Stability Analysis of Volterra Delay-Integro-Differential Equations and their Backward Differentiation Time-Discretization

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Report TW 344, August 2002

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Chengjian Zhang* Stefan Vandewalle*

Abstract

We study the analytic stability of initial value problems of a general class of systems of Volterra delay-integro-differential equations. Numerical methods based on the backward differentiation formulae and a linear compound quadrature formula are suggested. The nonlinear and linear stability conditions for the presented methods are derived.

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1 Introduction

Volterra delay-integro-differential equations (VDIDEs) arise widely in scientific fields such as biology, ecology, medicine and physics (cf. [6]). This class of equations plays an important role in modelling diverse problems of engineering and natural science, and hence have come to intrigue researchers in numerical computation and analysis. For initial value problems (IVPs) of VDIDEs

$$\begin{cases} y'(t) = f(t, y(t), \int_{t-\tau}^{t} g(t, s, y(s))ds), & t \in [t_0, +\infty), \\ y(t) = \varphi(t), & t \in [t_0 - \tau, t_0], \end{cases} \tag{1.1}$$

Baker and Ford dealt with linear stability and convergence of linear multistep methods with a given quadrature formula (cf. [1, 2]). For IVPs of neutral VDIDEs

$$\begin{cases} y'(t) = h(t, y(t)) + \int_{t-\tau}^{t} g(t, s, y(s), y'(s))ds, & t \in [t_0, T], \\ y(t) = \varphi(t), & t \in [t_0 - \tau, t_0], \end{cases} \tag{1.2}$$

Brunner [7] studied the attainable order of local superconvergence of continuous Volterra-Runge-Kutta methods. Enright and Hu [10] investigated convergence of explicit and implicit continuous-Runge-Kutta methods for (1.2). Furthermore, in [3] and [4], Baker and Tang extended the research to VDIDEs of the form

$$\begin{cases} y'(t) = f(t, y(t), \int_{t-\tau(t)}^{t} g(t, s, y(s))ds), & t \in [t_0, +\infty), \\ y(t) = \varphi(t), & t \in [\inf_{t \geq t_0}\{t_0 - \tau(t)\}, t_0] \end{cases} \tag{1.3}$$

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and

$$\begin{cases} y'(t) &= f(t, y(t), y(t-\tau(t)), \int_{-\infty}^{t} g(t,s,y(s))ds), \quad t \in [t_0, +\infty), \\ y(t) &= \varphi(t), \qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in (-\infty, t_0], \end{cases} \tag{1.4}$$

and obtained important analytic and numerical stability results for such equations.

In the areas of VDIDEs, a lot of open problems remain both in theory and in computation. Up to now, stability analysis for nonlinear VDIDEs has almost always been performed for the scalar case and has mostly been based on the use of the classical Lipschitz condition. In paper [10], a one-sided Lipschitz condition with respect to the unknown function $y(t)$ was used, yet the obtained results are still related to the classical Lipschitz condition. Hence, the previous analyses are mainly applicable and relevant for non-stiff problems.

In this paper, we consider a class of stiff VDIDEs. We investigate its analytical stability, and linear and nonlinear numerical stability. In §2, we introduce the class of problems that will be the subject of our study, and we present various examples. In §3, the analytic stability of this class of equations is dealt with. Some globally and asymptotically nonlinear stability results for the analytic solutions of the equations are obtained. From there several interesting linear stability propositions follow. In §4, a class of numerical methods for these problems is derived, based on the backward differentiation formula (BDF) time-discretization and a linear compound quadrature formula. Also, nonlinear numerical stability criteria for these methods are given. In §5, we derive the linear stability conditions of these numerical methods, using the concept of $A(\alpha)$-stability of the underlying BDF methods for ODEs. Finally, some conclusions are presented in §6.

2 A class of Volterra delay-integro-differential equations

We consider complex N-dimensional systems of VDIDEs with constant delay $\tau > 0$

$$\begin{cases} y'(t) &= f(t, y(t), G(t, y(t-\tau), \int_{t-\tau}^{t} g(t,s,y(s))ds)), \quad t \in [t_0, +\infty), \\ y(t) &= \varphi(t), \qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in [t_0-\tau, t_0], \end{cases} \tag{2.1}$$

where the mappings f, G, g and φ are smooth enough, such that system (2.1) has an unique smooth solution $y(t)$ and satisfies the conditions

$$\Re\langle f(t,y_1,z) - f(t,y_2,z), y_1 - y_2\rangle \leq \alpha\|y_1 - y_2\|^2, \tag{2.2}$$

$$\|f(t,y,z_1) - f(t,y,z_2)\| \leq \beta\|z_1 - z_2\|, \tag{2.3}$$

$$\|G(t,y_1,z_1) - G(t,y_2,z_2)\| \leq \sigma_1\|y_1 - y_2\| + \sigma_2\|z_1 - z_2\|, \tag{2.4}$$

$$\|g(t,s,z_1) - g(t,s,z_2)\| \leq \gamma\|z_1 - z_2\|, \quad (t,s) \in \mathbb{D}, \tag{2.5}$$

in which $t \in [t_0, +\infty)$; $\mathbb{D} = \{(t,s) : t \in [t_0, +\infty), s \in [t-\tau, t]\}$; $y, y_1, y_2, z, z_1, z_2 \in \mathbb{C}^N$; $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote a given inner product and the corresponding induced norm in the complex N-dimensional space $\mathbb{C}^N$, respectively. The constants $-\alpha, \beta, \sigma_1, \sigma_2$ and γ are given and nonnegative. Problems of type (2.1) with (2.2) -(2.5) will be called "problems of *class* $\mathbb{DI}(\alpha, \beta, (\sigma_1, \sigma_2), \gamma)$". Below, we present some examples in order to demonstrate the existence and the variety of such problems.

Example 2.1 The N-dimensional complex linear systems of the form

$$\begin{cases} y'(t) &= Ay(t) + By(t-\tau) + C\int_{t-\tau}^{t} y(s)ds + d(t), \quad t \in [t_0, +\infty), \\ y(t) &= \varphi(t), \qquad\qquad\qquad\qquad\qquad\qquad\quad t \in [t_0 - \tau, t_0] \end{cases} \tag{2.6}$$

belong to the class $\mathbb{DI}(\mu(A), 1, (\|B\|, \|C\|), 1)$, where A, B, C are $N \times N$ constant complex matrices, $d(t)$ is a given N-dimensional vector-valued function and $\mu(\cdot)$ denotes the logarithmic norm corresponding to the vectorial inner-product norm $\|\cdot\|$, i.e.,

$$\mu(A) = \max_{\xi \neq 0} \Re\langle \xi, A\xi\rangle / \|\xi\|^2.$$

Example 2.2 The nonlinear scalar system

$$\begin{cases} y'(t) &= -2y(t) + \frac{y(t-\tau)}{1+[y(t-\tau)]^2} + \int_{t-\tau}^{t} \frac{1}{[1+y^2(s)]^2}[\frac{-\exp(-s)}{(1+\ln^2(1+\exp(-s)))(1+\exp(-s))}]ds \\ & \quad + \ln(1+\exp(-t))[1 + \frac{\ln^2(1+\exp(-t))}{1+\ln^2(1+\exp(-t))}] - \frac{\exp(-t)}{1+\exp(-t)}, \quad t \in [\tau, +\infty), \\ y(t) &= \ln(1+\exp(-t)), \quad t \in [0, \tau] \end{cases} \tag{2.7}$$

belongs to the class $\mathbb{DI}(-2, 1, (1, 1), 2)$.

Example 2.3 The 2-dimensional nonlinear system for $t \in [0, +\infty)$

$$\begin{cases} \frac{d}{dt}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} &= -3\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} 0 & \sin t \\ \cos t & 0 \end{pmatrix}\begin{pmatrix} y_1(t-\frac{\pi}{4}) \\ y_2(t-\frac{\pi}{4}) \end{pmatrix} \\ & + \frac{1}{\sqrt{2}} \int_{t-\frac{\pi}{4}}^{t} \begin{pmatrix} \frac{(1+\sin^2 t)y_1^2(s)}{1+y_1^2(s)} \\ -\frac{(1+\cos^2 t)y_2^2(s)}{1+y_2^2(s)} \end{pmatrix} ds \\ & + \frac{\sqrt{2}}{16} \begin{pmatrix} 16\sin(t+\frac{\pi}{4}) - 2\sin 2t + 6\cos 2t - \pi - 4 + 16\sqrt{2}\sin t \\ 16\cos(t+\frac{\pi}{4}) - 2\sin 2t + 4\cos 2t + \pi + 4 + 16\sqrt{2}\cos t \end{pmatrix}, \\ \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} &= \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}, \quad t \in [-\frac{\pi}{4}, 0] \end{cases} \tag{2.8}$$

belongs to the class $\mathbb{DI}(-3, 1, (1, \frac{1}{\sqrt{2}}), 2)$ for the standard inner product and norm.

Moreover, it is apparent that system (1.1) is a special case of (2.1). As to neutral system (1.2), when introducing a new function $x(t)$ with $x(t) := y'(t)$, it can be rewritten as a regular equation for $t \in [t_0, +\infty)$, together with an appropriate initial condition,

$$\begin{cases} \frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} \frac{\partial h(t,y(t))}{\partial t} + x(t)\frac{\partial h(t,y)}{\partial y} + g(t,t,y(t),x(t)) \\ x(t) \end{pmatrix} \\ & \quad - \begin{pmatrix} g(t, t-\tau, y(t-\tau), x(t-\tau)) \\ 0 \end{pmatrix} + \int_{t-\tau}^{t} \begin{pmatrix} \frac{\partial g(t,s,y(s),x(s))}{\partial t} \\ 0 \end{pmatrix} ds. \\ \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} \varphi'(t) \\ \varphi(t) \end{pmatrix}, \quad t \in [t_0 - \tau, t_0]. \end{cases} \tag{2.9}$$

Hence, also (1.2) can be considered as a special case of (2.1).

3 Analytic stability of the systems

This section will focus on the analytic stability of equations of the form (2.1) which satisfy (2.2)-(2.5). For this, we will also need to consider system (2.1) with a different initial condition, i.e., $\psi(t)$ instead of $\varphi(t)$. Its solution will be denoted by $\tilde{y}(t)$,

$$\begin{cases} \tilde{y}'(t) &= f(t, \tilde{y}(t), G(t, \tilde{y}(t-\tau), \int_{t-\tau}^{t} g(t, s, \tilde{y}(s))ds)), \quad t \in [t_0, +\infty), \\ \tilde{y}(t) &= \psi(t), \qquad\qquad\qquad\qquad\qquad\qquad\quad t \in [t_0 - \tau, t_0]. \end{cases} \tag{3.1}$$

The Lemma bellow will play a key role in studying the analytic stability. It is a corollary of the generalized Halanay Theorem and its proof given by Baker and Tang [3].

Lemma 3.1 *If the scalar function $v(t)$ is continuous and nonnegative for $t \geq t_0 - \tau$, with*

$$\begin{cases} D_+ v(t) &\leq -Av(t) + B \sup_{t-\tau \leq s \leq t} v(s), \quad \forall t \in [t_0, +\infty), \\ v(t) &= |\Phi(t)|, \qquad\qquad\qquad\qquad t \in [t_0 - \tau, t_0], \end{cases} \tag{3.2}$$

where $D_+ v(t)$ denotes the right derivative of $v(t)$, $\Phi(t)$ is continuous and not identically vanishing for $t \in [t_0 - \tau, t_0]$, and A, B are nonnegative constants with $-A + B < 0$. Then,

$$v(t) \leq \max_{\theta \in [t_0 - \tau,\ t_0]} |\Phi(\theta)|, \quad \forall t \geq t_0 \quad \textit{and} \quad \lim_{t \to +\infty} v(t) = 0.$$

With Lemma 3.1, we can derive the following theorem.

Theorem 3.2 *Suppose that* (2.1) *and* (3.1) *belong to class* $\mathbb{DI}(\alpha, \beta, (\sigma_1, \sigma_2), \gamma)$ *with*

$$\alpha + \beta(\sigma_1 + \sigma_2 \gamma \tau) < 0. \tag{3.3}$$

Then, the following results hold

$$\|y(t) - \tilde{y}(t)\| \leq \max_{\theta \in [t_0 - \tau,\ t_0]} \|\varphi(\theta) - \psi(\theta)\|, \quad \forall t \geq t_0 \quad \textit{and} \quad \lim_{t \to +\infty} \|y(t) - \tilde{y}(t)\| = 0. \tag{3.4}$$

Proof. Write $Y(t) = y(t) - \tilde{y}(t)$. We first demonstrate the existence of the right derivative

$$\begin{aligned} D_+(\|Y(t)\|) &= \lim_{x \to 0^+} \frac{\|Y(t+x)\| - \|Y(t)\|}{x} \\ &= \lim_{x \to 0^+} \frac{\|Y(t+x)\| - \|Y(t) + xY'(t)\|}{x} + \lim_{x \to 0^+} \frac{\|Y(t) + xY'(t)\| - \|Y(t)\|}{x}. \end{aligned}$$

The first term on the right hand side equals zero. This follows from the fact that $\forall x > 0$,

$$0 \leq \left| \frac{\|Y(t+x)\| - \|Y(t) + xY'(t)\|}{x} \right| \leq \frac{\|Y(t+x) - Y(t) - xY'(t)\|}{x} = \frac{\|o(x)\|}{x}.$$

The second term on the right hand side exists, since the function

$$H(x) := \frac{\|Y(t) + xY'(t)\| - \|Y(t)\|}{x}$$

is monotone increasing and bounded w.r.t. $x > 0$. This infers the existence of $D_+(\|Y(t)\|)$.

For the remainder of the proof, we first remark that β, γ, σ_1 and σ_2 are nonnegative by (2.3)-(2.5); hence $\alpha < 0$ by (3.3). Now, we prove the conclusions of the theorem. We have

$$\begin{aligned}
\frac{d}{dt}(\|Y(t)\|^2) &= 2\Re\langle Y(t), Y'(t)\rangle \\
&= 2\Re\langle Y(t), f(t, y(t), G(t, y(t-\tau), \int_{t-\tau}^{t} g(t,s,y(s))ds)) \\
&\quad -f(t, \tilde{y}(t), G(t, y(t-\tau), \int_{t-\tau}^{t} g(t,s,y(s))ds))\rangle \\
&\quad +2\Re\langle Y(t), f(t, \tilde{y}(t), G(t, y(t-\tau), \int_{t-\tau}^{t} g(t,s,y(s))ds)) \\
&\quad -f(t, \tilde{y}(t), G(t, \tilde{y}(t-\tau), \int_{t-\tau}^{t} g(t,s,\tilde{y}(s))ds))\rangle
\end{aligned}$$

By conditions (2.2)-(2.5), the above can be bounded $\forall t \geq t_0$ as follows,

$$\begin{aligned}
\frac{d}{dt}(\|Y(t)\|^2) &\leq 2\alpha\|Y(t)\|^2 + 2\|Y(t)\|\|f(t, \tilde{y}(t), G(t, y(t-\tau), \int_{t-\tau}^{t} g(t,s,y(s))ds)) \\
&\quad -f(t, \tilde{y}(t), G(t, \tilde{y}(t-\tau), \int_{t-\tau}^{t} g(t,s,\tilde{y}(s))ds))\| \\
&\leq 2\alpha\|Y(t)\|^2 + 2\beta\|Y(t)\|\|G(t, y(t-\tau), \int_{t-\tau}^{t} g(t,s,y(s))ds) \\
&\quad -G(t, \tilde{y}(t-\tau), \int_{t-\tau}^{t} g(t,s,\tilde{y}(s))ds)\| \\
&\leq 2\alpha\|Y(t)\|^2 + 2\beta\|Y(t)\|[\sigma_1\|Y(t-\tau)\| + \sigma_2\gamma\int_{t-\tau}^{t}\|Y(s)\|ds] \\
&\leq 2\alpha\|Y(t)\|^2 + 2\beta\|Y(t)\|(\sigma_1 + \sigma_2\gamma\tau)\sup_{t-\tau\leq s\leq t}\|Y(s)\|. \qquad (3.5)
\end{aligned}$$

When $t \notin \mathbf{\Lambda} := \{t \in [t_0, +\infty) : \; Y(t) = 0\}$, it follows from (3.5) and the identity

$$D_+(\|Y(t)\|^2) = 2\|Y(t)\|D_+(\|Y(t)\|) \qquad (3.6)$$

that

$$D_+(\|Y(t)\|) \leq \alpha\|Y(t)\| + \beta(\sigma_1 + \sigma_2\gamma\tau)\sup_{t-\tau\leq s\leq t}\|Y(s)\|, \quad t \notin \mathbf{\Lambda}. \qquad (3.7)$$

When $t \in \mathbf{\Lambda}$, the formula for the right derivative becomes

$$D_+(\|Y(t)\|) = \lim_{x\to 0^+}\frac{\|Y(t+x)\|}{x} = \|Y'(t)\|. \qquad (3.8)$$

With conditions (2.3)-(2.5), and with the knowledge that $y(t) = \tilde{y}(t)$, we have the bound

$$\begin{aligned}
\|Y'(t)\| &\leq \beta\|G(t, y(t-\tau), \int_{t-\tau}^{t} g(t,s,y(s))ds) - G(t, \tilde{y}(t-\tau), \int_{t-\tau}^{t} g(t,s,\tilde{y}(s))ds)\| \\
&\leq \beta\|[\sigma_1\|Y(t-\tau)\| + \sigma_2\gamma\int_{t-\tau}^{t}\|Y(s)\|ds] \\
&\leq \beta(\sigma_1 + \sigma_2\gamma\tau)\sup_{t-\tau\leq s\leq t}\|Y(s)\| \; .
\end{aligned}$$

A combination with (3.8) leads to

$$D_+(\|Y(t)\|) \le \beta(\sigma_1 + \sigma_2\gamma\tau) \sup_{t-\tau \le s \le t} \|Y(s)\|, \quad t \in \mathbf{\Lambda}. \tag{3.9}$$

Hence, we may conclude that (3.7) also holds for $t \in \mathbf{\Lambda}$, and thus it holds for $\forall t \ge t_0$. Application of Lemma 3.1 then proves the conclusions of this theorem. □

Conclusions (3.4) in Theorem 3.2 imply that system (2.1) is globally stable and asymptotically stable, under the conditions of the theorem. In particular, when one of the constants β, σ_2 or γ equals zero, i.e., when both (2.1) and (3.1) are reduced to the corresponding ordinary or delay differential equations without the distributed delay, the conclusions of the theorem imply the stability of the reduced systems. This is consistent with similar results in Torelli [16] and Zenarro [17]. Next, we consider the stability behavior of the zero solution of the linear autonomous systems corresponding to (2.6)

$$\begin{cases} y'(t) &= Ay(t) + By(t-\tau) + C\int_{t-\tau}^{t} y(s)ds, \quad t \in [t_0, +\infty), \\ y(t) &= \varphi(t), \qquad\qquad\qquad\qquad\qquad t \in [t_0-\tau, t_0]. \end{cases} \tag{3.10}$$

System (3.10) also belongs to the class $\mathbb{DI}(\mu(A), 1, (\|B\|, \|C\|), 1)$, being an equation of type (2.6). From Theorem 3.2 one can simply obtain a stability result for system (3.10).

Corollary 3.3 *The solution $y(t)$ of system (3.10) with*

$$\mu(A) + \|B\| + \|C\|\tau < 0 \tag{3.11}$$

satisfies $\lim_{t\to+\infty} \|y(t)\| = 0$ *and* $\|y(t)\| \le \max_{t\in[t_0-\tau,t_0]} \|\varphi(t)\|, \ \forall t \ge t_0$.

In particular, when (3.10) becomes a scalar equation, condition (3.11) can be replaced by

$$\Re(A) + |B| + |C|\tau < 0. \tag{3.12}$$

4 The numerical method and its nonlinear stability

For the IVPs of ODEs of the form

$$\begin{cases} y'(t) &= f(t, y(t)), \quad t \in [t_0, +\infty), \\ y(t_0) &= y_0, \end{cases} \tag{4.1}$$

it is well known that backward differentiation formulae or BDF methods are quite effective. We denote such methods as

$$\rho(E)y_n = hf(t_{n+k}, y_{n+k}), \tag{4.2}$$

where $\rho(\xi) = \sum_{i=1}^{k} \alpha_i \xi^i$ is a polynomial with real coefficients α_i, subject to consistency conditions $\rho(1) = 0$ and $\rho'(1) = 1$; E denotes the shift operator: $Ey_n = y_{n+1}$. To distinguish the above methods from the induced methods that will be introduced in (4.3), the methods (4.2) will further be referred to as the *underlying BDF methods.*

The underlying BDF method together with a linear-compound-quadrature formula applied to system (2.1) generates the following numerical scheme for $n \geq 0$:

$$\left\{\begin{array}{rcl} \rho(E)y_n &=& hf(t_{n+k}, y_{n+k}, G(t_{n+k}, y_{n+k-m}, z_{n+k})), \\ z_{n+k} &=& h \sum\limits_{j=n+k-m+1}^{n+k} [\nu g(t_{n+k}, t_{j-1}, y_{j-1}) + (1-\nu) g(t_{n+k}, t_j, y_j)], \end{array}\right. \tag{4.3}$$

where ν is a parameter on interval $[0,1]$, $h = \frac{\tau}{m}$ is the stepsize, with m a given and positive integer, y_n and z_n are approximations to $y(t_n)$ and $z(t_n) := \int_{t_n-\tau}^{t_n} g(t_n, s, y(s))ds$, respectively, and $t_n = t_0 + nh$. In particular, when $-m \leq n \leq 0$, we set $y_n = \varphi(t_n)$ and $z_n = \int_{t_n-\tau}^{t_n} g(t_n, s, \varphi(s))ds$. An analogous numerical scheme, but related to (3.1), can be derived with $\tilde{y}_n$ and $\tilde{z}_n$ replacing y_n and z_n respectively. For $-m \leq n \leq 0$, we then set $\tilde{y}_n = \psi(t_n)$ and $z_n = \int_{t_n-\tau}^{t_n} g(t_n, s, \psi(s))ds$.

In order to study the numerical stability, we introduce some notational conventions:

$$\omega_n = y_n - \tilde{y}_n, \; W_n = (\omega_n, \omega_{n+1}, \ldots, \omega_{n+k-1}), \; \zeta_n = z_n - \tilde{z}_n,$$

$$\|U\| = \sqrt{\sum_{i=1}^{k} \|u_i\|^2} \;\text{ and }\; \|U\|_G = \sqrt{\sum_{i,j=1}^{k} g_{i,j}\langle u_i, u_j\rangle},$$

$\forall U = (u_1^T, u_2^T, \ldots, u_k^T)^T \in \mathbb{C}^{Nk}$, $u_i \in \mathbb{C}^N$, where $G = (g_{i,j}) \in \mathbb{C}^{k\times k}$ is a given real symmetric-positive-definite matrix. The following Lemma will be useful for presenting the results on numerical stability.

Lemma 4.1 *Suppose $B_i \geq 0$, $i = -m, -m+1, \ldots, -1, 0, 1, \ldots, n$. Then*

$$\sum_{i=0}^{n}\sum_{j=0}^{m} B_{i-j} \leq (m+1)\sum_{i=0}^{n} B_i + \frac{m(m+1)}{2} \max_{-m\leq i\leq -1}\{B_i\}, \qquad \forall n \geq 0. \tag{4.4}$$

Proof. First we rewrite the left hand side of (4.4) as follows

$$\begin{aligned} \sum_{i=0}^{n}\sum_{j=0}^{m} B_{i-j} &= \sum_{i=0}^{n} B_i + \sum_{j=1}^{m}\left(\sum_{i=j}^{n} B_{i-j} + \sum_{i=0}^{j-1} B_{i-j}\right) \\ &= \sum_{i=0}^{n} B_i + \sum_{j=1}^{m}\left(\sum_{i=0}^{n-j} B_i + \sum_{i=1}^{j} B_{-i}\right) \\ &= \sum_{j=0}^{m}\sum_{i=0}^{n-j} B_i + \sum_{j=1}^{m}\sum_{i=1}^{j} B_{-i}. \end{aligned}$$

Since $B_i \geq 0$, we obtain a bound from which (4.4) immediately follows,

$$\sum_{i=0}^{n}\sum_{j=0}^{m} B_{i-j} \leq \sum_{j=0}^{m}\sum_{i=0}^{n} B_i + \sum_{j=1}^{m} j \max_{-m\leq i\leq -1}\{B_i\}.$$

□

In Li [15], the concept of G-stability for ODE methods, proposed by Dahlquist [8], was expanded into that of $G(c,p,q)$-algebraic stability. Using this concept, Huang [13] dealt with the dissipativity of one-leg methods for a class of DDEs without distributed delay, and Zhang and Liao [18] investigated the stability of BDF reducible quadrature methods for a class of Volterra integral equations. In what follows, we will adopt this concept in order to evaluate the stability of methods (4.3). For simplicity, we only consider $G(c,p)$-algebraic stability, which is just $G(c,p,0)$-algebraic stability.

Definition 4.2 *Assume that c and p are real constants with $c > 0$. Assume there is a $k \times k$ real symmetric-positive-definite matrix G such that for any real sequence $\{a_i\}_{i=0}^{k}$, the following inequality holds*

$$A_1^T G A_1 - c A_0^T G A_0 \leq 2a_k(\rho(E)a_0) - p(a_k)^2,$$

where $A_i = (a_i, a_{i+1}, \ldots, a_{i+k-1})^T$ $(i = 0, 1)$. Then the underlying BDF method (4.2) is called $G(c,p)$-algebraically stable. A $G(1,0)-$algebraically stable method (4.2) is called $G-$stable.

G-stability is equivalent to A-stability (cf. [8]). Since the underlying 1- and 2-step BDF methods are A-stable, these underlying methods are G-stable too. The concept of $G(c,p)$-algebraic stability breaks through the well-known order obstacle, which states that an A-stable multistep method can be at most of second order (cf. [11]). For example, Li [15] pointed out that for all $c > 0$, there exists a $p := p(c) < 0$ and a diagonal matrix $G = diag(\frac{c^2}{4}, \frac{c}{2}, 1)$ such that the 3-step underlying BDF method is $G(c, p(c))$-algebraically stable.

Definition 4.3 *Method (4.3) is called globally stable for class $\mathbb{DI}(\alpha, \beta, (\sigma_1, \sigma_2), \gamma)$ if, when it is applied to problems (2.1) and (3.1) of this class, the solutions $\{y_n\}$ and $\{\tilde{y}_n\}$ satisfy*

$$\|y_n - \tilde{y}_n\| \leq M \max_{\min\{0,\ k-m\} \leq i \leq k-1} \|y_i - \tilde{y}_i\|, \quad \forall n \geq k,$$

where M is a positive constant depending only on $\alpha, \beta, \sigma_1, \sigma_2, \tau$ and the method. Furthermore, method (4.3) is called asymptotically stable for the class $\mathbb{DI}(\alpha, \beta, (\sigma_1, \sigma_2), \gamma)$ if

$$\lim_{n \to 0} \|y_n - \tilde{y}_n\| = 0.$$

Global stability of a method implies that the perturbation $\|y_n - \tilde{y}_n\|$ $(\forall n \geq k)$ of the numerical solution is controlled by the initial perturbation of both the system and the method,

$$\max_{\min\{0,\ k-m\} \leq i \leq k-1} \|y_i - \tilde{y}_i\| \leq \max\{\max_{\theta \in [t_0 - \tau,\ t_0]} \|\varphi(\theta) - \psi(\theta)\|, \ \max_{0 \leq i \leq k-1} \|y_i - \tilde{y}_i\|\}.$$

Theorem 4.4 *Suppose that the underlying BDF method (4.2) is $G(c,p)$-algebraically stable with $0 < c \leq 1$. Then, the induced method (4.3) is globally stable for the class $\mathbb{DI}(\alpha, \beta, (\sigma_1, \sigma_2), \gamma)$ and subject to the stability inequality*

$$\|y_{n+k} - \tilde{y}_{n+k}\| \leq \sqrt{\frac{k\lambda_{max}^G + \beta\tau(\sigma_1 + 2\sigma_2\gamma^2\tau^2)}{\lambda_{min}^G}} \max_{\min\{0,\ k-m\} \leq i \leq k-1} \|y_i - \tilde{y}_i\|, \tag{4.5}$$

for all $n \geq 0$, whenever

$$h[2\alpha + \beta(2\sigma_1 + \sigma_2) + 4\beta\sigma_2\gamma^2\tau^2] \leq p, \tag{4.6}$$

where λ_{min}^G and λ_{max}^G denote the minimum and the maximum eigenvalues of matrix G.

Proof. From the definition of $G(c,p)$-algebraic stability it follows that

$$\|W_{n+1}\|_G^2 - c\|W_n\|_G^2 \le 2\Re\langle \omega_{n+k}, \rho(E)\omega_n\rangle - p\|\omega_{n+k}\|^2. \tag{4.7}$$

With (2.2)-(2.5), we can bound the first term in the right hand side,

$$\begin{aligned}
2\Re\langle \omega_{n+k}, \rho(E)\omega_n\rangle &= 2\Re\langle \omega_{n+k}, h[f(t_{n+k}, y_{n+k}, G(t_{n+k}, y_{n+k-m}, z_{n+k}))\\
&\quad - f(t_{n+k}, \tilde{y}_{n+k}, G(t_{n+k}, y_{n+k-m}, z_{n+k}))]\rangle\\
&\quad + 2\Re\langle \omega_{n+k}, h[f(t_{n+k}, \tilde{y}_{n+k}, G(t_{n+k}, y_{n+k-m}, z_{n+k}))\\
&\quad - f(t_{n+k}, \tilde{y}_{n+k}, G(t_{n+k}, \tilde{y}_{n+k-m}, \tilde{z}_{n+k}))]\rangle\\
&\le 2h\alpha\|\omega_{n+k}\|^2 + 2h\beta\|\omega_{n+k}\|\|G(t_{n+k}, y_{n+k-m}, z_{n+k}) - G(t_{n+k}, \tilde{y}_{n+k-m}, \tilde{z}_{n+k})\|\\
&\le 2h\alpha\|\omega_{n+k}\|^2 + 2h\beta\|\omega_{n+k}\|[\sigma_1\|\omega_{n+k-m}\| + \sigma_2\|\zeta_{n+k}\|]\\
&\le h[2\alpha + \beta(\sigma_1+\sigma_2)]\|\omega_{n+k}\|^2 + h\beta\sigma_1\|\omega_{n+k-m}\|^2 + h\beta\sigma_2\|\zeta_{n+k}\|^2.
\end{aligned} \tag{4.8}$$

Also, we have that

$$\begin{aligned}
\|\zeta_{n+k}\| &\le h\sum_{j=n+k-m+1}^{n+k} \|\nu(g(t_{n+k}, t_{j-1}, y_{j-1}) - g(t_{n+k}, t_{j-1}, \tilde{y}_{j-1}))\\
&\quad + (1-\nu)(g(t_{n+k}, t_j, y_j) - g(t_{n+k}, t_j, \tilde{y}_j))\|\\
&\le h[\nu\sum_{j=n+k-m+1}^{n+k} \|g(t_{n+k}, t_{j-1}, y_{j-1}) - g(t_{n+k}, t_{j-1}, \tilde{y}_{j-1})\|\\
&\quad + (1-\nu)\sum_{j=n+k-m+1}^{n+k} \|g(t_{n+k}, t_j, y_j) - g(t_{n+k}, t_j, \tilde{y}_j)\|]\\
&\le h\gamma[\nu\sum_{j=n+k-m+1}^{n+k} \|y_{j-1} - \tilde{y}_{j-1}\| + (1-\nu)\sum_{j=n+k-m+1}^{n+k} \|y_j - \tilde{y}_j\|]\\
&\le h\gamma\sum_{j=n+k-m}^{n+k} \|y_j - \tilde{y}_j\|\\
&= h\gamma\sum_{j=0}^{m} \|\omega_{n+k-j}\|.
\end{aligned} \tag{4.9}$$

Substituting (4.8) and (4.9) into (4.7) yields an upper bound for $\|W_{n+1}\|_G^2 - c\|W_n\|_G^2$:

$$[h(2\alpha + \beta(\sigma_1+\sigma_2)) - p]\|\omega_{n+k}\|^2 + h\beta\sigma_1\|\omega_{n+k-m}\|^2 + h^3\beta\sigma_2\gamma^2(\sum_{j=0}^{m}\|\omega_{n+k-j}\|)^2. \tag{4.10}$$

According to the Cauchy inequality, we have

$$(\sum_{j=0}^{m}\|\omega_{n+k-j}\|)^2 \le (m+1)\sum_{j=0}^{m}\|\omega_{n+k-j}\|^2. \tag{4.11}$$

Combining (4.10) with (4.11) and using the fact that $0 < c \leq 1$, we get

$$\begin{aligned} \|W_{n+1}\|_G^2 &\leq \|W_n\|_G^2 + [h(2\alpha + \beta(\sigma_1 + \sigma_2)) - p]\|\omega_{n+k}\|^2 \\ &\quad + h\beta\sigma_1\|\omega_{n+k-m}\|^2 + h^3\beta\sigma_2\gamma^2(m+1)\sum_{j=0}^{m}\|\omega_{n+k-j}\|^2. \end{aligned} \tag{4.12}$$

An induction argument applied to (4.12) generates

$$\begin{aligned} \|W_{n+1}\|_G^2 &\leq \|W_0\|_G^2 + [h(2\alpha + \beta(\sigma_1 + \sigma_2)) - p]\sum_{i=0}^{n}\|\omega_{i+k}\|^2 \\ &\quad + h\beta\sigma_1\sum_{i=0}^{n}\|\omega_{i+k-m}\|^2 + h^3\beta\sigma_2\gamma^2(m+1)\sum_{i=0}^{n}\sum_{j=0}^{m}\|\omega_{i+k-j}\|^2. \end{aligned} \tag{4.13}$$

Also, setting $B_i = \|\omega_{i+k}\|$ in Lemma 4.1 and using (4.4) gives

$$\begin{aligned} \|W_{n+1}\|_G^2 &\leq \|W_0\|_G^2 + [h(2\alpha + \beta(\sigma_1 + \sigma_2)) - p]\sum_{i=0}^{n}\|\omega_{i+k}\|^2 \\ &\quad + h\beta\sigma_1\sum_{i=0}^{n}\|\omega_{i+k-m}\|^2 + h^3\beta\sigma_2\gamma^2(m+1)^2\sum_{i=0}^{n}\|\omega_{i+k}\|^2 \\ &\quad + h^3\beta\sigma_2\gamma^2[\frac{m(m+1)^2}{2}]\max_{-m\leq i\leq -1}\{\|\omega_{i+k}\|^2\}. \end{aligned}$$

Together with condition (4.6) and $mh = \tau$ $(m \geq 1)$, this leads to

$$\begin{aligned} \|W_{n+1}\|_G^2 &\leq \|W_0\|_G^2 + [h(2\alpha + \beta(\sigma_1 + \sigma_2) + h^2\beta\sigma_2\gamma^2(m+1)^2) - p]\sum_{i=0}^{n}\|\omega_{i+k}\|^2 \\ &\quad + h\beta\sigma_1\sum_{i=-m}^{n-m}\|\omega_{i+k}\|^2 + h^3\beta\sigma_2\gamma^2[\frac{m(m+1)^2}{2}]\max_{k-m\leq i\leq k-1}\{\|\omega_i\|^2\} \\ &\leq \|W_0\|_G^2 + [h(2\alpha + \beta(\mathbf{2}\sigma_1 + \sigma_2) + h^2\beta\sigma_2\gamma^2(m+1)^2) - p]\sum_{i=0}^{n}\|\omega_{i+k}\|^2 \\ &\quad + h\beta\sigma_1\sum_{i=-m}^{-1}\|\omega_{i+k}\|^2 + h^3\beta\sigma_2\gamma^2[\frac{m(m+1)^2}{2}]\max_{k-m\leq i\leq k-1}\{\|\omega_i\|^2\} \\ &\leq \|W_0\|_G^2 + [h(2\alpha + \beta(2\sigma_1 + \sigma_2) + 4(mh)^2\beta\sigma_2\gamma^2) - p]\sum_{i=0}^{n}\|\omega_{i+k}\|^2 \\ &\quad + (mh\beta\sigma_1 + 2(mh)^3\beta\sigma_2\gamma^2)\max_{k-m\leq i\leq k-1}\{\|\omega_i\|^2\} \\ &= \|W_0\|_G^2 + [h(2\alpha + \beta(2\sigma_1 + \sigma_2) + 4\beta\sigma_2\gamma^2\tau^2) - p]\sum_{i=0}^{n}\|\omega_{i+k}\|^2 \\ &\quad + (\tau\beta\sigma_1 + 2\beta\sigma_2\gamma^2\tau^3)\max_{k-m\leq i\leq k-1}\{\|\omega_i\|^2\} \\ &\leq \|W_0\|_G^2 + \beta\tau(\sigma_1 + 2\sigma_2\gamma^2\tau^2)\max_{k-m\leq i\leq k-1}\{\|\omega_i\|^2\}. \end{aligned} \tag{4.14}$$

This implies

$$\begin{aligned}\lambda^G_{min}\|\omega_{n+k}\|^2 &\leq \lambda^G_{max}\sum_{i=0}^{k-1}\|\omega_i\|^2 + \beta\tau(\sigma_1 + 2\sigma_2\gamma^2\tau^2)\max_{k-m\leq i\leq k-1}\{\|\omega_i\|^2\}. \\ &\leq [k\lambda^G_{max} + \beta\tau(\sigma_1 + 2\sigma_2\gamma^2\tau^2)]\max_{\min\{0,\ k-m\}\leq i\leq k-1}\{\|\omega_i\|^2\}.\end{aligned}$$

Therefore, inequality (4.5) is proved. □

Corollary 4.5 *Suppose that the underlying BDF method* (4.2) *is G-stable. Then, the induced method* (4.3) *is globally stable for the class* $\mathbb{DI}(\alpha, \beta, (\sigma_1, \sigma_2), \gamma)$ *and satisfies the stability inequality* (4.5) *whenever*

$$\beta(2\sigma_1 + \sigma_2) + 4\beta\sigma_2\gamma^2\tau^2 \leq -2\alpha. \tag{4.15}$$

A slight modification to the proof of Theorem 4.4 leads to asymptotic stability results.

Theorem 4.6 *Suppose that the underlying BDF method* (4.2) *is G(c,p)-algebraically stable with* $0 < c \leq 1$*. Then, the induced method* (4.3) *is asymptotically stable for the class* $\mathbb{DI}(\alpha, \beta, (\sigma_1, \sigma_2), \gamma)$ *whenever*

$$h[2\alpha + \beta(2\sigma_1 + \sigma_2) + 4\beta\sigma_2\gamma^2\tau^2] < p. \tag{4.16}$$

Proof. By a slight modification to the proof of (4.14), we get

$$\begin{aligned}&\|W_{n+1}\|^2_G + [p - h(2\alpha + \beta(2\sigma_1 + \sigma_2) + 4\beta\sigma_2\gamma^2\tau^2)]\sum_{i=0}^{n}\|\omega_{i+k}\|^2 \\ \leq\ &\|W_0\|^2_G + \beta\tau(\sigma_1 + 2\sigma_2\gamma^2\tau^2)\max_{k-m\leq i\leq k-1}\{\|\omega_i\|^2\},\quad \forall n \geq 0.\end{aligned} \tag{4.17}$$

The right hand side can be bounded by a constant. Hence, with (4.17) we deduce that $\lim_{n\to\infty}\|\omega_n\| = 0$, which completes the proof. □

Corollary 4.7 *Suppose that the underlying BDF method* (4.2) *is G-stable. Then, the induced method* (4.3) *is asymptotically stable for the class* $\mathbb{DI}(\alpha, \beta, (\sigma_1, \sigma_2), \gamma)$*, whenever*

$$\beta(2\sigma_1 + \sigma_2) + 4\beta\sigma_2\gamma^2\tau^2 < -2\alpha. \tag{4.18}$$

Theorem 4.8 *The stability condition* (4.18) *of method* (4.3) *is stronger than the stability condition* (3.3) *of system* (2.1) *when* $\beta \neq 0$ *and* $\sigma_2 \neq 0$*. When* $\beta = 0$ *or* $\sigma_2 = 0$*, the stability conditions of the method are consistent with these of the system.*

Proof. This follows immediately with the aid of the following inequality

$$\beta(2\sigma_1 + \sigma_2) + 4\beta\sigma_2\gamma^2\tau^2 = 2\beta(\sigma_1 + \sigma_2\gamma\tau) + 4\beta\sigma_2[(\gamma\tau - \frac{1}{4})^2 + \frac{3}{16}] \geq 2\beta(\sigma_1 + \sigma_2\gamma\tau).$$

□

Note that the above theorem implies that when system (2.1) is degenerated into an ODE system or a DDE system, the stability conditions of the methods and system are identical.

5 The linear stability of the methods

In this section, we adopt another approach to deal with the stability of methods (4.3) for linear systems (3.10). We will study what conditions can guarantee the solution y_n to satisfy $\lim_{n\to\infty} y_n = 0$. For non-distributed delay differential equations, this type of numerical asymptotic stability property has been widely investigated by many authors, see e.g. [5, 12] and their references. Here, we are interested in distributed delay systems of type (3.10).

First, we review the related approach for the underlying BDF method. Applying (4.2) to the scalar problem $y^{'}(t) = \lambda y(t)$ with $y(0) = y_0$, generates the following recursion scheme

$$\rho(E)y_n = \bar{h}y_{n+k}, \quad \text{where} \quad \bar{h} = h\lambda, \tag{5.1}$$

whose characteristic equation is given by

$$\rho(z) = \bar{h}z^k, \quad z \in \mathbb{C}. \tag{5.2}$$

Hence the absolute stability region of method (4.2) is

$$\mathbb{S}_{\mathbb{OBDF}} := \{\bar{h} \in \mathbb{C} : \quad (5.2) \Rightarrow |z| < 1\}. \tag{5.3}$$

Next, we turn to the linear stability of (4.3). Applying (4.3) to (3.10) yields, for $n \geq 0$,

$$\rho(E)y_n = h[Ay_{n+k} + By_{n+k-m} + h \sum_{j=n+k-m+1}^{n+k} (vy_{j-1} + (1-v)y_j)C]. \tag{5.4}$$

The characteristic equation of (5.4) is given by the expression

$$\det[\rho(z)I_N - hz^k(A + z^{-m}B + h\sum_{j=0}^{m-1}(vz^{-(j-1)} + (1-v)^{-j})C)] = 0. \tag{5.5}$$

This is equivalent to the equation

$$\prod_{i=1}^{N}[\rho(z) - h\lambda_i(Q(z))z^k] = 0, \quad z \in \mathbb{C}, \tag{5.6}$$

where $\lambda_i(Q(z))$ denotes the i-th eigenvalue of matrix

$$Q(z) := A + z^{-m}B + h\sum_{j=0}^{m-1}[vz^{-(j-1)} + (1-v)z^{-j}]C.$$

Theorem 5.1 *The induced method* (4.3) *for problem* (3.10) *satisfies* $\lim_{n\to\infty} y_n = 0$ *if*

$$h\lambda_i(Q(\xi)) \in \mathbb{S}_{\mathbb{OBDF}}, \quad \textit{for} \quad i = 1, 2, \ldots, N \ \textit{and} \ \forall \xi \in \mathbb{C} : |\xi| \geq 1. \tag{5.7}$$

Proof. By the theory on difference equations (cf. [14]), we have that $\lim_{n\to\infty} y_n = 0$ if the following implication holds

$$(5.6) \Rightarrow |z| < 1. \tag{5.8}$$

Assume that (5.8) doesn't hold, then there exists certain $\tilde{z}: |\tilde{z}| \geq 1$ such that

$$\prod_{i=1}^{N}[\rho(\tilde{z}) - h\lambda_i(Q(\tilde{z}))\tilde{z}^k] = 0.$$

This implies that there exists an i, with $1 \leq i \leq N$, for which

$$\rho(\tilde{z}) - h\lambda_i(Q(\tilde{z}))\tilde{z}^k = 0. \quad (5.9)$$

Combining (5.7) with (5.9) leads to $|\tilde{z}| < 1$. This contradicts our earlier assumption, hence (5.8) must hold. □

With Theorem 5.1 and the concept of A(α)-stability we have the following corollary.

Corollary 5.2 *The induced method* (4.3) *for problem* (3.10) *satisfies* $\lim_{n\to\infty} y_n = 0$, *if the underlying BDF method* (4.2) *is* $A(\alpha)$*-stable for some* $\alpha \in (0, \frac{\pi}{2})$ *and*

$$|\arg[-\lambda_i(Q(\xi))]| < \alpha, \quad \textit{for} \;\; i = 1, 2, \ldots, N \;\textit{and}\; \forall\xi : |\xi| \geq 1. \quad (5.10)$$

It is well-known that when $k = 3, 4, 5, 6$, the underlying BDF methods (4.2) are A(α)-stable for different $\alpha \in (0, \frac{\pi}{2})$ (cf. [11]). Hence, by Corollary 5.2 we know that the induced methods (4.3) for the problem (3.10), with $k = 3, 4, 5, 6$, satisfy $\lim_{n\to\infty} y_n = 0$ whenever (3.10) is subject to (5.10).

With Theorem 5.1 and the concept of A-stability, we can derive one more corollary.

Corollary 5.3 *The induced method* (4.3) *for problem* (3.10) *satisfies* $\lim_{n\to\infty} y_n = 0$, *if the underlying BDF method* (4.2) *is* A*-stable and*

$$\Re\{\lambda_i(Q(\xi))\} < 0, \quad \textit{for} \;\; i = 1, 2, \ldots, N \;\textit{and}\; \forall\xi : \; |\xi| \geq 1. \quad (5.11)$$

Moreover, under conditions (3.11) and $|\xi| \geq 1$, it holds for $i = 1, 2, \ldots, N$ by the properties of the logarithmic norm (cf. [9]) that

$$\begin{aligned}
\Re\{\lambda_i(Q(\xi))\} &= \Re\{\lambda_i[A + \xi^{-m}B + h\sum_{j=0}^{m-1}[v\xi^{-(j-1)} + (1-v)\xi^{-j}]C]\} \\
&\leq \mu[A + \xi^{-m}B + h\sum_{j=0}^{m-1}[v\xi^{-(j-1)} + (1-v)\xi^{-j}]C] \\
&\leq \mu(A) + \|\xi^{-m}B\| + h\|\sum_{j=0}^{m-1}[v\xi^{-(j-1)} + (1-v)\xi^{-j}]C\| \\
&\leq \mu(A) + |\xi|^{-m}\|B\| + h\sum_{j=0}^{m-1}[v|\xi|^{-(j-1)} + (1-v)|\xi|^{-j}]\|C\|
\end{aligned}$$

Therefore,

$$\Re\{\lambda_i(Q(\xi))\} \leq \mu(A) + \|B\| + mh\|C\| = \mu(A) + \|B\| + \tau\|C\| < 0. \quad (5.12)$$

Hence, substituting (3.11) for (5.11) in Corollary 5.3 yields our final corollary.

Corollary 5.4 *The induced method* (4.3) *for problem* (3.10) *satisfies* $\lim_{n\to\infty} y_n = 0$*, if the underlying BDF method* (4.2) *is A-stable and condition* (3.11) *holds.*

Since the underlying BDF methods (4.2) with $k = 1, 2$ are A-stable, by Corollary 5.3 and Corollary 5.4 we conclude that the induced methods (4.3) for the problem (3.10), with $k = 1, 2$, satisfy $\lim_{n\to\infty} y_n = 0$ whenever condition (5.11) or (3.11) holds.

6 Conclusion

The study of numerical stability is quite important for evaluating a numerical method. In this paper, we first studied the analytic stability of a class of nonlinear IVPs of N-dimensional VDIDEs. Then, for this class of IVPs, we constructed a class of numerical methods based on BDF methods and a linear compound quadrature formula. We obtained nonlinear and linear stability conditions for the presented methods. In fact, it is obvious that all the results we derived can be modified to handle one-leg methods and equations with multiple delays, such as

$$\begin{cases} y'(t) &= f(t, y(t), G(t, y(t-\tau_1), \ldots, y(t-\tau_{r-1}), y(t-\tau_r), \ \int_{t-\tau_1}^{t} g_1(t,s,y(s))ds), \ldots, \\ & \quad \int_{t-\tau_{r-1}}^{t} g_{r-1}(t,s,y(s))ds), \int_{t-\tau_r}^{t} g_r(t,s,y(s))ds)), \quad t \in [t_0, +\infty), \\ y(t) &= \varphi(t), \ t \in [t_0 - \max_{1\le i\le r} \tau_i, \ t_0]. \end{cases}$$

As pointed out earlier, there remain a lot of open problems in the theory and computation of VDIDEs. In the future, we plan to study the numerical stability of other numerical methods, especially for VDIDEs with time-depending delays, and we will work on convergence aspects and the effective implementation of those methods.

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