An analysis of delay-dependent stability for ordinary and partial differential equations with fixed and distributed delays

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Report TW 341, June 2002

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Abstract

This paper is concerned with the study of the stability of ordinary and partial differential equations with both fixed and distributed delays, and with the study of the stability of discretisations of such differential equations. We start with a delay-dependent asymptotic stability analysis of scalar ordinary differential equations with real coefficients. We study the exact stability region of the continuous problem as a function of the parameters of the model. Next, it is proved that a time discretisation based on the trapezium rule can preserve the asymptotic stability for the considered set of test problems. In the second part of the paper, we study delay partial differential equations. The stability region of the fully continuous problem is analysed first. Then, a semi-discretisation in space is applied. It is shown that the spatial discretisation leads to a reduction of the stability region when the standard second-order central difference operator is employed to approximate the diffusion operator. Finally we consider the delay-dependent stability of the fully discrete problem, where the partial differential equation is discretised both in space and in time. Some numerical examples are given to confirm the theoretical results.

Keywords : integro-differential equations, delay partial differential equations, stability region, trapezium rule
AMS(MOS) Classification : 65L20, 65M10, 65M20

AN ANALYSIS OF DELAY-DEPENDENT STABILITY FOR ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS WITH FIXED AND DISTRIBUTED DELAYS

CHENGMING HUANG* AND STEFAN VANDEWALLE†

Abstract. This paper is concerned with the study of the stability of ordinary and partial differential equations with both fixed and distributed delays, and with the study of the stability of discretisations of such differential equations. We start with a delay-dependent asymptotic stability analysis of scalar ordinary differential equations with real coefficients. We study the exact stability region of the continuous problem as a function of the parameters of the model. Next, it is proved that a time discretisation based on the trapezium rule can preserve the asymptotic stability for the considered set of test problems. In the second part of the paper, we study delay partial differential equations. The stability region of the fully continuous problem is analysed first. Then, a semi-discretisation in space is applied. It is shown that the spatial discretisation leads to a reduction of the stability region when the standard second-order central difference operator is employed to approximate the diffusion operator. Finally we consider the delay-dependent stability of the fully discrete problem, where the partial differential equation is discretised both in space and in time. Some numerical examples are given to confirm the theoretical results.

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1. Introduction.

The stability of discretisation methods plays an important role in the numerical solution of delay differential equations. In the last twenty years, many papers have dealt with this topic and a significant number of important results have been found. For a description of state-of-the-art, we refer the reader to the papers by Baker [2, 3] and Zennaro [18] and to the extensive bibliography therein.

One of the interesting problems in stability analysis is the investigation of the delay-dependent stability region of numerical methods. For ordinary differential equations (ODEs) with a fixed delay, this subject was first discussed by Cryer [7], Al Mutib [1], Van der Houwen and Sommeijer [16] and Calvo and Grande [6]. More recently, a series of results have been obtained by Guglielmi [9, 10, 11], Guglielmi and Hairer [12, 13] and Maset [15]. For ODEs with a distributed delay, a numerical investigation was given by Baker and Ford [4]. For ODEs with both fixed and distributed delays, Koto [14] recently investigated the delay-independent stability of Runge-Kutta methods. We have not found any result on delay-dependent stability of such equations. The numerical solution of partial differential equations (PDEs) with delays has only recently attracted the attention of several authors. Zubik-Kowal and Vandewalle [20] have studied a special class of iterative methods, i.e., waveform relaxation methods, for solving semi-discrete delay PDEs. Their paper is not concerned with stability issues of the discretisation, but rather with the convergence of the iterative ODE solution method. Zubik-Kowal [19] has discussed the contractivity of θ-methods under the assumption that the semi-discretised system with a fixed delay is asymptotically stable for every positive delay.

In the first part of this paper, we discuss the numerical stability of real coefficient

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scalar ODEs with both fixed and distributed delays. On the basis of the investigation of the analytical stability region, the stability region of the trapezium rule is derived. It is proved that the trapezium rule preserves many features of the analytical stability region, including asymptotic stability. In the second part of this paper, we consider the numerical delay-dependent stability of PDEs of parabolic type, with both fixed and distributed delays. We derive the analytical stability region of the PDE and also investigate the stability of the semi-discrete and fully discrete problems. To this end we employ standard second-order central differences to discretise the spatial derivative and the trapezium rule to discretise the time derivative. Our result shows that the spatial discretisation leads to a reduction of the size of the stability region. This is different from the case of PDEs without delays. Hence, also the corresponding full discretisation based on the use of the trapezium rule cannot completely preserve the asymptotic stability of the original PDE.

This paper is organised as follows. In §2, we study the scalar ODE case for problems with fixed and distributed delays. In §3, we investigate the stability of a class of linear PDE-problems, and the stability of the semi-discrete and fully discrete cases. In §4, some numerical examples are given to support the theory. Finally, we draw some conclusions in §5.

2. Stability of ordinary delay differential equations.

2.1. Stability region of the continuous problem.

2.1.1. Study of the imaginary roots of the characteristic equation. We study the asymptotic stability of the solution to the equation

$$y'(t) = \alpha y(t) + \beta y(t-\tau) + \gamma \int_{-\tau}^{0} y(t+s)ds \ , \quad t > 0 \ , \tag{2.1}$$

where $\alpha, \beta, \gamma \in \mathbb{R}, \tau \in \mathbb{R}^+$ and $y(t) = g(t)$ on $[-\tau, 0]$. The results obtained for this very general equation will in §2.1.3 be specialised towards the cases of fixed delay problems ($\gamma = 0$), distributed delay problems ($\beta = 0$) and pure delay problems ($\alpha = 0$). By looking at solutions of the form $y(t) = \exp(\lambda t)$ we are led to the characteristic equation

$$\lambda = \alpha + \beta \exp(-\tau\lambda) + \gamma \int_{-\tau}^{0} \exp(\lambda s)ds \ . \tag{2.2}$$

The asymptotic stability of the zero solution to (2.1) is equivalent to the condition that all the roots of algebraic equation (2.2) have negative real parts. Below, we analyse these roots, following the ideas in the work by Diekmann et al. [8, Ch.XI]. Writing $\lambda = \mu + i\nu$ with $\mu, \nu \in \mathbb{R}$, we find two real equations

$$\mu = \alpha + \beta \exp(-\tau\mu)\cos(\tau\nu) + \gamma \int_{-\tau}^{0} \exp(\mu s)\cos(\nu s)ds \ , \tag{2.3}$$

$$\nu = -\beta \exp(-\tau\mu)\sin(\tau\nu) + \gamma \int_{-\tau}^{0} \exp(\mu s)\sin(\nu s)ds \ , \tag{2.4}$$

for the real and imaginary parts of the eigenvalue λ. Now we concentrate on roots which lie on the imaginary axis, the so-called critical roots, i.e., with $\mu = 0$. With also $\nu = 0$, one finds that there exists a root $\lambda = 0$ if the parameters α, β and γ satisfy

$$\alpha + \beta + \gamma\tau = 0 \ . \tag{2.5}$$

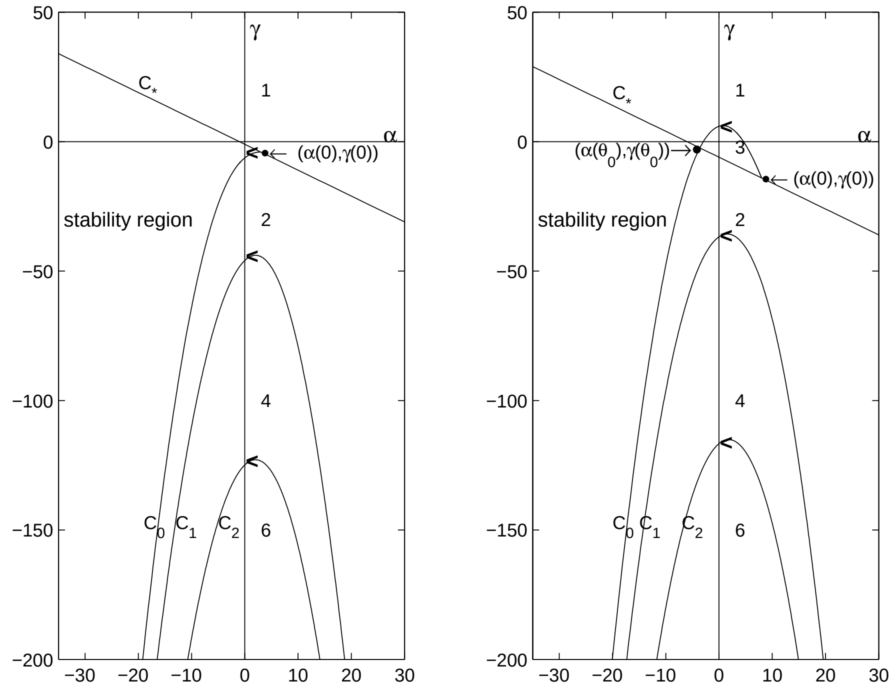

FIG. 2.1. *The curves $C_k(\beta)$ in the $(\alpha,\gamma)-$plane for $\tau = 1$ (left: $\beta = 1$; right: $\beta = 6$). The arrows along the curves refer to the direction of increasing θ. The numbers in the different regions bordered by the curves indicate the number of roots in the right half-plane.*

For a given value of β this set can be interpreted as a line in the (α,γ)-plane. This line will be denoted by $C_*(\beta)$. Setting $\theta = \tau\nu$, we find that the following parameter sets $(\alpha(\theta), \beta, \gamma(\theta))$ with $\theta \neq \pm 2k\pi,\ k = 1, 2, \ldots$, lead to a purely imaginary root λ:

$$\alpha(\theta) = \beta + \frac{\theta\sin\theta}{\tau(1-\cos\theta)} \quad \text{and} \quad \gamma(\theta) = \frac{-\theta(\theta+\beta\tau\sin\theta)}{\tau^2(1-\cos\theta)} \ . \tag{2.6}$$

Since these functions have singularities for $\theta = \pm 2k\pi$, we introduce intervals

$$J_k = (2k\pi, 2(k+1)\pi) \tag{2.7}$$

and denote by $C_k(\beta)$ the curve in the (α,γ)-plane for a fixed β-value, parameterised by θ for $\theta \in J_k$. We will adhere to the convention that $C_0(\beta)$ also contains the limit point for $\theta \to 0$:

$$(\alpha(0), \gamma(0)) = (\beta + \frac{2}{\tau}, \frac{-2(1+\beta\tau)}{\tau^2}) \ . \tag{2.8}$$

The functions in (2.6) are even functions in θ, which reflects the fact that the complex roots appear in complex conjugate pairs. Hence, we may restrict further attention to $k \geq 0$, i.e., $\nu \geq 0$. As an illustration, the curves $C_k(\beta)$ are shown in Fig. 2.1 for two different values of β and for $\tau = 1$.

We will now analyse some important properties of these curves, in a series of four (rather technical) lemmas. Those will show that the configuration and features of the curves in Fig. 2.1 are typical. We show that the curves $C_k(\beta)$ do not intersect, and that they can be ordered according to k. We determine their limiting values, and consider the location of curve $C_*(\beta)$ versus the $C_0(\beta)$-curve.

LEMMA 2.1. *The curves $C_k(\beta)$ do not intersect.*

Proof. Suppose that there exist $\theta_1 \in J_k$ and $\theta_2 \in J_l$ with $k \neq l$ such that $\alpha(\theta_1) = \alpha(\theta_2)$ and $\gamma(\theta_1) = \gamma(\theta_2)$, or

$$\frac{\theta_1 \sin\theta_1}{1-\cos\theta_1} = \frac{\theta_2 \sin\theta_2}{1-\cos\theta_2} \quad \text{and} \quad \frac{\theta_1^2 + \beta\tau\theta_1 \sin\theta_1}{1-\cos\theta_1} = \frac{\theta_2^2 + \beta\tau\theta_2 \sin\theta_2}{1-\cos\theta_2} \ . \tag{2.9}$$

Substituting the first equality into the second one yields

$$\frac{1-\cos\theta_1}{\theta_1^2} = \frac{1-\cos\theta_2}{\theta_2^2}. \tag{2.10}$$

Multiplying both sides by the corresponding sides of the left equality in (2.9), and after squaring, one arrives at

$$\frac{\sin^2\theta_1}{\theta_1^2} = \frac{\sin^2\theta_2}{\theta_2^2}. \tag{2.11}$$

Subtracting (2.11) from (2.10) yields

$$\frac{\cos\theta_1(\cos\theta_1 - 1)}{\theta_1^2} = \frac{\cos\theta_2(\cos\theta_2 - 1)}{\theta_2^2}. \tag{2.12}$$

Equations (2.10) and (2.12) lead to $\cos\theta_1 = \cos\theta_2$. Hence, (2.10) can only be satisfied when $\theta_1 = \theta_2$, which contradicts our earlier assumption. □

LEMMA 2.2. *The curve $C_k(\beta)$ intersects the line $\alpha = \beta$ exactly once. The γ-coordinate γ_k of the intersection satisfies $\gamma_{k+1} < \gamma_k$.*

Proof. When $\theta \in J_k$, the equality $\alpha(\theta) = \beta$ implies $\theta = (2k+1)\pi$. Hence,

$$\gamma_k = \frac{-(2k+1)^2\pi^2}{2\tau^2} \ ,$$

which satisfy $\gamma_{k+1} < \gamma_k$. This completes the proof. □

LEMMA 2.3.

$$\lim_{\theta\to 2k\pi+0} \alpha(\theta) \to +\infty, \lim_{\theta\to 2k\pi-0} \alpha(\theta) \to -\infty, \lim_{\theta\to 2k\pi\pm 0} \gamma(\theta) \to -\infty \ , \quad k = 1, 2, \ldots$$

LEMMA 2.4. *The curve $C_0(\beta)$ intersects $C_*(\beta)$ at $(\alpha(0), \gamma(0))$. If $\beta\tau > 2$ it also intersects at an other point, $(\alpha(\theta_0), \gamma(\theta_0))$, with θ_0 defined in (2.13).*

Proof. For any given $\beta\tau > 2$, there exists a unique $\theta_0 \in (0, 2\pi)$ such that

$$\beta\tau = \frac{\theta_0(\theta_0 - \sin\theta_0)}{2 - 2\cos\theta_0 - \theta_0 \sin\theta_0}. \tag{2.13}$$

A straightforward computation shows that the corresponding point $(\alpha(\theta_0), \gamma(\theta_0))$ lies on the line $C_*(\beta)$. □

2.1.2. Number of roots in the right half-plane. Based on equations (2.3) and (2.4) one easily verifies the following result which guarantees that the roots with positive real parts are located in a compact region, with size depending on the problem parameters.

LEMMA 2.5. *Let λ be a root of characteristic equation (2.2) with* $\mathrm{Re}\lambda > 0$. *Then*

$$\mathrm{Re}\lambda < \alpha + |\beta| + |\gamma\tau| \quad \text{and} \quad \mathrm{Im}\lambda < |\beta| + |\gamma\tau| \ .$$

From the continuity of the roots as a fuction of the coefficients of the characteristic equation (see, e.g., [8]), we may conclude that the number of roots in the right half-plane is constant in the regions in the (α, γ)-plane bounded by $C_k(\beta)$ and the line $C_*(\beta)$. It is easy to verify that all roots of (2.2) are in the left half-plane when $\alpha = -|\beta| - 1, \gamma = 0$. Now we study the crossing of a pair of imaginary roots into the right half-plane, and determine the direction of the corresponding trajectory in the (α, γ)-plane. To this end, we define two functions based on (2.3) and (2.4):

$$G_1(\alpha, \gamma, \mu, \nu) = -\mu + \alpha + \beta \exp(-\tau\mu) \cos(\tau\nu) + \gamma \int_{-\tau}^{0} \exp(\mu s) \cos(\nu s) ds$$

$$G_2(\alpha, \gamma, \mu, \nu) = -\nu - \beta \exp(-\tau\mu) \sin(\tau\nu) + \gamma \int_{-\tau}^{0} \exp(\mu s) \sin(\nu s) ds \ .$$

Calculating the Jacobian matrix M defined by

$$M = \begin{bmatrix} \frac{\partial G_1}{\partial \alpha} & \frac{\partial G_1}{\partial \gamma} \\ \frac{\partial G_2}{\partial \alpha} & \frac{\partial G_2}{\partial \gamma} \end{bmatrix}_{\mu=0} ,$$

and computing its determinant, one finds

$$\det M = \frac{\cos \tau\nu - 1}{\nu} < 0 \ .$$

Now a result from [8] (Prop. 2.13, Chapter XI) can be applied.

LEMMA 2.6. *The critical roots enter the right half-plane for parameters sets in the (α, γ) parameter region to the left of the critical curve, when we follow this curve in the direction of increasing θ, whenever* $\det M < 0$ *and to the right when* $\det M > 0$.

Hence, the critical roots move into the right half-plane when moving away in the parameter space to the left of $C_k(\beta)$, with "left" as determined w.r.t. a counter clock-wise tracking of $C_k(\beta)$. In Fig. 2.1, the number of roots in the right half-plane is indicated for each separate parameter region. As a consequence of the above analysis, we are able to state the main result of this subsection.

THEOREM 2.7. *The zero solution of (2.1) is asymptotically stable, i.e., all the roots of (2.2) have negative real parts, iff*

(a) $\alpha < \beta + \frac{2}{\tau}$ *and (b)* $\tau(\alpha + \beta) < -\gamma\tau^2 < \frac{\theta(\theta + \beta\tau \sin\theta)}{1 - \cos\theta}$

where θ is the root of $\alpha = \beta + \frac{\theta \sin\theta}{\tau(1 - \cos\theta)}$ *such that* $\theta \in (0, 2\pi)$.

Proof. The stability region lies below $C_*(\beta)$ and above $C_0(\beta)$. Therefore, the results immediately follow from (2.5), (2.6) and (2.8). ◻

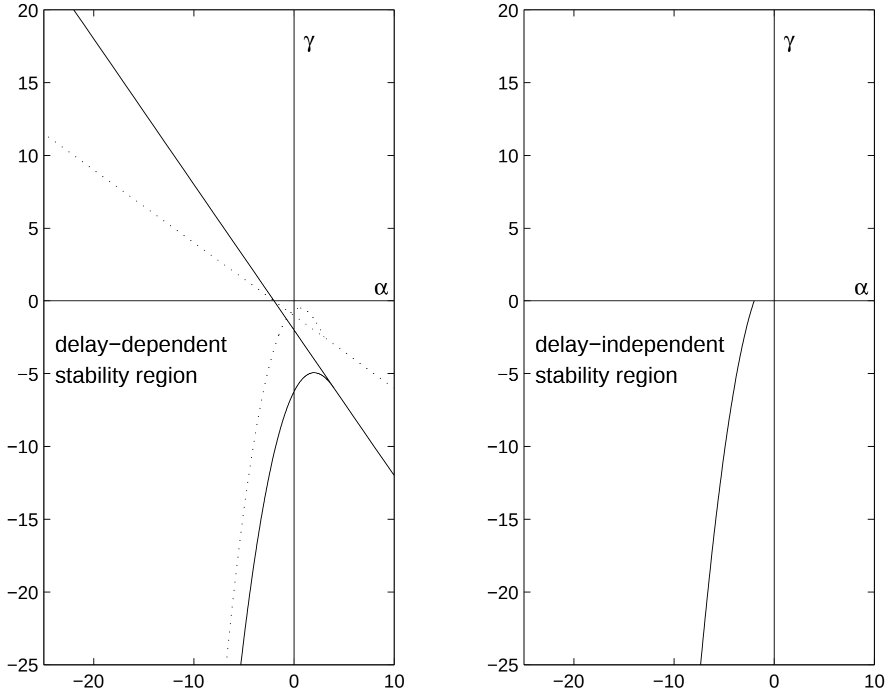

FIG. 2.2. *Delay-dependent stability region for various τ-values (left: $\tau = 1$ (solid) and $\tau = 2$ (dotted)) and delay-independent stability region (right) for $\beta = 2$.*

Theorem 2.7 can be seen as a generalization of the delay-independent stability condition derived in [14] by Koto for problem (2.1). We consider the case $\gamma \neq 0$. If condition (a) and the left inequality of condition (b) are to hold for every $\tau > 0$, we must have that $\alpha < 0$ and $\gamma < 0$. Squaring the equality $\alpha = \beta + \frac{\theta \sin\theta}{\tau(1-\cos\theta)}$, and substitution into the right inequality of condition (b) leads to an equivalent inequality,

$$-\gamma < \frac{\alpha^2 - \beta^2}{2} + \frac{\theta^2(2 - 2\cos\theta - \sin^2\theta)}{2\tau^2(1-\cos\theta)^2} \ .$$

If this is to hold independent of τ we must have that $-\gamma \leq \frac{\alpha^2-\beta^2}{2}$. Hence, we have derived a result, originally due to Koto, as a corollary of our stability theorem.

COROLLARY 2.8. *Consider the case $\gamma \neq 0$. The zero solution of (2.1) is asymptotically stable independent of the delay τ, iff*

$$\alpha < 0 \ , \quad \gamma < 0 \ , \quad -\gamma \leq \frac{\alpha^2 - \beta^2}{2} \ .$$

Because we consider the delay-dependent stability in this paper, the stability regions we obtain are larger than those in [14]. The difference between the delay-dependent and the delay-independent stability regions for the case $\beta = 2$ is graphically illustrated in Fig. 2.2.

Our analysis of the stability of delay PDEs in §3 will require some more detailed information about the curves $C_k(\beta)$. Accurate knowledge of the location of extremal values will be important, and, in particular, also knowledge about the monotonicity.

This will be investigated next. It is easily verified that $\alpha(\theta)$ is strictly decreasing with respect to θ in every $J_k, k = 0, 1, 2, \ldots$. The $\gamma(\theta)$-function on the other hand satisfies

$$\gamma'(\theta) = \frac{-\theta(2 - 2\cos\theta - \theta\sin\theta) + \beta\tau(1 - \cos\theta)(\theta - \sin\theta)}{\tau^2(1 - \cos\theta)^2},$$

which shows that $\gamma(\theta)$ has exactly one extremum with respect to θ along the curves $C_k(\beta)$ when $k > 0$. The more intricate and important case $\theta \in J_0$ is now considered separately. When $\beta\tau \le 1$, the function $\gamma(\theta)$ is strictly decreasing. When $\beta\tau > 1$, there exists a unique $\bar{\theta}_0 \in (0, 2\pi)$ such that $\gamma'(\bar{\theta}_0) = 0$, namely the root of

$$\beta\tau = \frac{\bar{\theta}_0(2 - 2\cos\bar{\theta}_0 - \bar{\theta}_0\sin\bar{\theta}_0)}{(1 - \cos\bar{\theta}_0)(\bar{\theta}_0 - \sin\bar{\theta}_0)}.$$

Hence, $\gamma(\theta)$ has one extremum $\gamma(\bar{\theta}_0)$. This value $\bar{\theta}_0$ satisfies

$$\alpha(\bar{\theta}_0) = \frac{\bar{\theta}_0(2 - 2\cos\bar{\theta}_0 - \sin^2\bar{\theta}_0)}{\tau(1 - \cos\bar{\theta}_0)(\bar{\theta}_0 - \sin\bar{\theta}_0)} > 0.$$

Define the constant α_0 as follows

$$\alpha_0 = \begin{cases} \alpha(0), & \text{if } \beta\tau \le 1, \\ \alpha(\bar{\theta}_0), & \text{otherwise.} \end{cases} \tag{2.14}$$

Therefore, for any given $\beta \in \mathbb{R}$, $C_0(\beta)$ is strictly monotonic in the (α, γ)-plane when $\alpha \le \alpha_0$. In particular, $C_0(\beta)$ is strictly monotonic in the left half of the (α, γ)-plane. This leads to the following results which will be used in §3.

THEOREM 2.9. *Suppose all roots of (2.2) have negative real parts for given* α, $\beta, \gamma \in \mathbb{R}$, *with* $\alpha \le \alpha_0$ *for* α_0 *defined in (2.14). Then for any* $x \le \alpha$, *all roots of the following equations have negative real parts too,*

$$\lambda = x + \beta\exp(-\tau\lambda) + \gamma\int_{-\tau}^{0}\exp(\lambda s)ds\ . \tag{2.15}$$

COROLLARY 2.10. *Suppose all roots of (2.2) have negative real parts for given* $\alpha \le 0, \beta, \gamma \in \mathbb{R}$. *Then for any* $x \le \alpha$, *all roots of (2.15) have negative real parts too.*

2.1.3. Some special cases. In the following, we consider some special cases and we show that our framework enables to rederive some specialised results from the delay equation literature. First we consider a differential equation with a distributed delay,

$$\begin{cases} y'(t) = \alpha y(t) + \gamma\int_{-\tau}^{0} y(t+s)ds, & t > 0, \\ y(t) = g(t), t \in [-\tau, 0]. \end{cases} \tag{2.16}$$

This is the basic equation studied by Baker and Ford in [4]. A direct application of Theorem 2.7 for $\beta = 0$ allows us to regain a result given in the above reference.

COROLLARY 2.11. *The zero solution of (2.16) is asymptotically stable iff*

(a) $\alpha\tau < 2$ *and* *(b)* $\tau\alpha < -\gamma\tau^2 < \frac{\theta^2}{1-\cos\theta}$

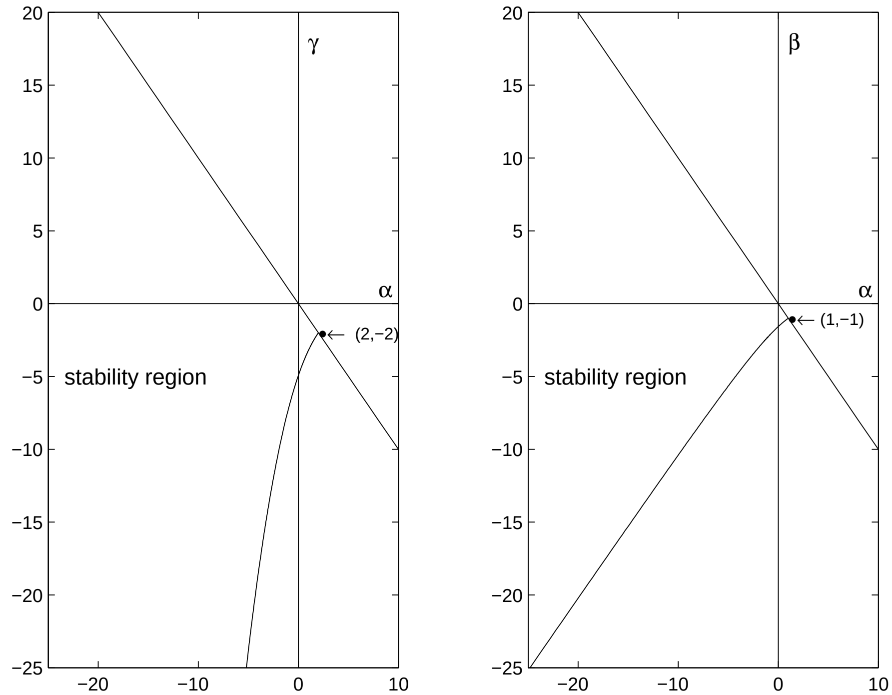

FIG. 2.3. *Stability region of (2.16) (left) and stability region of (2.17) (right) for* $\tau = 1$.

where θ *is the root of* $\alpha = \frac{\theta \sin\theta}{\tau(1-\cos\theta)}$ *such that* $\theta \in (0, 2\pi)$.

The stability region of equation (2.16) for $\tau = 1$ is shown in the left picture of Fig. 2.3. Next we study the case of an equation with a fixed delay, i.e., with $\gamma = 0$,

$$\begin{cases} y'(t) = \alpha y(t) + \beta y(t-\tau), \quad t > 0 \ , \\ y(t) = g(t), t \in [-\tau, 0] \ . \end{cases} \tag{2.17}$$

This is a classical model equation in the delay equation literature. In this case, condition (b) in Theorem 2.7 simplifies to

$$\alpha + \beta < 0 \quad \text{and} \quad \theta + \beta\tau \sin\theta > 0 \ , \tag{2.18}$$

where θ is the root of

$$\alpha = \beta + \frac{\theta \sin\theta}{\tau(1 - \cos\theta)} \tag{2.19}$$

such that $\theta \in (0, 2\pi)$. If $\alpha \le \beta$, i.e. $\theta \in [\pi, 2\pi)$, from the left inequality in (2.18) and (2.19) it follows that

$$2\beta + \frac{\theta \sin\theta}{\tau(1 - \cos\theta)} < 0$$

which guarantees that the right inequality of (2.18) holds. Therefore, we have the well known result (see, e.g., Diekmann et al. [8]).

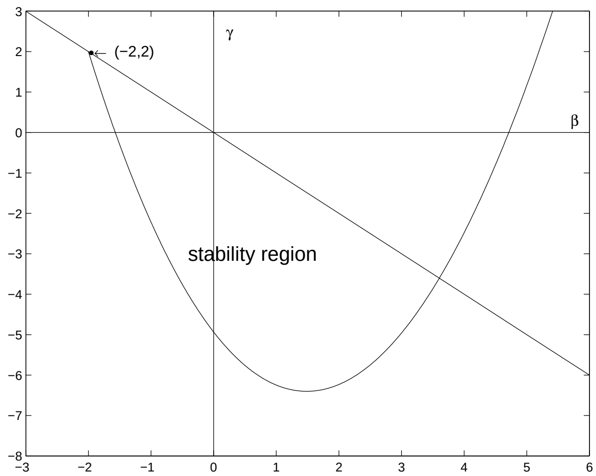

FIG. 2.4. *Stability region of the analytical solution of (2.20) for $\tau = 1$.*

COROLLARY 2.12. *The zero solution of (2.17) is asymptotically stable iff*
(a) $\alpha\tau < 1$ *and* *(b)* $\tau\alpha < -\tau\beta < \frac{\theta}{\sin\theta}$
where θ *is the root of* $\theta\cos\theta = \tau\alpha\sin\theta$ *such that* $\theta \in (0, \pi)$.

The stability region of (2.17) for $\tau = 1$ is shown in the right picture of Fig. 2.3. Finally, the case of a pure delay equation is considered, i.e., with $\alpha = 0$,

$$\begin{cases} y'(t) = \beta y(t-\tau) + \gamma \int_{-\tau}^{0} y(t+s)ds, \quad t > 0 \ , \\ y(t) = g(t), t \in [-\tau, 0] \ . \end{cases} \tag{2.20}$$

For the case $\gamma = 0$, a numerical stability analysis of this equation has first been given by Cryer [7] and by van der Houwen and Sommeijer [16]. Equation (2.20) has a finite region of asymptotic stability. An application of Theorem 2.7 leads to the corollary.

COROLLARY 2.13. *The zero solution of (2.20) is asymptotically stable iff*
(a) $\beta\tau + 2 > 0$ *and* *(b)* $\beta\tau < -\gamma\tau^2 < \frac{-\theta^2 cos\theta}{1-\cos\theta}$
where θ *is the root of* $\beta = \frac{-\theta\sin\theta}{\tau(1-\cos\theta)}$ *such that* $\theta \in (0, 2\pi)$.

The stability region of (2.20) for $\tau = 1$ is drawn in Fig. 2.4.

2.2. Stability region of the trapezium rule discretization.

2.2.1. Study of the unit modulus roots of the characteristic equation.

We apply the trapezium rule with constant and consistent stepsize $h = \tau/m$ to (2.1), with m a positive integer. We denote by y_{n-m} the approximation to $y(t_n - \tau)$ and use the repeated trapezium rule to approximate the integral term. This leads to

$$\begin{aligned} y_{n+1} - y_n = &\frac{h}{2}[\alpha y_{n+1} + \beta y_{n+1-m} + \frac{\gamma h}{2}(\sum_{j=0}^{m-1} y_{n+1-j} + \sum_{j=1}^{m} y_{n+1-j})] \\ &+\frac{h}{2}[\alpha y_n + \beta y_{n-m} + \frac{\gamma h}{2}(\sum_{j=0}^{m-1} y_{n-j} + \sum_{j=1}^{m} y_{n-j})] \ . \end{aligned} \tag{2.21}$$

The characteristic equation of the above difference equation is given by

$$1 - z^{-1} = \frac{h}{2}(1+z^{-1})[\alpha + \beta z^{-m} + \frac{\gamma h}{2}(1+z^{-1}) \sum_{j=0}^{m-1} z^{-j}] \ . \tag{2.22}$$

The numerical solution of (2.21) is asymptotically stable for any initial values if and only if all roots z of the characteristic equation (2.22) satisfy $|z| < 1$. Since z in (2.22) depends continuously on α, β and γ, we apply the so-called boundary locus technique (cf. [4, 5]) to analyse the roots of (2.22). This technique has been successfully used to analyse the numerical delay-dependent stability region for ODEs with a fixed delay (cf. [9, 10, 11, 12, 13, 15]), and for ODEs with distributed delay (cf. [4]).

Similar to the case of the continuous system in §2.1, we fix β and consider the stability region in the (α, γ)-plane. To distinguish from the continuous case, we will use "barred" notation to denote any quantities or functions related to the discrete case. For example, we use $\bar{\alpha}$ and $\bar{\gamma}$ as parameters for the curves on which the characteristic equation (2.22) has at least one root z on the unit circle. Let $z^{-1} = \exp(i\varphi)$. Because α, β and γ are real, we can restrict our analysis to $\varphi \in [0, \pi]$. The value $z = 1$, i.e. $\varphi = 0$, gives the line $\bar{C}_*(\beta)$ with equation $\bar{\alpha} + \beta + \bar{\gamma}\tau = 0$, which is identical to the line $C_*(\beta)$ in the case of the continuous problem. Obviously, $z = -1$, i.e. $\varphi = \pi$, is not a root of (2.22). It is easily verified that, except for $z = 1$, any other z satisfying $z^m = 1$ is not a root of (2.22). For other φ-values, we have

$$\frac{-i \sin\varphi}{1+\cos\varphi} = \frac{h}{2}\left[\bar{\alpha} + \beta(\cos m\varphi + i \sin m\varphi) + \frac{\bar{\gamma}h}{2}\,\frac{i(1+\cos\varphi)}{\sin\varphi}(1 - \cos m\varphi - i \sin m\varphi)\right] .$$

Separating real and imaginary parts yields

$$\begin{aligned} 0 &= \bar{\alpha} + \beta \cos m\varphi + \frac{\bar{\gamma}h}{2}\,\frac{(1+\cos\varphi)\sin m\varphi}{\sin\varphi} \ , \\ \frac{-\sin\varphi}{1+\cos\varphi} &= \frac{h}{2}\left[\beta \sin m\varphi + \frac{\bar{\gamma}h}{2}\,\frac{(1+\cos\varphi)(1-\cos m\varphi)}{\sin\varphi}\right] . \end{aligned}$$

Solving for $\bar{\alpha}$ and $\bar{\gamma}$, one obtains

$$\bar{\alpha}(\varphi) = \beta + \frac{2m \sin\varphi \sin m\varphi}{\tau(1+\cos\varphi)(1-\cos m\varphi)} \ , \tag{2.23}$$

$$\bar{\gamma}(\varphi) = \frac{-2m\sin\varphi(2m\sin\varphi + \beta\tau\sin m\varphi(1+\cos\varphi))}{\tau^2(1+\cos\varphi)^2(1-\cos m\varphi)} \ . \tag{2.24}$$

Define the intervals

$$\bar{J}_k = (\frac{2k\pi}{m}, \frac{2(k+1)\pi}{m}), k = 0, 1, \ldots, \lfloor \frac{m}{2} \rfloor - 1,$$

where the $\lfloor x \rfloor$ stands for the integer part of x. When m is odd, we additionally define

$$\bar{J}_{\lfloor \frac{m}{2} \rfloor} = (\frac{(m-1)\pi}{m}, \pi) \ .$$

For every fixed β, we denote by $\bar{C}_k(\beta)$ the curve in the (α, γ)-plane parameterized by φ for $\varphi \in \bar{J}_k$, with the convention that $\bar{C}_0(\beta)$ also contains the limit point for $\varphi \to 0$:

$$(\bar{\alpha}(0), \bar{\gamma}(0)) = (\beta + \frac{2}{\tau}, \frac{-2(1+\beta\tau)}{\tau^2}) \ . \tag{2.25}$$

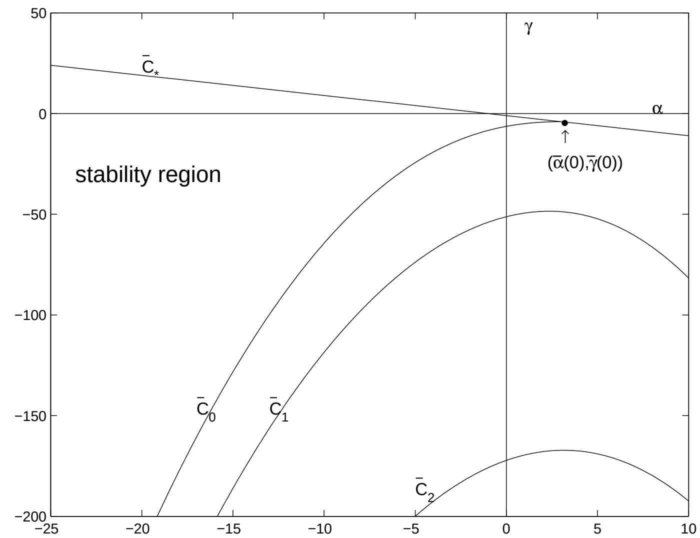

FIG. 2.5. *The curves $\bar{C}_k(\beta)$ in the $(\alpha,\gamma)-$plane for $\beta = 1, \tau = 1$ and $m = 12$.*

Note that this limit point is identical to the point $(\alpha(0),\gamma(0))$ derived earlier in (2.8). For illustration purposes, some curves $\bar{C}_k(\beta)$ are shown in Fig. 2.5 for $\beta = 1, \tau = 1$ and $m = 12$. Now we prove some results similar to the ones derived earlier in §2.1.

LEMMA 2.14. *The curves $\bar{C}_k(\beta)$ do not intersect.*

Proof. Suppose that there exist $\varphi_1 \in \bar{J}_k$ and $\varphi_2 \in \bar{J}_l$ with $k \neq l$ such that $\bar{\alpha}(\varphi_1) = \bar{\alpha}(\varphi_2)$ and $\bar{\gamma}(\varphi_1) = \bar{\gamma}(\varphi_2)$. These equalities lead to the following two identities

$$\frac{\sin\varphi_1 \sin m\varphi_1}{(1+\cos\varphi_1)(1-\cos m\varphi_1)} = \frac{\sin\varphi_2 \sin m\varphi_2}{(1+\cos\varphi_2)(1-\cos m\varphi_2)} \tag{2.26}$$

$$\frac{\sin^2\varphi_1}{(1+\cos\varphi_1)^2(1-\cos m\varphi_1)} = \frac{\sin^2\varphi_2}{(1+\cos\varphi_2)^2(1-\cos m\varphi_2)} \ . \tag{2.27}$$

Squaring (2.26) and dividing the result by (2.27) yields

$$\frac{\sin^2 m\varphi_1}{1-\cos m\varphi_1} = \frac{\sin^2 m\varphi_2}{1-\cos m\varphi_2} \ ,$$

which leads to $\cos m\varphi_1 = \cos m\varphi_2$. Then, from (2.27) it follows that

$$\frac{\sin^2\varphi_1}{(1+\cos\varphi_1)^2} = \frac{\sin^2\varphi_2}{(1+\cos\varphi_2)^2} \ .$$

As the positive function $\frac{\sin\varphi}{1+\cos\varphi}$ is strictly monotonically increasing when $\varphi \in (0,\pi)$, one finds $\varphi_1 = \varphi_2$, which contradicts the initial assumption. □

One can further prove very similar lemmata to those that have been proven in §2.1. We consider, e.g., the intersection of the curves $\bar{C}_k(\beta)$ and the line $\bar{\alpha} = \beta$ when $m > 1$. The equality $\bar{\alpha}(\varphi_k) = \beta$ for $\varphi_k \in \bar{J}_k, k = 0,1,\ldots,\lfloor\frac{m}{2}\rfloor - 1$, implies

$$\varphi_k = \frac{(2k+1)\pi}{m} \quad \text{and} \quad \bar{\gamma}(\varphi_k) = \frac{-2m^2\sin^2\varphi_k}{\tau^2(1+\cos\varphi_k)^2} \ .$$

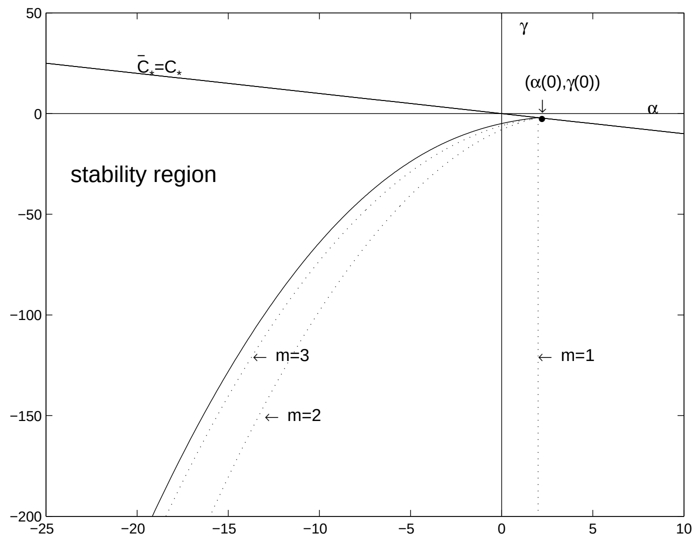

FIG. 2.6. *Curve $C_0(\beta)$ (solid) and $\bar{C}_0(\beta)$-curves for different values of m (dotted), for $\beta = 0$.*

Therefore, $\bar{\gamma}(\varphi_{k+1}) < \bar{\gamma}(\varphi_k)$, which shows the curves $\bar{C}_k(\beta)$, $k = 0, 1, \ldots, \lfloor \frac{m}{2} \rfloor - 1$, can be ordered in a natural way. If m is odd, it is easily verified that $\bar{C}_{\lfloor \frac{m}{2} \rfloor}(\beta)$ does not intersect the line $\bar{\alpha} = \beta$, since

$$\lim_{\varphi \to \pi} \bar{\alpha}(\varphi) = \beta + 2m^2/\tau \quad \text{and} \quad \lim_{\varphi \to \pi} \bar{\gamma}(\varphi) = -\infty \ .$$

2.2.2. Stability region of the discrete scheme. Based on the continuity of the roots, one may state that the number of roots outside the unit disk is constant inside the regions of the (α, γ)-space separated by the $\bar{C}_k(\beta)$-curves and the line $\bar{C}_*(\beta)$. It is easy to verify that all roots of (2.22) are in the unit disk when $\alpha = -|\beta| - 1, \gamma = 0$. Hence, we have the stability result given below. It states that the difference equation is asymptotically stable for the sets of parameters (α, γ) inside the region bounded by the line $\bar{C}_*(\beta)$ and the curve $\bar{C}_0(\beta)$ for the particular value of m. This is illustrated in Fig. 2.6 where the boundary curves $\bar{C}_0(\beta)$ are drawn for $\beta = 0$ and for different m.

THEOREM 2.15. *Difference equation (2.21) is asymptotically stable for $m = 1$ iff*
(a) $\alpha < \beta + \frac{2}{\tau}$ *and* *(b)* $\alpha + \beta < -\gamma\tau$.
The equation is asymptotically stable for $m > 1$ iff
(c) $\alpha < \beta + \frac{2}{\tau}$ *and*
(d) $(\alpha + \beta)\tau < -\gamma\tau^2 < \frac{-2m \sin\frac{\varphi}{m}(2m\sin\frac{\varphi}{m} + \beta\tau\sin\varphi(1+\cos\frac{\varphi}{m}))}{(1+\cos\frac{\varphi}{m})^2(1-\cos\varphi)}$
where φ is the root of $\alpha = \beta + \frac{2m\sin\frac{\varphi}{m}\sin\varphi}{\tau(1+\cos\frac{\varphi}{m})(1-\cos\varphi)}$ *such that* $\varphi \in (0, 2\pi)$.

Some more detailed features of the curve $\bar{C}_0(\beta)$ for the case $m > 1$ will be required in our study of fully discrete delay PDE problems. The following analyses are similar to the ones in §2.1.2, but technically somewhat more complicated. One can show that $\bar{\alpha}(\varphi)$ is strictly decreasing with respect to $\varphi \in \bar{J}_0$. The function $\bar{\gamma}(\varphi)$ on the other hand, for $\varphi \in \bar{J}_0$, satisfies

$$\bar{\gamma}'(\varphi) = \frac{2m\beta\tau(m\sin\varphi - \sin m\varphi)}{\tau^2(1+\cos\varphi)(1-\cos m\varphi)} - \frac{4m^2\sin\varphi(2-2\cos m\varphi - m\sin\varphi\sin m\varphi)}{\tau^2(1+\cos\varphi)^2(1-\cos m\varphi)^2}.$$

From this, one can derive that $\bar{\gamma}(\varphi)$ is strictly decreasing when $\beta\tau \le \frac{m^2+2}{m^2-1}$. When $\beta\tau > \frac{m^2+2}{m^2-1}$, there exists a unique $\bar{\varphi}_0 \in (0, 2\pi/m)$ such that

$$\beta\tau = \frac{2m\sin\bar{\varphi}_0(2-2\cos m\bar{\varphi}_0 - m\sin\bar{\varphi}_0\sin m\bar{\varphi}_0)}{(1-\cos m\bar{\varphi}_0)(1+\cos\bar{\varphi}_0)(m\sin\bar{\varphi}_0 - \sin m\bar{\varphi}_0)},$$

which gives

$$\bar{\alpha}(\bar{\varphi}_0) = \frac{2m\sin\bar{\varphi}_0(2-2\cos m\bar{\varphi}_0 - \sin^2 m\bar{\varphi}_0)}{\tau(1+\cos\bar{\varphi}_0)(1-\cos m\bar{\varphi}_0)(m\sin\bar{\varphi}_0 - \sin m\bar{\varphi}_0)} > 0 \ .$$

Therefore, $\bar{C}_0(\beta)$ has exactly one extremum in the right half-plane. Let

$$\bar{\alpha}_0 = \begin{cases} \beta + \frac{2}{\tau} & \text{if } m = 1 \ , \\ \bar{\alpha}(0) & \text{if } \beta\tau \le \frac{m^2+2}{m^2-1} \text{ and } m > 1 \ , \\ \bar{\alpha}(\bar{\varphi}_0) & \text{otherwise} \ . \end{cases} \tag{2.28}$$

Then, for any given $\beta \in \mathbb{R}$ and $m > 1$, $\bar{C}_0(\beta)$ is strictly monotonic in (α, γ)-space when $\alpha \le \bar{\alpha}_0$. In particular, $\bar{C}_0(\beta)$ is strictly monotonic in the left half of the (α, γ)-plane. This leads to the following results which will be used in the next section.

THEOREM 2.16. *Suppose all roots z of (2.22) lie inside the unit disk for given $\alpha, \beta, \gamma \in \mathbb{R}$, with $\alpha \le \bar{\alpha}_0$ for $\bar{\alpha}_0$ defined in (2.28). Then for any $x \le \alpha$, all roots z of the following equation lie in the unit disk too,*

$$1 - z^{-1} = \frac{h}{2}(1+z^{-1})[x + \beta z^{-m} + \frac{\gamma h}{2}(1+z^{-1})\sum_{j=0}^{m-1} z^{-j}] \ . \tag{2.29}$$

COROLLARY 2.17. *Suppose all roots z of (2.22) lie in the unit disk for given $\alpha \le 0, \beta, \gamma \in \mathbb{R}$. Then for any $x \le \alpha$, all roots z of (2.29) lie in the unit disk too.*

2.2.3. Relation between the curves $C_k(\beta)$ and $\bar{C}_k(\beta)$. We are now in a position to derive some relations between the boundary locus curves $C_k(\beta)$ of the continuous problem and the curves $\bar{C}_k(\beta)$ of the discrete problem. These relations will lead immediately to the central result of the present section, which states that the stability region of the continuous problem is contained in the stability region of the discrete problem.

LEMMA 2.18. *$C_0(\beta)$ and $\bar{C}_0(\beta)$ intersect only at the point $(\beta + \frac{2}{\tau}, \frac{-2(1+\beta\tau)}{\tau^2})$.*

Proof. If $m = 1$, then $\bar{\alpha}(\varphi) = \beta + \frac{2}{\tau}$. Considering the fact that

$$\frac{\theta\sin\theta}{1-\cos\theta} < 2 \quad \text{for} \quad \theta \in (0, 2\pi) \ ,$$

we have for $\alpha(\theta)$ from (2.6) that $\alpha(\theta) < \beta + \frac{2}{\tau}$ for $\theta \in (0, 2\pi)$.

Next, we deal with the case $m > 1$. Suppose there exist $m\varphi, \theta \in (0, 2\pi)$ such that $\bar{\alpha}(\varphi) = \alpha(\theta)$ and $\bar{\gamma}(\varphi) = \gamma(\theta)$, i.e.,

$$\frac{2m \sin\varphi \sin m\varphi}{(1+\cos\varphi)(1-\cos m\varphi)} = \frac{\theta \sin\theta}{1-\cos\theta} \tag{2.30}$$

$$\frac{4m^2 \sin^2\varphi}{(1+\cos\varphi)^2(1-\cos m\varphi)} = \frac{\theta^2}{1-\cos\theta} \ . \tag{2.31}$$

Squaring (2.30) and dividing the result by (2.31), we obtain

$$\frac{\sin^2 m\varphi}{1-\cos m\varphi} = \frac{\sin^2\theta}{1-\cos\theta} \ ,$$

which gives $\cos m\varphi = \cos\theta$. From (2.31) it follows then that

$$\theta^2 = \frac{4m^2 \sin^2\varphi}{(1+\cos\varphi)^2} > (m\varphi)^2,$$

where we have used that $m \geq 2$ such that $\varphi \in (0, \pi)$. Thus, $\cos m\varphi = \cos\theta$ implies

$$\theta + m\varphi = 2\pi \ , \ \theta \in (\pi, 2\pi) \ , \ m\varphi \in (0, \pi) \ .$$

Hence, the left side of (2.30) is positive and the right side is negative, which is impossible. This completes the proof. □

LEMMA 2.19. *Curve $C_0(\beta)$ lies above curve $\bar{C}_0(\beta)$ in the (α, γ)-plane for $m > 1$.*

Proof. The lemma follows from Lemma 2.18 combined with, for $m > 1$,

$$\alpha(\pi) = \bar{\alpha}(\frac{\pi}{m}) = \beta \quad \text{and} \quad \gamma(\pi) = \frac{-\pi^2}{2\tau^2} > \frac{-2m^2 \sin^2\frac{\pi}{m}}{\tau^2(1+\cos\frac{\pi}{m})^2} = \bar{\gamma}(\frac{\pi}{m}) \ . \ \square$$

THEOREM 2.20. *The trapezium rule (2.21) preserves the asymptotic stability of problem (2.1) if α, β, γ satisfy the conditions of Theorem 2.7.*

Proof. Suppose α, β and γ satisfy the conditions in Theorem 2.7. Then difference equation (2.21) is asymptotically stable. This follows immediately from Lemma 2.19 and from the fact that $C_*(\beta) = \bar{C}_*(\beta)$. □

Specialising Theorem 2.20 to the case $\gamma = 0$, one regains the stability result for the trapezium rule as obtained by Guglielmi in [10]. A further algebraic investigation reveals that for any fixed $\theta \in J_k$ with $k \geq 0$,

$$\lim_{m\to\infty} (\bar{\alpha}(\theta/m) - \alpha(\theta)) = O(m^{-2}) \ \text{ and } \ \lim_{m\to\infty} (\bar{\gamma}(\theta/m) - \gamma(\theta)) = O(m^{-2}) \ .$$

With $O(m^{-2}) = O(h^2)$, the above result can be formulated as follows.

THEOREM 2.21. *Curve $\bar{C}_k(\beta)$ is an $O(h^2)$ approximation to curve $C_k(\beta)$.*

Fig. 2.7 shows the approximation of $\bar{C}_k(0)$ to $C_k(0)$ for $m = 12$ and $\tau = 1$.

3. Stability of delay partial differential equations.

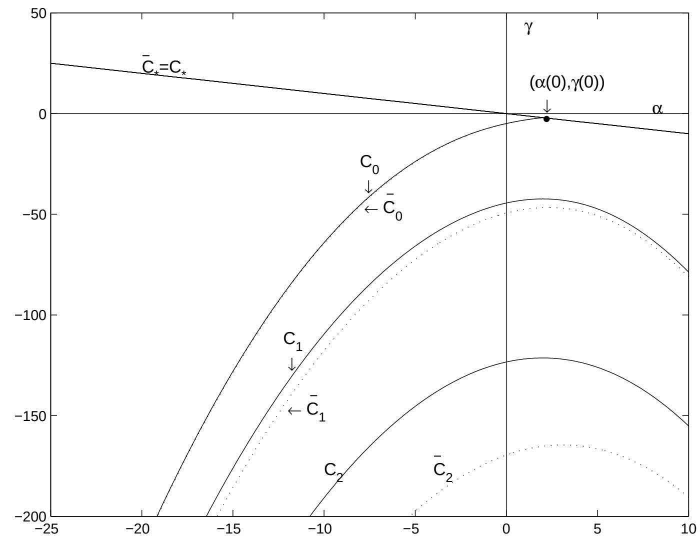

FIG. 2.7. *The curves $\bar{C}_k(\beta)$ and $C_k(\beta)$ for $\beta = 0$, $\tau = 1$ and $m = 12$. $\bar{C}_0$ and C_0 almost overlap.*

3.1. Stability of the continuous problem. Consider the two-dimensional parabolic functional-differential equation with both fixed and distributed delays,

$$\frac{\partial u}{\partial t} = a(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + bu(t-\tau, x, y) + c\int_{-\tau}^{0} u(t+s, x, y)ds \ , \tag{3.1}$$

defined on the square domain $(x, y) \in \Omega = [0, L] \times [0, L]$, for $t > 0$, and supplemented with initial and boundary conditions,

$$u(t, x, y) = g(t, x, y) \ , \quad t \in [-\tau, 0] \ , \quad (x, y) \in \Omega$$
$$u(t, x, y) = 0 \ , \quad t > 0 \ , \quad (x, y) \in \partial\Omega \ .$$

Here $a, b, c \in \mathbb{R}$; L and τ are positive constants. The characteristic equations of the above problem are given for $k, l = 1, 2, \ldots$, by

$$\lambda = -a[(\frac{k\pi}{L})^2 + (\frac{l\pi}{L})^2] + b\exp(-\lambda\tau) + c\int_{-\tau}^{0} \exp(\lambda s)ds \ , \tag{3.2}$$

see, e.g., [17, Ch. III]. The asymptotic stability of (3.1) is equivalent to the condition that all the roots of each of the equations (3.2) have negative real parts. This condition can be studied using the theory from §2, setting the parameters α, β, γ as :

$$\alpha = -a[(\frac{k\pi}{L})^2 + (\frac{l\pi}{L})^2] \ , \quad \beta = b \ , \quad \gamma = c \ .$$

The conditions of Theorem 2.7 can be reformulated for the present problem as

$$-a[(\frac{k\pi}{L})^2 + (\frac{l\pi}{L})^2] < b + \frac{2}{\tau} \quad \text{for} \quad k, l = 1, 2, \ldots, \tag{3.3}$$

$$-a[(\frac{k\pi}{L})^2 + (\frac{l\pi}{L})^2] + b < -c\tau \quad \text{for} \quad k, l = 1, 2, \ldots, \tag{3.4}$$

$$-c\tau^2 < \frac{\theta_{kl}(\theta_{kl} + b\tau\sin\theta_{kl})}{1 - \cos\theta_{kl}} \quad \text{for} \quad k, l = 1, 2, \ldots, \tag{3.5}$$

where θ_{kl} is the root of

$$-a[(\frac{k\pi}{L})^2 + (\frac{l\pi}{L})^2] = b + \frac{\theta_{kl}\sin\theta_{kl}}{\tau(1-\cos\theta_{kl})} \tag{3.6}$$

such that $\theta_{kl} \in (0, 2\pi)$. By using the corollary to Theorem 2.9 one verifies the following result.

THEOREM 3.1. *The zero solution of (3.1) is asymptotically stable iff*
(a) $a \geq 0$,
(b) $-2a\pi^2/L^2 < b + 2/\tau$, *and*
(c) $(-2a\pi^2/L^2 + b)\tau < -c\tau^2 < \frac{\theta(\theta+b\tau\sin\theta)}{1-\cos\theta}$,
where θ *is the root of* $-2a\pi^2/L^2 = b + \frac{\theta\sin\theta}{\tau(1-\cos\theta)}$ *such that* $\theta \in (0, 2\pi)$.

Proof. The "only if" part follows from (3.3) - (3.5) and the fact that (3.3) implies $a \geq 0$. The "if" part follows from Corollary 2.10 which implies that the stability is determined by the equation corresponding to the largest value of α, i.e., the one corresponding to $k = l = 1$. □

Next we consider some special cases. First, we deal with the case of a parabolic partial differential equation with a distributed delay, i.e., the case $b = 0$,

$$\frac{\partial u}{\partial t} = a(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + c\int_{-\tau}^{0} u(t+s, x, y)ds \ . \tag{3.7}$$

A direct application of Theorem 3.1 to (3.7) yields the following result.

COROLLARY 3.2. *The zero solution of (3.7) is asymptotically stable iff*
(a) $a \geq 0$ *and* *(b)* $-2\pi^2\tau a/L^2 < -c\tau^2 < \frac{\theta^2}{1-\cos\theta}$
where θ *is the root of* $\theta\sin\theta = -2\tau a\pi^2(1-\cos\theta)/L^2$ *such that* $\pi \leq \theta < 2\pi$.

For $L = \pi$ and $\tau = 1$, the stability region of (3.7) in the (a, c)-plane is shown in the left picture of Fig. 3.1. The right picture of that figure shows the stability region for $a = 1/2$ and $c = -1$, in the (τ, L)-plane.

The next example is a parabolic PDE with a fixed delay, i.e., with $c = 0$,

$$\frac{\partial u}{\partial t} = a(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + bu(t-\tau, x, y) \ . \tag{3.8}$$

A similar application of the general theorem leads to the following corollary.

COROLLARY 3.3. *The zero solution of Equation (3.8) is asymptotically stable iff*
(a) $a \geq 0$ *and* *(b)* $-2\pi^2 a/L^2 < -b < \frac{\theta}{\tau\sin\theta}$
where θ *is the root of* $\theta\cos\theta = -2\tau a\pi^2\sin\theta/L^2$ *such that* $\pi/2 \leq \theta < \pi$.

For $L = \pi$ and $\tau = 1$, the stability region of the zero solution to (3.8) is shown in the (a, b)-plane in the left picture of Fig. 3.2. The right picture of that figure shows the stability region in the (τ, L)-plane for fixed $a = 1/2$ and $b = -1$.

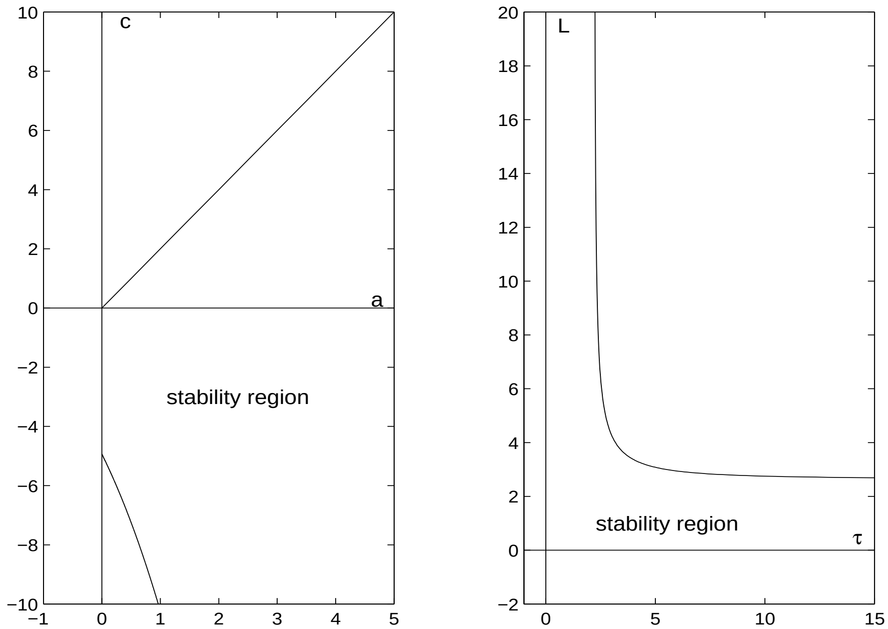

FIG. 3.1. *Stability region of distributed delay PDE (3.7). Left:* (a, c)*-plane with* $L = \pi$ *and* $\tau = 1$*. Right:* (τ, L)*-plane with* $a = 1/2$ *and* $c = -1$*.*

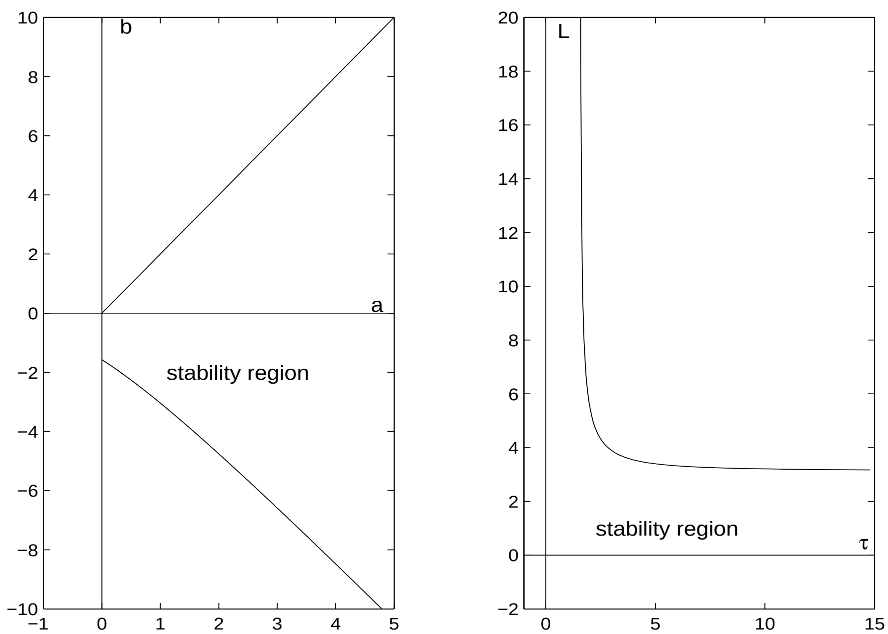

FIG. 3.2. *Stability region of fixed delay PDE (3.8). Left:* (a, b)*-plane with* $L = \pi$ *and* $\tau = 1$*. Right:* (τ, L)*-plane with* $a = 1/2$ *and* $b = -1$*.*

3.2. Stability of the semi-discrete PDE problem. We discretize the spatial variables (x, y) into $(N+2)\times(N+2)$ discrete values ($N > 0$) with a constant stepsize $\Delta x = \Delta y = L/(N+1)$, so that $x_i = i\Delta x,\ y_j = j\Delta y$ for $i, j = 0, 1, 2, \ldots, N+1$. Using the standard central difference operator to approximate the Laplacian we obtain a system of delay ODEs of the form

$$u'_{ij}(t) = a\left(\frac{u_{i+1,j}(t) + u_{i-1,j}(t) + u_{i,j+1}(t) + u_{i,j-1}(t) - 4u_{ij}(t)}{\Delta x^2}\right)$$
$$+b\,u_{ij}(t-\tau) + c\int_{-\tau}^{0} u_{ij}(t+s)ds, \qquad i, j = 1, \ldots, N\ , \tag{3.9}$$

where $u_{ij}(t)$ denotes an approximation to $u(t, x_i, y_j)$. Let

$$U(t) = (u_{11}(t), \ldots, u_{1N}(t), u_{21}(t), \ldots, u_{2N}(t), \ldots\ldots, u_{N1}(t), \ldots, u_{NN}(t))^T.$$

Considering the homogeneous Dirichlet conditions and the fact that $\Delta x = \Delta y$, one can write the semi-discrete system compactly as

$$U'(t) = \frac{a(N+1)^2}{L^2}(A\otimes I_N + I_N\otimes A)U(t) + bU(t-\tau) + c\int_{-\tau}^{0} U(t+s)ds\ , \tag{3.10}$$

where I_N stands for the $N \times N$ identity matrix and

$$A = \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & -2 \end{bmatrix}_{N\times N} .$$

The asymptotic stability of the zero solution to (3.10) is equivalent to the condition that all the roots of the characteristic equation

$$\det\left(\left((\lambda - b\exp(-\lambda\tau) - c\int_{-\tau}^{0}\exp(\lambda s)ds\right) I_{N^2} - \frac{a(N+1)^2}{L^2}(A\otimes I_N + I_N\otimes A)\right) = 0$$

have negative real parts. The eigenvalues of A are given by

$$\lambda_i^A = -4\sin^2\frac{i\pi}{2(N+1)}\ ,\quad i = 1, \ldots, N.$$

Therefore, the eigenvalues of the matrix

$$\frac{(N+1)^2}{L^2}(A\otimes I_N + I_N\otimes A) \tag{3.11}$$

are given by

$$\lambda_{ij} = \frac{-4(N+1)^2}{L^2}(\sin^2\frac{i\pi}{2(N+1)} + \sin^2\frac{j\pi}{2(N+1)})\ ,\quad i, j = 1, \ldots, N. \tag{3.12}$$

As a consequence, the asymptotic stability of system (3.10) is equivalent to the condition that all the roots of each of the following equations have negative real parts

$$\lambda - a\lambda_{ij} - b\exp(-\lambda\tau) - c\int_{-\tau}^{0}\exp(\lambda s)ds = 0\ ,\quad i, j = 1, \ldots, N\ . \tag{3.13}$$

A direct application of Corollary 2.10 reveals the conditions for stability.

THEOREM 3.4. *Assume $a \geq 0$. Then the zero solution of the semi-discrete delay PDE (3.9) is asymptotically stable iff*

(a) $a\lambda_{11} < b + 2/\tau$ *and* *(b)* $(a\lambda_{11} + b)\tau < -c\tau^2 < \frac{\theta(\theta + b\tau \sin\theta)}{1-\cos\theta}$

where θ is the root of $a\lambda_{11} = b + \frac{\theta \sin\theta}{\tau(1-\cos\theta)}$, $\theta \in (0, 2\pi)$ and λ_{11} is defined in (3.12).

Proof. Based on Corollary 2.10, one can see that the stability is determined by the equation corresponding to the largest value of α, i.e., the eigenvalue λ_{11}. □

The semi-discretization of delay PDE (3.8) leads to a set of delay ODEs

$$u'_{ij} = a\left(\frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij}}{\Delta x^2}\right) + bu_{ij}(t-\tau) , \tag{3.14}$$

for $i, j = 1, \ldots, N$. Analysis of this set of equations yields the next result.

COROLLARY 3.5. *Assume $a \geq 0$. Then the zero solution of the semi-discrete delay PDE (3.14) is asymptotically stable iff*

$$a\lambda_{11} < -b < \frac{\theta}{\tau \sin\theta}$$

where θ is the root of $\theta\cos\theta = a\lambda_{11}\tau\sin\theta$ such that $\pi/2 \leq \theta < \pi$ and λ_{11} is defined in (3.12).

The semi-discretisation of the distributed delay PDE (3.7) gives

$$u'_{ij} = a\left(\frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij}}{\Delta x^2}\right) + c\int_{-\tau}^{0} u_{ij}(t+s)ds , \tag{3.15}$$

for $i, j = 1, \ldots, N$. A direct application of Theorem 3.4 leads to the next corollary.

COROLLARY 3.6. *Assume $a \geq 0$. Then the zero solution of the semi-discrete delay PDE (3.15) is asymptotically stable iff*

$$a\lambda_{11}\tau < -c\tau^2 < \frac{\theta^2}{1-\cos\theta}$$

where θ is the root of $\theta\sin\theta = a\lambda_{11}\tau(1-\cos\theta)$ such that $\pi \leq \theta < 2\pi$.

It is interesting to compare the stability region of the original PDE with that of the semi-discretized set of ODEs. The following theorem shows that, when $a > 0$, the stability region of the latter is smaller than the stability region of the former. This is different from the case of PDEs without delays. Here we do not draw the pictures of these stability regions because they can hardly be distinguished from their counterparts in the continuous case. Besides, both regions converge to each other when $\Delta x \to 0$.

THEOREM 3.7. *Assume $a > 0$. Then the asymptotic stability region of (3.9) is a subset of the asymptotic stability region of (3.1).*

Proof. Based on Corollary 2.10, one can see that the stability of (3.9) is determined by equation (3.13) with $i = j = 1$ and the stability of (3.1) is determined by equation (3.2) with $k = l = 1$. A repeated application of Corollary 2.10 combined with $-2\pi^2/L^2 < \lambda_{11} < 0$ and $a > 0$ leads to the conclusion. □

3.3. Stability of the fully discrete PDE problem. An application of the trapezium rule to (3.9) leads to the scheme

$$
\begin{aligned}
\frac{u_{ij}^{n+1}-u_{ij}^{n}}{\Delta t} = \frac{a}{2}&\left(\frac{u_{i+1,j}^{n+1}+u_{i-1,j}^{n+1}+u_{i,j+1}^{n+1}+u_{i,j-1}^{n+1}-4u_{ij}^{n+1}}{\Delta x^2}\right)\\
&+\frac{b}{2}u_{ij}^{n+1-m}+\frac{c\Delta t}{4}\Big(\sum_{k=0}^{m-1}u_{ij}^{n+1-k}+\sum_{k=1}^{m}u_{ij}^{n+1-k}\Big)\\
&+\frac{a}{2}\left(\frac{u_{i+1,j}^{n}+u_{i-1,j}^{n}+u_{i,j+1}^{n}+u_{i,j-1}^{n}-4u_{ij}^{n}}{\Delta x^2}\right)\\
&+\frac{b}{2}u_{ij}^{n-m}+\frac{c\Delta t}{4}\Big(\sum_{k=0}^{m-1}u_{ij}^{n-k}+\sum_{k=1}^{m}u_{ij}^{n-k}\Big),
\end{aligned}
\tag{3.16}
$$

for $i,j=1,2,\ldots,N$; the time stepsize is given by $\Delta t=\tau/m$, each u_{ij}^n denotes an approximation to $u(n\Delta t, i\Delta x, j\Delta y)$. An analysis similar to the one leading to Theorem 3.4 in combination with Theorem 2.15 and Corollary 2.17 leads to the next theorem.

THEOREM 3.8. *Assume $a \geq 0$. Let λ_{11} be defined by (3.12). When $m=1$, the difference equation (3.16) is asymptotically stable iff*

(a) $a\lambda_{11} < b+\frac{2}{\tau}$ *and* *(b)* $a\lambda_{11}+b < -c\tau$.

When $m>1$, the difference equation (3.16) is asymptotically stable iff

(c) $a\lambda_{11} < b+\frac{2}{\tau}$ *and*

(d) $(a\lambda_{11}+b)\tau < -c\tau^2 < \frac{-2m\sin\frac{\varphi}{m}(2m\sin\frac{\varphi}{m}+b\tau\sin\varphi(1+\cos\frac{\varphi}{m}))}{(1+\cos\frac{\varphi}{m})^2(1-\cos\varphi)}$

where φ is the root of $a\lambda_{11} = b+\frac{2m\sin\frac{\varphi}{m}\sin\varphi}{\tau(1+\cos\frac{\varphi}{m})(1-\cos\varphi)}$ *such that* $\varphi\in(0,2\pi)$.

This scheme fully preserves the stability of ODE system (3.9). It is easily seen, however, that the fully discrete scheme (3.16) cannot completely preserve the asymptotic stability of the underlying PDE system (3.1). In fact, we have the following general result.

THEOREM 3.9. *Any consistent time discretization scheme applied to the spatial discretization (3.9) cannot completely preserve the asymptotic stability of (3.1).*

Proof. We will prove the result by constructing a counterexample. To this end we consider parameters $a,b,c\in\mathbb{R}$ such that

$$a\lambda_{11}+b+c\tau = 0\ , \tag{3.17}$$

and show that there always exist such parameters that are outside the fully discrete stability region, yet inside the fully continuous stability region.

Let $V\in\mathbb{R}^{N^2}$ be the eigenvector corresponding to the eigenvalue λ_{11} of the matrix (3.11). Consider the initial value problem (3.10) with the initial value condition $U(t)=V, t\in[-\tau,0]$. Then, with any a,b, and c satisfying (3.17), the exact solution of (3.9) on interval $[0,+\infty)$ is $U(t)=V$. Therefore, the numerical solution of any consistent scheme is always V too. Hence, every (a,b,c) determined by (3.17) is always out of the asymptotic stability region of the fully discrete scheme.

On the other hand,

$$0>\lambda_{11}>\frac{-2\pi^2}{L^2}, \tag{3.18}$$

which implies that there exists a part of the sets of parameters (a, b, c) determined by (3.17) which belongs to the stability region of (3.1). In fact, the limiting values

$$\lim_{\theta\to 2\pi-0}(\theta + b\tau\sin\theta) = \theta > 0 \quad \text{and} \quad \lim_{\theta\to 2\pi-0}\frac{\theta\sin\theta}{\tau(1-\cos\theta)} = -\infty \ ,$$

imply that there exists $\hat{\theta} \in (\pi, 2\pi)$ such that

$$\hat{\theta} + b\tau\sin\hat{\theta} > 0 \quad \text{and} \quad -2a\pi^2/L^2 = b + \frac{\hat{\theta}\sin\hat{\theta}}{\tau(1-\cos\hat{\theta})}, \tag{3.19}$$

where $a > \frac{|b|+1}{-\lambda_{11}} > 0$. We further choose $c = -(a\lambda_{11} + b)/\tau$, such that $c > 0$ and a, b and c satisfy (3.17). With (3.18), it is then possible to verify

$$-2a\pi^2/L^2 < -(|b|+1) < b + 2/\tau \quad \text{and} \quad (-2a\pi^2/L^2 + b)\tau < (a\lambda_{11} + b)\tau = -c\tau^2.$$

Considering $c > 0$ and (3.19), we conclude that a, b and c satisfy the conditions of Theorem 3.1: they belong to the stability region of (3.1). This completes the proof. □

REMARK 3.1. The above proof shows that a reduction of the time stepsize cannot improve the stability of the scheme. No matter how small Δt, one can always find a set of parameters for which the PDE is asymptotically stable, while the fully discrete system is not asymptotically stable.

4. Numerical examples. In this section we present some numerical examples illustrating the difference between the stability regions of the continuous and the discretised systems. For simplicity, we consider the one-dimensional equation on $x \in [0, \pi]$, and we set the delay $\tau = 1$,

$$\frac{\partial}{\partial t}u(t,x) = a\frac{\partial^2}{\partial x^2}u(t,x) + bu(t-1,x) + c\int_{-1}^{0} u(t+s,x)ds \ , \tag{4.1}$$

with $u(t,x) = g(x), t \le 0, x \in (0,\pi)$ and $u(t,x) = 0, t > 0, x \in \{0,\pi\}$.

The problem is discretised with central differences in space with space stepsize $\Delta x = \pi/(N+1)$ and with the trapezium rule in time, with time stepsize $\Delta t = 1/m$. We set the initial values $u_i^k = 1, i = 1, 2, \ldots, N, k = -m, -m+1, \ldots, 0$. We consider the norm of the numerical solution at time-point $T = 1000$ as a function of Δt and Δx. Here we use the weighted discrete L_2-norm to measure the numerical solution:

$$\|u\| = \sqrt{\sum_{j=1}^{N}(u_j)^2\Delta x}, \quad u = (u_1, u_2, \ldots, u_N)^T \in \mathbb{R}^N.$$

First we consider a fixed delay PDE problem with parameters $a = 8, b = -8.4772$, and $c = 0$. From Theorem 3.1 it follows that the analytical solution is asymptotically stable. The norm of the numerical solution is given in Table 4.1. The column of values for $\Delta x = \pi/17$, i.e., $N = 16$ will be explained by means of Fig. 4.1 (left). The row of values for $\Delta t = 1/8$, i.e., $m = 8$, will be explained by means of Fig. 4.1 (right). Next, we also consider a distributed delay PDE problem with parameters $a = 10, b = 0, c = 9.999$. From Theorem 3.1 it follows that the analytical solution is asymptotically stable too. The norm of the numerical solution is given in Table 4.2.

These numerical examples confirm our theoretical findings. In the case of delay PDEs, the trapezium rule can only preserve the stability region of the semi-discrete

TABLE 4.1
The norm of the numerical solution of (4.1) at $T = 1000$ as a function of Δt and Δx for $a = 8, b = -8.4772, c = 0$.

	$\Delta x = \pi/9$	$\Delta x = \pi/17$	$\Delta x = \pi/33$	$\Delta x = \pi/65$
$\Delta t = 1/2$	1.4739E-04	8.4839E-07	3.2096E-04	0.0157
$\Delta t = 1/4$	15.1095	0.4321	0.0540	0.0279
$\Delta t = 1/8$	573.3108	7.9030	1.2107	0.6878
$\Delta t = 1/16$	5.1889E+03	10.2455	2.7399	1.8089

TABLE 4.2
The norm of the numerical solution of (4.1) at $T = 1000$ as a function of Δt and Δx for $a = 10, b = 0, c = 9.999$.

	$\Delta x = \pi/9$	$\Delta x = \pi/17$	$\Delta x = \pi/33$	$\Delta x = \pi/65$
$\Delta t = 1/8$	2.9882E+07	154.2481	4.7490	1.8679
$\Delta t = 1/16$	2.9867E+07	154.2417	4.7490	1.8679

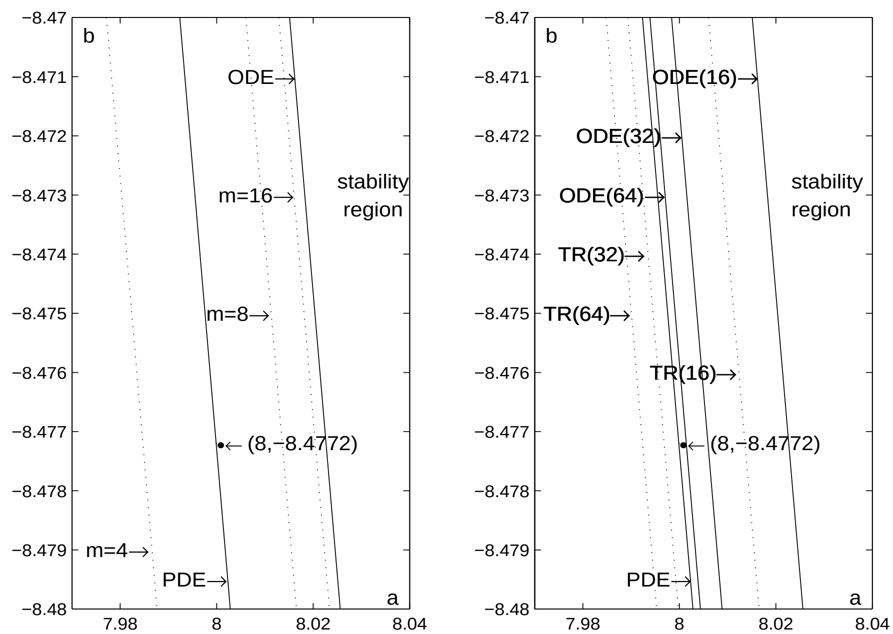

FIG. 4.1. *Local boundaries of the stability regions of (3.1) and the discrete systems with $c = 0$. The left is for $N = 16$ and for different m (dotted). The right is for $m = 8$ and for different N (solid: semi-discrete systems; dotted: fully-discrete systems).*

system which is smaller than that of the original PDE. The reduction of the time stepsize leads to a better approximation to the stability region of the semi-discrete system such that the stability region of the fully discrete scheme is further reduced. This is illustrated in Fig. 4.1 where the local boundary curves of the stability regions of the scheme for $c = 0$ and $N = 16$ and for different m are drawn in the left picture. Obviously, the point (8,-8.4772) is located in the stability region of PDE. When $m = 4$ (i.e. $\Delta t = 1/4$), it is also in the stability region of the fully discrete scheme. When $m = 8$ or 16, however, it is out of the stability region of the scheme. From the picture it is obvious to see that a smaller time stepsize leads to a smaller stability region. This is in accordance with the numerical results in Table 4.1. However, the reduction of the spatial stepsize is apparently useful to improve the stability; it leads to a better approximation to the stability region of the original PDE. This is shown in the right picture where the local boundary curves of the stability regions of the semi-discrete and fully-discrete ($m = 8$) systems for different N are drawn.

The results of Table 4.2 are different from the ones in Table 4.1. This is because we selected a point (a, c) near the boundary curve C_*, which depends only on the spatial stepsize but which is independent of the time stepsize. The changes of the time stepsize do not affect the stability. A spatial stepsize reduction, however, leads to a bigger stability region of the scheme, as is illustrated in the table.

5. Concluding remarks. In this work a novel stability property of numerical methods for differential equations with both fixed and distributed delays has been introduced. A complete stability analysis has been accomplished for the trapezium rule with respect to the real coefficient test problems of both ordinary and partial differential equations. A positive result is that the time discretization based on the trapezium rule can preserve the asymptotic stability of underlying ODE test problems. A negative result is that the spatial discretization based on standard central difference unavoidably leads to a reduction of the asymptotic stability region of the correponding PDE problem. Any consistent full discretization based on central difference cannot completely preserve the asymptotic stability of the original delay PDE.

Acknowledgements. The authors would like to thank Dr. Tatyana Luzyanina for many valuable discussions.

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