

**Global stabilization of multiple
integrators with time-delay and input
constraints**

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Report TW 325, May 2001



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Abstract

In this paper we construct parametrized controllers for the semi-global and global asymptotic stabilization of a multiple integrator with time-delay and constraints on the input variable. The stability results are semi-global in the delay.

Keywords : delay equations, stabilization, input constraints, low-gain control

1 Introduction

We study the global stabilization of linear systems with time-delay and constraints on the input variable,

$$\dot{y} = Ay + B\text{sat}(u(t - \tau)), \quad y \in \mathbb{R}^n, \quad u \in \mathbb{R}, \quad (1)$$

by means of static state feedback, $u = F(y)$.

When $\tau = 0$, it is well known that the system (1) can only be globally asymptotically stabilized when the pair (A, B) is controllable and A has all its eigenvalues in the *closed* left half plane, see [10] and the references therein. Moreover both in [10], where a saturation design is used, and in [8], in the context of low-gain design, procedures were given for the explicit construction of a globally stabilizing controller.

In the literature on delay equations the (robust) stabilization of the system

$$\dot{y} = Ay + A_d y(t - \tau) + B \text{sat}(u(t)), \quad (2)$$

has been widely studied, see e.g. [5, 9, 11, 12]. At one hand the uncontrolled system is more general than in (1), since it contains a delayed term. This leads to a lot of technicalities in the analysis, but on the other hand the stabilizability problem itself is facilitated by the fact that the control input is not delayed. Further, research up to now has been focussed on the derivation of sufficient conditions for (robust) local or global stabilization in a Lyapunov framework. Typically, such conditions are expressed by the feasibility of an LMI or the solvability of an ARE, on which also the construction of a controller relies. However in the context of global stabilization, few attention has been paid to the structural requirements on the controller and to the study of the feasibility of such LMIs or the solvability of such AREs in the important case where the uncontrolled system has its rightmost eigenvalues on the imaginary axis. Note for instance that in that case, *nonlinear* feedback is generally needed for global stabilization, even when $\tau = 0$, see [10].

In this paper we assume that (1) has its rightmost eigenvalues on the imaginary axis for $u = 0$. We restrict ourselves to the simple, yet practically important case where the system (1) corresponds to the multiple integrator,

$$\dot{y}_1 = y_2, \quad \dot{y}_2 = y_3, \dots, \dot{y}_{n-1} = y_n, \quad \dot{y}_n = \text{sat}(u(t - \tau)), \quad (3)$$

and construct (semi)-globally stabilizing controllers..

The structure of the paper is as follows. First we study the local and semi-global stabilization of (3) with linear low-gain feedback and briefly discuss an alternative. Then we consider the global stabilization problem.

Preliminaries: The state of the delay equation

$$\dot{y} = f(y, y(t - \tau)), \quad y \in \mathbb{R}^n, \quad f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \quad (4)$$

at time t can be described as a vector $y(t) \in \mathbb{R}^n$ or as a function segment y_t defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-\tau, 0].$$

When the right hand side of (4) is continuous and Lipschitz in both of its arguments, a solution is uniquely defined by specifying as initial condition a function segment y_0 , where $y_0 \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$, the Banach space of continuous functions mapping the delay-interval $[-\tau, 0]$ into \mathbb{R}^n and equipped with the supremum-norm $\|\cdot\|_s$. Since also $y_t \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ for all $t \geq 0$, the delay equation can be considered as an evolutionary equation over this space. Therefore delay equations form a special class of functional differential equations [6, 7].

Stability definitions are analogous to the ODE case, see for instance [6, Section V]. When $f(0, 0) = 0$ in (4), the zero solution is asymptotically stable when the following conditions hold simultaneously:

1. Stability: *for all $\epsilon > 0$ there is a $\delta > 0$ such that $|y(t)| \leq \epsilon$ for all $t \geq 0$ and $y_0 \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ with $\|y_0\|_s \leq \delta$.*
2. Attractivity: *there is a $\Delta > 0$ such that $\lim_{t \rightarrow \infty} y(t) = 0$ for all $y_0 \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ with $\|y_0\|_s \leq \Delta$.*

The zero solution is globally asymptotically stable when $\Delta = \infty$. It is delay-independent asymptotically stable when the asymptotic stability property holds for all $\tau \geq 0$. For a linear delay equation $\dot{y} = Ay + A_d y(t - \tau)$, (global) asymptotic stability is equivalent with the fact that all eigenvalues, i.e. the roots of the characteristic equation,

$$\det(\lambda I - A - A_d e^{-\lambda\tau}) = 0,$$

are in the open left half plane. When the delay equation is parametrized,

$$\dot{y} = f(y, y(t - \tau), \epsilon), \quad \epsilon > 0, \quad f(0, 0, \epsilon) = 0, \quad (5)$$

its zero solution is semi-globally asymptotically stable in y (and semi-globally in the delay τ), iff for all $R > 0$ (and $\forall \bar{\tau} > 0$), there is a $\bar{\epsilon} > 0$ such that for all $\epsilon < \bar{\epsilon}$ (and $\forall \tau \leq \bar{\tau}$) it is asymptotically stable with the attractivity property holding for $\|y_0\|_s \leq R$. For functions $\eta, T : \mathbb{R}^+ \times \mathcal{C}([-\tau, 0], \mathbb{R}^n) \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ and a number $p \in \mathbb{N}$, we say that

$$\eta = O(\epsilon^p), \quad t \geq T \quad (6)$$

w.r.t. the solutions of (5) when there exists constants $\bar{\epsilon}, \gamma > 0$ such that

$$\begin{aligned} |\eta(\tau, y_0, \epsilon)| &\leq \gamma \cdot \epsilon^p, \quad \forall t \geq T(\tau, y_0, \epsilon), \\ \forall \epsilon &\leq \bar{\epsilon}, \quad \forall y_0 \in \mathcal{C}([-\tau, 0], \mathbb{R}^n). \end{aligned} \quad (7)$$

For a constant $\bar{\tau} > 0$, we have $\eta = O(\epsilon^p)$, $\tau \leq \bar{\tau}$, $t \geq T$, when (7) holds uniformly in $\tau \in [0, \bar{\tau}]$, i.e. with $\bar{\epsilon}$ and γ independent of τ .

The delay equations considered in this paper originate from a stabilization problem of the form

$$\dot{y} = f(y, u(t - \tau)), \quad u = g(y), \quad (8)$$

and we analyse the stability of the closed-loop system,

$$\dot{y} = f(y, g(y(t - \tau))), \quad y_0 \in \mathcal{C}([-\tau, 0], \mathbb{R}^n). \quad (9)$$

However for (8) it is more natural to take as initial conditions $u(t) = 0$ for $t \in [-\tau, 0]$ and $y(0) \in \mathbb{R}^n$. Then (8) is equivalent with

$$\begin{cases} \dot{y} = f(y, 0) & y(0) \in \mathbb{R}^n \quad t \in [0, \tau], \\ \dot{y} = f(y, g(y(t - \tau))) & t \geq \tau. \end{cases} \quad (10)$$

However, due to the time-invariance of (9), stability of (10) is implied by the stability of (9).

2 Linear low-gain feedback

In this section we consider the stabilization of (3) with the parametrized family of linear state feedback laws

$$u = K(\epsilon)^T y = -k_1 \epsilon^n y_1 - k_2 \epsilon^{n-1} y_2 - \dots - k_n \epsilon y_n, \quad \epsilon > 0, \quad (11)$$

where we assume that the polynomial $\lambda^n + \sum_{i=1}^n k_i \lambda^{i-1}$ is hurwitz. Since $\|K(\epsilon)\| \rightarrow 0$ as $\epsilon \rightarrow 0$, the control laws (11) correspond to low-gain feedback.

Instrumental to the derivation of the stabilizability properties of (11) is the similarity transformation $z = T(\epsilon)y$, defined by

$$z_i = \epsilon^{n-i+1} y_i, \quad i = 1 \dots n, \quad (12)$$

and the re-scaling of time

$$t_{(\text{new})} = \epsilon t_{(\text{old})}, \quad (13)$$

which allow to transform the linearized closed loop system into

$$\dot{z} = Az + BK^T z(t - \epsilon\tau), \quad (14)$$

where $K = [-k_1 \quad -k_2 \dots -k_n]^T$. Because of the assumption on $K(\epsilon)$ in (11), matrix $\bar{A} \triangleq A + BK^T$ is hurwitz. Hence for all $Q > 0$, there exists a $P > 0$ satisfying the Riccati equation

$$\bar{A}^T P + P \bar{A} = -Q \quad (15)$$

and

$$V = z^T P z = y^T T(\epsilon)^T P T(\epsilon) y \triangleq y^T P(\epsilon) y \quad (16)$$

is a Lyapunov function for the closed loop system when $\tau = 0$.

2.1 Local stability and performance

Because $A + BK^T$ is Hurwitz, the system (14) is asymptotically stable for small values of $\epsilon\tau$. Consequently we have

Theorem 2.1 *The system (3) can be locally asymptotically stabilized, semi-globally in the delay with the control law (11), as $\epsilon \rightarrow 0$.*

Note however that delay-independent stabilization is not possible, since A is not Hurwitz [6]. Intuitively the semi-global stabilizability in the delay is explained by the fact that, whatever the value of the delay, it becomes negligible compared to the system's dynamics as $\epsilon \rightarrow 0$.

Remark: In singular perturbation theory, the opposite phenomenon may occur: when for instance the fast dynamics of a slow-fast system are described by $\epsilon\dot{y} = -y(t - \tau)$ instability occurs for small values of ϵ , see [4].

Despite of the good stabilizability properties of the controller (11), the input delay puts severe restrictions on its performance, in contrast to the ODE-case, where ϵ performs a scaling of the closed-loop eigenvalues. From (14), it follows that for a fixed value of τ , increasing ϵ ultimately leads to instability and hence there exists an optimal value of ϵ , maximizing the exponential decay rate of the solutions. As an illustration, the rightmost eigenvalues of the system (3)-(11) are shown in Figure 1 for $n = 2$ and $K = [-2 \ -2]^T$, in function of ϵ and τ . The numerical calculations were done using the publically available software package DDE-BIFTOOL [2].

2.2 Semi-global stabilization

In this subsection we will show that the control law (11) also achieves semi-global stability in y . Because of Theorem 2.1, it is sufficient to prove that for any bounded set of initial conditions, the input does not saturate along the closed-loop solutions when ϵ is sufficiently small. For this we first construct a compact subset of \mathbb{R}^n containing these solutions, and then we guarantee that inside a larger set, input saturation is not possible. This approach is inspired by [8].

In the ODE-case, bounds on the solutions can be derived based on the fact that a level set of an appropriate Lyapunov function forms the boundary of a positively invariant set. For the delay equation (14) however, the set

$$S = \{z \in \mathbb{R}^n : V(z) = z^T P z \leq c\}, \quad (17)$$

with c an arbitrary constant, is not positively invariant since A is not Hurwitz. Therefore it is always possible to construct an initial condition $z_0 \subseteq S$ with $z(0) \in \partial S$ such that along the solution,

$$\lim_{t \rightarrow 0^+} \dot{V}(t) = z(0)^T \underbrace{(A^T P + P A)}_{\text{indefinite}} z(0) + 2z(0)^T P B K^T z(-\epsilon\tau) > 0.$$

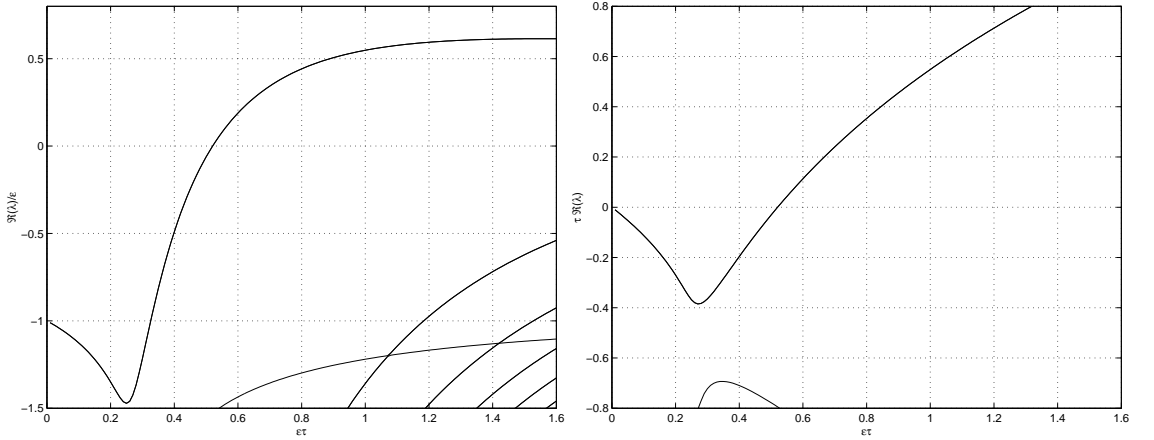


Figure 1: Rightmost eigenvalues of the double integrator controlled with (11) as a function of ϵ and τ . $K = [-2 \ -2]^T$. The maximal achievable exponential decay of the closed loop solutions, $\approx e^{-0.38/\tau}$, is obtained for $\epsilon\tau \approx 0.27$.

Note that when $\epsilon\tau \rightarrow 0$, such initial condition necessarily has arbitrarily large time-derivatives, which cannot occur for $t \in [0, \epsilon\tau]$, since in that time-interval the derivative is bounded by $\sup_{z_1, z_2 \in S} (|A||z_1| + |BK^T||z_2|)$. This motivates us to extend the definition of a positively invariant set to

Definition 2.1 For the delay equation $\dot{z} = Az + A_d z(t - \tau)$ a set $D \subset \mathbb{R}^n$ is 1-positively invariant iff $z_0 \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$ and $z(t) \in D, \forall t \in [-\tau, \tau]$ implies $z(t) \in D, \forall t \geq 0$.

Alternative a Lipschitz condition could be required on the initial conditions, as in [1, Subsection V.6].

We now give conditions on ϵ under which the set (17) is 1-positively invariant for the system (14) or equivalently the set $S(\epsilon) = \{y \in \mathbb{R}^n : y^T P(\epsilon)y \leq c\}$ is 1-positively invariant for the closed loop system (3)-(11). Therefore we first rewrite (14) as

$$\dot{z} = \bar{A}z + BK^T(z(t - \epsilon\tau) - z(t)),$$

where the second term can be interpreted as a perturbation of the asymptotically stable ODE $\dot{z} = \bar{A}z$. When a solution would leave the set S for the first time at $t \geq \epsilon\tau$ we have,

$$\begin{aligned} \dot{V}(t) &\leq -z^T Qz + 2|z| |PBK^T| |z(t - \epsilon\tau) - z(t)| \\ &\leq -z^T Qz + 2|z| |PBK^T| \epsilon\tau |\dot{z}(\theta)|, \quad \theta \in [t - \epsilon\tau, t] \\ &\leq -z^T Qz + 2|z| |PBK^T| \epsilon\tau (|Az(\theta)| + |BK^T z(\theta - \epsilon\tau)|) \\ &\leq -\lambda_{\min}(Q)|z|^2 + 2|PBK^T| \epsilon\tau (|A| + |BK^T|) \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} |z(t)|^2. \end{aligned}$$

The set (17) is 1-positively invariant when $\dot{V}(t) \leq 0$. This is satisfied when

$$\epsilon\tau \leq \frac{\lambda_{\min}(Q)}{2|PBK^T|(|A| + |BK^T|)} \frac{\lambda_{\min}(P)}{\lambda_{\max}(P)}. \quad (18)$$

In order to check the input constraints, we calculate the critical level set of (16), i.e. the minimal level set on which $u = K(\epsilon)^T y$ reaches the value ± 1 , by solving

$$\min_{K^T T(\epsilon)y=1} y^T P(\epsilon)y = \min_{K^T z=1} z^T Pz. \quad (19)$$

Denote the minimum by α . Then the critical level set is given by

$$y^T P(\epsilon)y = \alpha,$$

and since $\lim_{\epsilon \rightarrow 0} P(\epsilon) = 0$, any compact set in the state space lies inside the critical level set when ϵ is sufficiently small.

We now return to the semi-global stabilization of (3). For an arbitrary delay value and any compact set of initial conditions y_0 , there exists a compact set Ω_2 such that $y(t) \in \Omega_2, \forall t \in [-\tau, \tau]$. By taking ϵ sufficiently small, the critical level set contains Ω_2 , its interior is 1-positively invariant because of (18), and the linearized system, describing the closed-loop dynamics inside the critical level set, is asymptotically stable because of Theorem 2.1. Hence we have proven the following result:

Theorem 2.2 *The system (3) can be semi-globally asymptotically stabilized in both y and the delay with a feedback of the form (11), as $\epsilon \rightarrow 0$.*

2.3 Alternatives

From the above analysis, it follows that a small value of ϵ guarantees large regions of attraction of the zero solution, while larger values lead to improved (local) performance. However, as shown in the previous section, the maximal achievable exponential decay rate of the solutions is limited. This restriction of (delayed) state feedback could at first sight be removed by first compensating the delay with a static prediction. With the low-gain distributed delay feedback law,

$$u = K(\epsilon)^T \hat{y}(t, t + \tau) = K(\epsilon)^T \left(e^{A\tau} y(t) + \int_0^\tau e^{A(\tau-\theta)} B u(t + \theta - \tau) d\theta \right), \quad (20)$$

where $\hat{y}(t, t + \tau)$ is the prediction of the state y over one delay interval, the characteristic equation of the closed loop system is given by

$$\det(\lambda I - A - BK(\epsilon)^T) = 0.$$

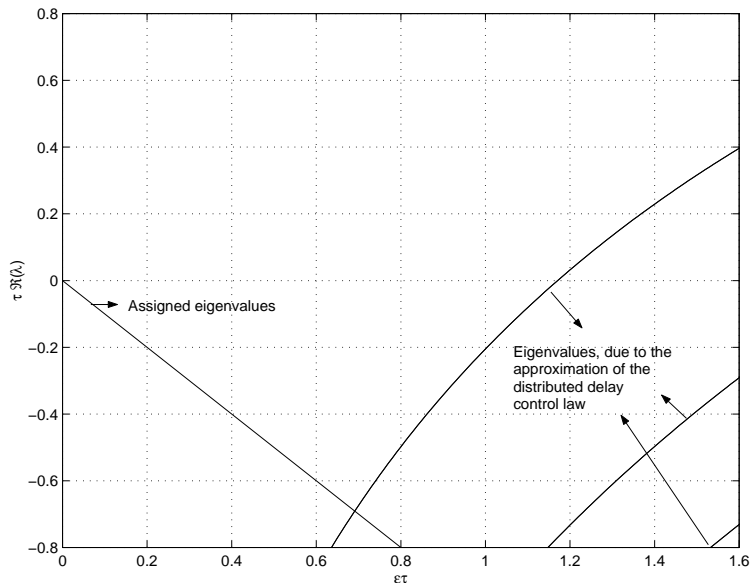


Figure 2: Rightmost eigenvalues of the double integrator controlled with the distributed delay feedback law(20) with $K = [-2 \ -2]^T$, in function of ϵ and τ . The assigned eigenvalues are $(-1 \pm j)\epsilon$. The other eigenvalues are caused by an approximation of the integral term in with a finite sum. The maximal achievable exponential decay of the closed loop solutions is approx. $e^{-0.69/\tau}$.

Now there are only n closed loop eigenvalues which can be assigned arbitrarily in the complex plane. However in order to implement the control law (20), a numerical approximation of the integral is needed (e.g. using a quadrature rule), and in [3, 14] it is shown that even an *arbitrarily* small approximation error introduces an essential spectrum which may affect both stability and performance. In [3] it is also explained how this essential spectrum can be numerically calculated. As we now illustrate with an example, the practical limitations on stability and performance are qualitatively the same as for pure state feedback. In Figure 2, the rightmost eigenvalues of the system (3)-(20) with $n = 2$ and $K = [-2 \ -2]^T$ are shown in function of ϵ and τ . Theoretically only the assigned eigenvalues $(-1 \pm j)\epsilon$ occur. The other eigenvalues are caused by the approximation of the integral with a finite sum. They limit the maximal achievable exponential decay rate and lead to instability for large values of ϵ , a situation qualitatively comparable to the state feedback case.

3 Global stabilization

In order to achieve global stability in y , nonlinear feedback is necessary. A first possibility consists of gain scheduling, i.e. by using the control law (11) where ϵ depends on y and is thus continuously adapted along the solutions. This approach is not further worked out in this paper.

Alternatively an adaptation to the time-delay case can be made of the so-called saturation design, initiated by the famous paper of Teel [13], which motivated many researchers to consider the explicit construction of globally asymptotically stable feedback laws. In saturation design the control law typically consists of a (non)linear combination of saturated linear functions.

For the multiple integrator with time-delay, we have the following result:

Theorem 3.1 *The feedback*

$$u = -\epsilon \operatorname{sat} x_n - \epsilon^2 \operatorname{sat} \frac{x_{n-1}}{\epsilon} - \epsilon^3 \operatorname{sat} \frac{x_{n-2}}{\epsilon^2} - \dots - \epsilon^n \operatorname{sat} \frac{x_1}{\epsilon^{n-1}}, \quad (21)$$

where

$$x_k = \sum_{j=k}^n \epsilon^{n-j} \binom{n-k}{n-j} y_j, \quad k = 1 \dots n,$$

stabilizes the system (3) globally in y and semi-globally in the delay τ , as $\epsilon \rightarrow 0$.

Despite of some technicalities, the structure of the proof is analogous to the proof of Theorem 2.1 of [13]. For an arbitrary solution of the closed loop system, first the existence is proven of a finite time $T > 0$ such that for $\forall t \geq T$, all saturation functions operate in their linear region and hence the closed loop system is linear. Asymptotic stability of the linearized system follows from an application of Theorem 2.1. Typical for the time-delay case considered in this paper is that the control law (21) is constructed in such a way that its linearization still depends on ϵ , which is necessary for achieving semi-global asymptotic stability in the delay.

Proof. With the change of coordinates,

$$x_k = \sum_{j=k}^n \epsilon^{n-j} \binom{n-k}{n-j} y_j \quad k = 1 \dots n, \quad (22)$$

whose inverse is characterized by

$$y_k = \frac{1}{\epsilon^{n-k}} \sum_{j=k}^n (-1)^{j-k} \binom{n-k}{n-j} x_j \quad k = 1 \dots n, \quad (23)$$

the system (3) and the control law (21) are transformed into

$$\begin{cases} \dot{x}_1 = \epsilon x_2 + \dots + \epsilon x_n + u(t - \tau) \\ \dot{x}_2 = \epsilon x_3 + \dots + \epsilon x_n + u(t - \tau) \\ \vdots \\ \dot{x}_{n-1} = \epsilon x_n + u(t - \tau) \\ \dot{x}_n = u(t - \tau) \end{cases} \quad (24)$$

and

$$u = -\epsilon \text{sat} x_n - \epsilon^2 \text{sat} \frac{x_{n-1}}{\epsilon} - \epsilon^3 \text{sat} \frac{x_{n-2}}{\epsilon^2} - \dots - \epsilon^n \text{sat} \frac{x_1}{\epsilon^{n-1}}. \quad (25)$$

For an arbitrary constant $\bar{\tau} > 0$, we now show that the closed-loop system (24)-(25) is globally asymptotically stable, $\forall \tau \in [0, \bar{\tau}]$, when ϵ is small. Thereby we make use of the lemmas in the appendix, which describe the dependence of the solutions of (24)-(25) on ϵ .

Obviously there exists a finite time $T_{n+1}(\tau, x_0, \epsilon)$ such that, along the solutions of (24)-(25),

$$\dot{x}_n = -\epsilon \text{sat} x_n(t - \tau) + O(\epsilon^2), \quad \tau \leq \bar{\tau}, \quad t \geq T_{n+1}. \quad (26)$$

By an application of Lemma A.1 and Lemma A.2, $\exists T_n(\tau, x_0, \epsilon)$ such that for $t \geq T_n$, the saturation function in (26) operates in its linear region and $x_n = O(\epsilon)$. Consequently $u = O(\epsilon^2)$. By applying Lemma A.3 $n - 1$ times, the existence is guaranteed of a finite time T_1 such that $x_k = O(\epsilon^n)$, $\tau \leq \bar{\tau}$, $t \geq T_1$ for $k = 1, n$. Hence when ϵ is sufficiently small, the saturation functions in (24)-(25) always operate in their linear region after a finite time.

For $t \geq T_1$, the control law is simplified to

$$u = -\epsilon(x_1 + \dots + x_n),$$

or using transformation (23),

$$u = -\sum_{k=1}^n \binom{n}{k-1} \epsilon^{n-k+1} y_k, \quad (27)$$

i.e. a control law of the form (11). With $K(\epsilon)^T y$ equal to (27), all eigenvalues of the closed-loop system (3)-(11) lie for $\tau = 0$ at $\lambda = -\epsilon$ or equivalently, all eigenvalues of (14) at -1 . By Theorem 2.1, this results in asymptotic stability for $\tau \leq \bar{\tau}$, when ϵ is small. \square

4 Conclusions

In this paper we constructed parametrized state feedback controllers for the semi-global and global stabilization of a multiple integrator with time-delay and input constraints.

Remarkably all results were semi-global in the delay. This is due to the fact that when the controller gain tends to zero, the time-delay becomes negligible compared to system's dynamics. Mathematically this is exploited in the transformation (12)-(13), the so-called small peaking dilatation, which was the basis for the analysis of linear low-gain controllers, and in transformation (22), which was used in the construction of the globally stabilizing controllers.

Although simple static state feedback controllers are robust and retain their good stabilizability properties, the time-delay introduces limitations on their performance. However we have illustrated with an example that for complicated distributed-delay controllers, which compensate the effect of the input delay with a prediction, the limitations on the closed-loop performance may be qualitatively the same when small implementation errors are taken into account.

Further research concerns the global stabilization of (1) when matrix A has pure imaginary eigenvalues, and the extension to the case where the uncontrolled system is described by delay differential equations.

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A Technical Lemmas

We state and prove some technical lemma's on the behavior of the solutions of (24)-(25) as a function of ϵ .

Lemma A.1 *Assume that the j -th component ($j = 1, n$) of the solutions of the system (24)-(25) satisfies*

$$\dot{x}_j = -\epsilon^k \text{sat} \frac{x_j(t - \tau)}{\epsilon^{k-1}} + \eta, \quad (28)$$

where

$$\eta = O(\epsilon^p), \quad \tau \leq \bar{\tau}, \quad t \geq T_1 \quad \text{with } p > k > 1.$$

Then there exists a constant $\bar{\epsilon} > 0$ with the following property:

$\forall \tau \leq \bar{\tau}, \forall x_0 \in \mathcal{C}([- \tau, 0], \mathbb{R}^n), \forall \epsilon \leq \bar{\epsilon}$, there is a finite time $T_2(\tau, x_0, \epsilon)$ such that the saturation function in (28) is in its linear region for $t \geq T_2$.

Proof. $\exists \gamma > 0$ such that, along an arbitrary solution, $|\eta| < \gamma \epsilon^p$ for $t \geq T_1$ provided $\tau \leq \bar{\tau}$. As long as $|x_j(t - \tau)| \geq \gamma \epsilon^{p-1}$, we have $x_j(t - \tau) \dot{x}_j(t) < 0$. As a consequence, there always

exists a finite time $T_2 > 0$ such that $|x_j(T_2)| \leq \gamma\epsilon^{p-1}$. Using $|\dot{x}_j| \leq 2\epsilon^k$, $t \geq T_1$, we obtain that for $t \geq T_2$, $|x_j(t)| \leq \gamma\epsilon^{p-1} + 2\epsilon^k\tau \leq \gamma\epsilon^{p-1} + 2\epsilon^k\bar{\tau} \leq \epsilon^{k-1}$, because $x_j(t-\tau)\dot{x}_j(t) < 0$ whenever $x_j(t) > \gamma\epsilon^{p-1}$ for a time-interval larger than τ . Consequently the saturation function operates in its linear region for $t \geq T_2$. Note that this is only valid when ϵ is sufficiently small, but a threshold $\bar{\epsilon}$ can be chosen independently of τ and the initial condition. \square

Lemma A.2 *Assume that the j -th component ($j = 1, n$) of the solutions of the system (24)-(25) satisfies*

$$\dot{x}_j = -\epsilon x_j(t - \tau) + \eta,$$

where

$$\begin{aligned} \eta &= O(\epsilon^p), \quad \tau \leq \bar{\tau}, \quad t \geq T_1, \\ x_j &= O(\epsilon^{p-2}), \quad \tau \leq \bar{\tau}, \quad t \geq T_1, \quad \text{with } p > 2. \end{aligned}$$

Then there exists a finite time $T_2(\tau, x_0, \epsilon)$ such that $x_j = O(\epsilon^{p-1})$, $\tau \leq \bar{\tau}$, $t \geq T_2$.

Proof. $\exists \gamma_1, \gamma_2 > 0$ such that along any solution, $|x_j(t)| < \gamma_1\epsilon^{p-2}$, $|\eta| \leq \gamma_2\epsilon^p$ for $t \geq T_1$ and hence $|\dot{x}_j| \leq 2\gamma_1\epsilon^{p-1}$. Since $x_j(t-\tau)\dot{x}_j(t) < 0$ when $|x_j(t-\tau)| \geq \gamma_2\epsilon^{p-1}$ there exists a finite time T_2 such that $|x_j(T_2)| \leq \gamma_2\epsilon^{p-1}$ and $|x_j(t)| \leq \gamma_2\epsilon^{p-1} + 2\gamma_1\epsilon^{p-1}\bar{\tau}$, $\forall t \geq T_2$, provided ϵ is sufficiently small. A threshold $\bar{\epsilon}$ can be determined independently of the delay and the initial condition. \square

Lemma A.3 *Assume that for the system (24)-(25), there exists a function $T_k(\tau, x_0, \epsilon)$ such that*

$$\begin{aligned} x_k &= O(\epsilon^{n-k+1}) \\ &\vdots \\ x_n &= O(\epsilon^{n-k+1}), \quad \tau \leq \bar{\tau}, \quad t \geq T_k, \\ u &= O(\epsilon^{n-k+2}) \end{aligned}$$

with $2 \leq k \leq n$. Then $\exists T_{k-1}(\tau, x_0, \epsilon)$ such that

$$\begin{aligned} x_{k-1} &= O(\epsilon^{n-k+2}) \\ &\vdots \\ x_n &= O(\epsilon^{n-k+2}), \quad \tau \leq \bar{\tau}, \quad t \geq T_{k-1}. \\ u &= O(\epsilon^{n-k+3}) \end{aligned}$$

Proof. For $t \geq T_k$, the $(k-1)$ -th equation of (24)-(25), can be written as

$$\begin{aligned} \dot{x}_{k-1} &= -\epsilon^{n-k+2} \text{sat} \frac{x_{k-1}}{\epsilon^{n-k+1}} + \epsilon[x_k - x_k(t-\tau)] + \epsilon[x_{k+1} - x_{k+1}(t-\tau)] + \dots \\ &\quad + \epsilon[x_n - x_n(t-\tau)] + O(\epsilon^{n-k+3}). \end{aligned} \quad (29)$$

Because for $l \geq k$,

$$\begin{aligned} |x_l(t) - x_l(t - \tau)| &\leq \int_{t-\tau}^t |\dot{x}_l| ds \\ &\leq \bar{\tau} \sup_{s \in [t-\tau, t]} |\epsilon x_{l+1}(s) + \dots + \epsilon x_n(s) + u(s - \tau)| \\ &= O(\epsilon^{n-k+2}), \end{aligned}$$

when $t \geq T_k + \tau$, equation (29) can be rewritten as

$$\dot{x}_{k-1} = -\epsilon^{n-k+2} \text{sat} \frac{x_{k-1}}{\epsilon^{n-k+1}} + O(\epsilon^{n-k+3}). \quad (30)$$

An application of Lemma A.1 and Lemma A.2 yields that after a finite time, the saturation function in (30) is in its linear region (provided ϵ is small) and

$$x_{k-1} = O(\epsilon^{n-k+2}).$$

Using this new estimate, it can be derived that

$$\begin{aligned} \dot{x}_k &= \epsilon[x_{k+1} - x_{k+1}(t - \tau)] + \dots + \epsilon[x_n - x_n(t - \tau)] \\ &\quad - \epsilon x_k(t - \tau) - \epsilon x_{k-1}(t - \tau) + O(\epsilon^{n-k+3}) \\ &= -\epsilon x_k(t - \tau) + O(\epsilon^{n-k+3}), \end{aligned}$$

and, by Lemma A.2, after a finite time we have:

$$x_k = O(\epsilon^{n-k+2}).$$

Continuing with the same arguments, there exists a finite time T_{k-1} such that

$$x_{k-1} = O(\epsilon^{n-k+2}), \dots, x_n = O(\epsilon^{n-k+2})$$

and as a consequence, $u = O(\epsilon^{n-k+3})$, provided ϵ is small. Once more, a threshold on ϵ can be determined independently of τ and x_0 . \square

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