a contradiction if

$$K \geq \frac{3}{2 \sin(\theta_{\Delta^*}/2) \sin(\theta_{\Delta^*})^{2m_i+4}},$$

or, by Lemma 3.2, if

$$K \geq \frac{3}{2 \sin(\theta_{\Delta} \sin(\theta_{\Delta^*})/8) \sin(\theta_{\Delta} \sin(\theta_{\Delta^*})/4)^{2m_i+4}}.$$

It is easy to see that the molecule number $m_i$ can be bounded by

$$m_i \leq \frac{2\pi}{\theta_{\Delta^*}}. \quad (27)$$

This proves the theorem. \hspace{1cm} □

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