# Cubature formulae that are exact for trigonometric polynomials

*I.P. Mysovskikh*
*edited by R. Cools & H.J. Schmid*

*Report TW 324, May 2001*

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**Abstract**

This paper contains results on the approximate evaluation of integrals

$$I(f) = \frac{1}{\text{mes } \Omega} \int_{\Omega} f(x) w(x) \, dx$$

by cubature formulae that are exact for trigonometric polynomials. Its contents corresponds to talks given by Prof. I.P. Mysovskikh in June 2000 at Universität Erlangen-Nürnberg (Germany), Katholieke Universiteit Leuven (Belgium) and at University Dortmund (Germany).



**Keywords :** Cubature formulas, lower bounds, fully symmetric, consistency conditions

**AMS(MOS) Classification :** Primary : 65D32



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# Ivan Petrovich Mysovskikh

Иван Петрович Мысовских

**Short curriculum vitae**

(July 2000)

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Ivan Petrovich Mysvovskikh was born at Tyumenskaya Oblast on November 25, 1921. He is professor at the St. Peterburg State-University and was visiting professor at Jilin University (China). His research interests include the solution of non-linear functional equations, error estimates for the solution of integral equations and interpolatory cubature formulae. He is the author of about 80 scientific publications. This includes a textbook on "*Numerical Methods*" of which the 1st edition appeared in 1962 (in Russian, Chinese and English) and the enlarged 2nd edition appeared in 1998. He also wrote a monograph on "*Interpolatory Cubature Formulae*" which describes the known results up to its year of publication, 1981.

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This report contains the text of the talks presented by Prof. I.P. Mysovskikh in June 2000 at the Universität Erlangen-Nürnberg (Germany), the Katholieke Universiteit Leuven (Belgium) and at the University Dortmund (Germany). We would like to thank Frau Jutta Zintchenko, Erlangen, for translating the Russian text and producing a first TEX version of the text. At the end this report contains a list of his publications. Our aim was to compile a complete list. We have the feeling however that this list is still incomplete.

This year Professor Mysovskikh is celebrating his 80-th birthday.
On behalf of the cubature-community we are expressing our best wishes.

Ronald Cools & Hans Joachim Schmid

# Cubature Formulae that are Exact for Trigonometric Polynomials

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(edited by R. Cools & H.J. Schmid †)

Notes accompanying a talk presented in June 2000

This paper contains results on the approximate evaluation of integrals

$$I(f) = \frac{1}{\text{mes } \Omega} \int_\Omega f(x)w(x)\,dx \qquad (1)$$

by cubature formulae that are exact for trigonometric polynomials. Here $x = (x_1, x_2, \dots, x_n)$, $\Omega \subset \mathbb{R}^n$ is the domain of integration with interior points and $w(x)$ is a weight function.
For later use we present necessary information about trigonometric polynomials in $n$ variables $x = (x_1, x_2, \dots, x_n)$. The trigonometric polynomial

$$e^{i(\alpha,x)} = e^{i(\alpha_1 x_1 + \cdots + \alpha_n x_n)} \qquad (2)$$

is called a trigonometric monomial, where $i$ is the imaginary unit and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is a vector with integer coordinates. The number $||\alpha||_1 = |\alpha_1| + |\alpha_2| + \ldots + |\alpha_n|$ is called the degree of this monomial. A monomial is called even(odd) if $||\alpha||_1$ is even(odd). The product of monomials of identical (different) parities is an even(odd) monomial.
We denote by $\tau(n, m)$ the cardinality of monomials (2) of degree $m$. It is clear that $\tau(n, m)$ is the number of integer solutions of the equation

$$|\alpha_1| + |\alpha_2| + \ldots + |\alpha_n| = m. \qquad (3)$$

If $m = 0$, then the number of solutions is one: $\tau(n, 0) = 1$.
Let $m > 0$. Then the solutions of (3) can be separated into $n$ blocks: the $s$-th block, $s = 1(1)n$, contains all solutions with $s$ nonzero coordinates. We want to determine the number of solutions in the $s$-th block. The $s$ nonzero coordinates can be distributed among $n$ positions in $C_n^s = n!/[(n-s)!s!]$ different ways.
Let us find out how many solutions of equation (3) correspond to every possibility of $s$ nonzero coordinates. Consider, for instance, the case when the nonzero coordinates occupy the first $s$

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positions. If we assume that all $s$ coordinates are positive, then equation (3) is of the form

$$\alpha_1 + \alpha_2 + \ldots + \alpha_s = m. \tag{4}$$

If there are $\alpha_j \geq 0,\ j = 1(1)s$, then the number of solutions of equation (4) is the number of algebraic monomials of degree $m$ in $s$ variables.
Let us introduce new unknowns $\beta_j = \alpha_j - 1,\ j = 1(1)s$, which are nonnegative integers and satisfy the equation

$$\beta_1 + \beta_2 + \ldots + \beta_s = m - s. \tag{5}$$

The number of solutions of equations (4) and (5) is the same. Henceforth we will use the notation $M(m,n) = \frac{(m+n)!}{m!n!}$, where $M(m,n)$ is the number of algebraic monomials in $n$ variables of degree not greater than $m$ and $M(m-1,n)$ is the number of algebraic monomials of degree $m$.
The number of nonnegative integer solutions of equation (5) is

$$M(s-1, m-s) = \frac{(m-1)!}{(s-1)!(m-s)!} = C_{m-1}^{s-1}.$$

We get the same number of solutions under any different distribution of $s$ positive coordinates. Thus, the number of solutions of equation (3) which have $s$ positive coordinates is equal to $C_n^s C_{m-1}^{s-1}$. Since every coordinate of the solution may be both positive and negative, the number of all solutions of equation (3) in the $s$-th block is equal to $C_n^s C_{m-1}^{s-1} 2^s$. The number of all solutions of equation (3) for $m > 0$ is given by

$$\tau(n,m) = \sum_{s=1}^{n} C_n^s C_{m-1}^{s-1} 2^s. \tag{6}$$

Let $t(n,k)$ be the number of all trigonometric monomials of degree at most $k$ in $n$ variables. We have

$$\begin{aligned} t(n,k) &= 1 + \sum_{m=1}^{k} \tau(n,m) = 1 + \sum_{m=1}^{k} \sum_{s=1}^{n} C_n^s C_{m-1}^{s-1} 2^s \\ &= 1 + \sum_{s=1}^{n} C_n^s 2^s \sum_{m=1}^{k} C_{m-1}^{s-1}. \end{aligned} \tag{7}$$

Since

$$\begin{aligned} \sum_{m=1}^{k} C_{m-1}^{s-1} &= \sum_{m=s}^{k} C_{m-1}^{s-1} = \sum_{m=s}^{k} \frac{(m-1)!}{(s-1)!(m-s)!} \\ &= \sum_{m=s}^{k} M(m-s, s-1) = \sum_{t=0}^{k-s} M(t, s-1), \end{aligned}$$

and $M(t, s-1)$ is the number of algebraic monomials of degree $t$ in $s$ variables, the last sum is the number of linearly independent algebraic monomials of degree $k-s$ in $s$ variables: $M(k-s, s)$. Thus, we have

$$\sum_{m=1}^{n} C_{m-1}^{s-1} = \frac{k!}{s!(k-s)!} = C_k^s$$

and equation (7) can be written in the form

$$t(n,k) = \sum_{s=0}^{n} C_n^s C_k^s 2^s. \tag{8}$$

It is not difficult to see that $t(n,k) = t(k,n)$ holds. In fact, the sum

$$t(k,n) = \sum_{s=0}^{k} C_k^s C_n^s 2^s$$

coincides for $k > n$ with the sum in (8) by virtue of $C_n^s = 0$ for $s > n$. If $k < n$, these sums also coincide, since $C_k^s = 0$ for $s > k$. The function $\tau(n,m)$ is unsymmetric. We prove that $m\,\tau(n,m) = n\,\tau(m,n)$. Indeed, by virtue of (6)

$$\begin{aligned}
\tau(n,m) &= \sum_{s=1}^{n} \frac{n!}{s!\,(n-s)!}\;\frac{(m-1)!}{(s-1)!\,(m-s)!}\,2^s \\
&= \frac{n}{m}\sum_{s=1}^{n} \frac{(n-1)!}{(s-1)!(n-s)!}\;\frac{m!}{s!(m-s)!}\,2^s \\
&= \frac{n}{m}\sum_{s=1}^{n} C_m^s C_{n-1}^{s-1} 2^s = \frac{n}{m}\,\tau(m,n).
\end{aligned}$$

From the definition of $\tau(l,m)$ it follows that

$$\tau(l,m) = t(l,m) - t(l,m-1).$$

Assuming $m \geq 2$ we insert into this equality $l = 0$. Since $t(0,m) = t(0,m-1)$ we obtain

$$\tau(0,m) = 0, \text{ if } m \geq 2. \tag{9}$$

We prove the equality

$$\tau(l,m) + \tau(l-1,m) = 2t(l-1,m). \tag{10}$$

The value $\tau(l,m)$ is the number of integer solutions of $|\alpha_1| + |\alpha_2| + \ldots + |\alpha_l| = m$. The set of all solutions is the union of the solutions of the equations

$$|\alpha_1| + \ldots + |\alpha_{l-1}| = m - |\alpha_l|,\ |\alpha_l| = 0(1)m. \tag{11}$$

The number of solutions of equation (11) for $|\alpha_l| = 0$ is $\tau(l-1,m)$. If $|\alpha_l| = j \neq 0$, then $\alpha_l$ is of the form $\pm j$, and the number of solutions of equation (11) is $2\tau(l-1,m-j)$. The number of all solutions of equation (11) for $|\alpha_l| = 1(1)m$ is

$$\sum_{j=1}^{m} 2\tau(l-1,m-j) = 2\sum_{s=0}^{m-1} \tau(l-1,s) = 2t(l-1,m-1).$$

Thus,

$$\tau(l,m) = \tau(l-1,m) + 2t(l-1,m-1).$$

Adding $\tau(l-1,m)$ to the left and right hand side of this equation, we obtain by virtue of (10)

$$2\tau(l-1,m) + 2t(l-1,m-1) = 2t(l-1,m).$$

**Lemma 1.** The number of monomials (2) of degree not greater than $k$ and of parity identical to $k$ is equal to $\tau(k+1,n)/2$.

**Proof.** We re-write equality (10) by replacing $l$ by $l+1$,

$$\tau(l+1,m)+\tau(l,m)=2t(l,m),$$

and subtract equation (10),

$$\tau(l+1,m)-\tau(l-1,m)=2[t(l,m)-t(l-1,m)]=2\tau(m,l).$$

We get

$$2\tau(m,l)=\tau(l+1,m)-\tau(l-1,m).$$

For $m=n$ this relation is of the form

$$2\tau(n,l)=\tau(l+1,n)-\tau(l-1,n). \tag{12}$$

We assume that $k$ is odd. Using equation (12) we find the number of odd monomials (2) of degree not greater than $k$. Inserting $l=2r-1$ in equation (12) and summing both sides with respect to $r$ we obtain

$$2\sum_{r=1}^{(k+1)/2}\tau(n,2r-1)=\sum_{r=1}^{(k+1)/2}[\tau(2r,n)-\tau(2r-2,n)]=\tau(k+1,n)-\tau(0,n).$$

The statement of the lemma is true for $n=1$; the number of odd monomials in one variable of degree not greater than $k$ is equal $k+1=\tau(k+1,1)/2$. Hence we can assume that $n\geq 2$, consequently, $\tau(0,n)=0$, and we get

$$\sum_{r=1}^{(k+1)/2}\tau(n,2r-1)=\tau(k+1,n)/2.$$

If $k$ is even, then the number of even monomials of degree not greater than $k$ is equal to the number of all monomials of degree not greater than $k$ minus the number of odd monomials of degree not greater than $k-1$:

$$t(k,n)-\frac{\tau((k-1)+1,n)}{2}=\frac{1}{2}\left[2t(k,n)-\tau(k,n)\right]=\frac{1}{2}\tau(k+1,n).$$

Here we use equation (10). ■

We cite the expressions for $\tau(n,m)$ and $t(n,m)$ for $m=1,2,3$.

$$\tau(n,1)=2n,\ \tau(n,2)=2n^2,\ \tau(n,3)=2n(2n^2+1)/3\ ,$$

$$t(n,1)=1+2n,\ t(n,2)=1+2n+2n^2,\ t(n,3)=(3+8n+6n^2+4n^3)/3.$$

The trigonometric monomials (2) of second degree in three variables are

$$e^{i2x_1},\ e^{i2x_2},\ e^{i2x_3},\ e^{i(x_1+x_2)},\ e^{i(x_1+x_3)}, e^{i(x_2+x_3)}, e^{i(x_1-x_2)},\ e^{i(x_1-x_3)},\ e^{i(x_2-x_3)}.$$

To these 9 monomials, in addition, 9 complex conjugate ones have to be added, thus we obtain $18 = \tau(3,2)$ monomials.
We will enumerate the trigonometric monomials in (2) by positive integers denoted by $\varphi_j(x)$, $j = 1, 2, \ldots$ We assume that the numbering is consistent with the degree of the monomials: monomials of lower degree are assigned smaller numbers, while monomials of the same degree are numbered arbitrarily. Hence, $\{\varphi_j(x)\}_{j=1}^{t(n,k)}$ contains all monomials of degree not greater than $k$.
Since $\Omega$ is a set in $\mathbb{R}^n$ with interior points the trigonometric monomials $\{\varphi_j(x)\}_{j=1}^{\infty}$ are linearly independent over $\Omega$.

**Lemma 2.** For any natural number $r$ there exist points $x^{(j)} \in \Omega,\ j = 1(1)r$, such that the matrix

$$A_r = \left[\varphi_1(x^{(j)}),\ \varphi_2(x^{(j)}),\ \ldots, \varphi_r(x^{(j)})\right]_{j=1}^{r}$$

is nonsingular.

**Proof.** For any point $x^{(1)} \in \Omega$ we have $\varphi_1(x^{(1)}) \neq 0$ since $\varphi_1(x) = 1$. We assume that there exist the points $x^{(j)} \in \Omega,\ j = 1(1)s$, such that matrix

$$A_s = \left[\varphi_1(x^{(j)}), \ldots,\ \varphi_s(x^{(j)})\right]_{j=1}^{s}$$

is nonsingular. We prove that there exists a point $x^{(s+1)} \in \Omega$ such that $\det A_{s+1} \neq 0$. We consider the determinant

$$\Delta(x) = \begin{vmatrix} \varphi_1(x^{(1)}) & \varphi_2(x^{1)}) & \ldots & \varphi_s(x^{(1)}) & \varphi_{s+1}(x^{(1)}) \\ \varphi_1(x^{(2)}) & \varphi_2(x^{(2)}) & \ldots & \varphi_s(x^{(2)}) & \varphi_{s+1}(x^{(2)}) \\ \vdots & \vdots & & \vdots & \vdots \\ \varphi_1(x^{(s)}) & \varphi_2(x^{(s)}) & \ldots & \varphi_s(x^{(s)}) & \varphi_{s+1}(x^{(s)}) \\ \varphi_1(x) & \varphi_2(x) & \ldots & \varphi_s(x) & \varphi_{s+1}(x) \end{vmatrix}.$$

Expanding this determinant with respect to the elements of the last row, we obtain

$$\Delta(x) = b_1\varphi_1(x) + \ldots + b_s\varphi_s(x) + b_{s+1}\varphi_{s+1}(x),$$

where $b_{s+1} = \det A_s \neq 0$. If $\Delta(x) = 0$ for any $x \in \Omega$, then the monomials $\varphi_l(x),\ l = 1(1)s+1$, are linearly dependent over $\Omega$. This is impossible. Hence, there exists a point $x^{(s+1)} \in \Omega$ such that $\Delta(x^{(s+1)}) = \det A_{s+1} \neq 0$. ■

We make the following assumptions concerning the weight function $w(x)$. The moments

$$\int_\Omega \varphi_j(x)w(x)dx,\ j = 1, 2, 3, \ldots \tag{13}$$

exist and

$$w(x) \geq 0,\ x \in \Omega,\ \int_\Omega w(x)dx > 0. \tag{14}$$

**Theorem 1.** There exists a cubature formula

$$I(f) \cong \sum_{j=1}^{N} C_j f(x^{(j)}), \tag{15}$$

which is exact for all trigonometric polynomials of degree at most $m$ with $N = t(n,m)$ nodes belonging to $\Omega$.

**Proof.** By Lemma 2 for $r = t(n,m)$ there exist the points $x^{(j)} \in \Omega,\ j = 1(1)t(n,m)$ such that the matrix

$$A_{t(n,m)} = \left[\varphi_1(x^{(j)}), \ldots, \varphi_{t(n,m)}(x^{(j)})\right]_{j=1}^{t(n,m)}$$

is nonsingular. The degree of accuracy of cubature formula (15) is $m$, if

$$\sum_{j=1}^{t(n,m)} C_j \varphi_k(x^{(j)}) = \int_\Omega \varphi_k(x) w(x) dx,\ k = 1(1)t(n,m).$$

We obtain a linear algebraic system for $C_j,\ j = 1(1)t(n,m)$. The matrix of this system is $A^T_{t(n,m)}$ and $\det A^T_{t(n,m)} \neq 0$. It follows that the system has a unique solution, and, consequently, the cubature formula (15) exists and is unique. ■

In Theorem 1 we did not use condition (14).

**Theorem 2.** The number of nodes of a cubature formula (15) which is exact for all trigonometric polynomials of degree at most $m$ satisfies

$$N \geq \kappa := t(n,k), \tag{16}$$

where $k = \lfloor m/2 \rfloor$ is the integer part of $m/2$.

**Proof.** Let us consider the matrix

$$\left[\varphi_1(x^{(j)}),\ \ldots,\ \varphi_{t(n,k)}(x^{(j)})\right]_{j=1}^{N}. \tag{17}$$

The columns of this matrix are linearly independent. Otherwise, we could construct a nonzero trigonometric polynomial of degree at most $k$

$$t(x) = \sum_{j=1}^{t(n,k)} d_j \varphi_j(x),$$

vanishing at the nodes of the cubature formula (15), i.e. $t(x^{(j)}) = 0,\ j = 1(1)N$. We consider the trigonometric polynomial $t(x)\overline{t(x)}$. Here and further the bar denotes the complex conjugate. The degree of $t(x)\overline{t(x)} = |t(x)|^2$ is at most $2k \leq m$. Its cubature formula (15) is not exact, since the integral is positive and the cubature sum is zero. Inequality (16) follows from the linear independency of the columns in (17). ■

These results have been obtained in [1].

We assume that $w(x)$ and $\Omega$ satisfy

$$I(e^{i(\alpha,x)}) = 0, \tag{18}$$

whenever $||\alpha||_1 = |\alpha_1| + \ldots + |\alpha_n|$ is an odd number.

**Theorem 3.** Let the cubature formula (15) be exact for trigonometric polynomials of degree not greater than $2k+1$ and let $w(x)$ and $\Omega$ satisfy conditions (13), (14), (18). Then the number of nodes of this formula satisfies the inequality

$$N \geq \tau(k+1, n). \tag{19}$$

The proof of Theorem 3 is given in [2]. The special case $\Omega = [0, 2\pi]^n$, $w(x) = 1$ has been treated before in [1,3].
The integral (1) defines a scalar product in the vector space $\mathcal{T}$ of trigonometric polynomials

$$(\varphi, \psi) = I(\varphi\overline{\psi}) = \frac{1}{\text{mes }\Omega} \int_\Omega \varphi(x)\overline{\psi(x)}w(x)dx, \ \varphi, \psi \in \mathcal{T}. \tag{20}$$

We orthonormalize the system of trigonometric monomials $\varphi_j(x), \ j = 1, 2, \ldots,$ with respect to the scalar product (20). This orthonormal system will be denoted by $F_j(x), \ j = 1, 2, \ldots,$ where $(F_r, F_s) = \delta_{rs}, \ r, s = 1, 2, \ldots$
We introduce the following trigonometric polynomial in $2n$ variables, where $u = (u_1 \ldots, u_n)$, $x = (x_1, \ldots, x_n)$,

$$K_k(u, x) = \sum_{j=1}^{\kappa} \overline{F_j(u)} F_j(x), \ \kappa = t(n, k); \tag{21}$$

its degree in both variables $u$ and $x$ is $k$. This is the reproducing kernel for the vector space $\mathcal{T}_k$ of trigonometric polynomials of degree not greater than $k$ with scalar product (20).
Indeed, if $t(x) \in T_k$, then

$$\begin{aligned}(K_k(u, x), t(x)) &= \frac{1}{\text{mes }\Omega} \int_\Omega \sum_{j=1}^{\kappa} \overline{F_j(u)} F_j(x) \overline{t(x)} w(x) dx \\ &= \sum_{j=1}^{\kappa} \overline{F_j(u)} \ \overline{(t(x), F_j(x))} = \overline{t(u)}.\end{aligned} \tag{22}$$

The conditions (13) and (14) are important for the definition of the scalar product (20), condition (18) was not yet used. Suppose now that integral (1) satisfies the conditions (13), (14) and (18). Then the following statement holds.
A trigonometric polynomial of degree $m$ that is orthogonal with respect to the scalar product (20) has the same parity as $m$ (i.e., it contains only monomials with the parity identical to that of $m$). Let $\tilde{\mathcal{T}}_k$ denote the subspace of $\mathcal{T}$, which contains all trigonometric polynomials of degree not greater than $k$ that have the same parity as $k$ . The reproducing kernel for the space $\tilde{\mathcal{T}}_k$ with scalar product (20) may be written in the form

$$\tilde{K}_k(u, x) = \sum_{j=1}^{\kappa}{}' \ \overline{F_j(u)} \, F_j(x), \ \kappa = t(n, k), \tag{23}$$

where the prime on the summation sign indicates that the summation is taken over all $F_j(x)$ having the same parity as $k$.
The method of reproducing kernel for the construction of cubature formulae being exact for algebraic polynomials is well-known, cf. [4,5]. The question of the extension of the method of reproducing kernel to the trigonometric case is considered in [6]. The reproducing kernels, which are defined in the spaces $\mathcal{T}_k$ and $\tilde{\mathcal{T}}_k$ with scalar product (20), are given by means of (21) and (23).

**Theorem 4.** For the existence of a cubature formula (15) with $N = \kappa$ exact for trigonometric polynomials of degree not greater than $2k$, it is necessary and sufficient that the following conditions are satisfied.

$$K_k(x^{(r)}, x^{(s)}) = b_r \delta_{rs},\ r, s = 1(1)\kappa,\ \kappa = t(n, k). \tag{24}$$

If the conditions (24) are satisfied, the coefficients in formula (15) are defined by

$$C_r^{-1} = b_r = K_k(x^{(r)}, x^{(r)}) = \sum_{j=1}^{\kappa} |F_j(x^{(r)})|^2,\ r = 1(1)\kappa. \tag{25}$$

The proof of Theorem 4 is given in [6]. The proof is the same as that of Theorem 10.2 in [4, p.212].
The equalities (24) are the basis for the application of the method of reproducing kernel in the trigonometric case. If $a^{(1)}$ is a node of a cubature formula in Theorem 4, then the remaining nodes satisfy by virtue of (24) $K_k(a^{(1)}, x) = 0$. If there are $n$ nodes $a^{(j)} = 1(1)n$, then we get a system of equations for the unknowns $x_1, \ldots, x_n$;

$$K_k(a^{(l)}, x) = 0,\ l = 1(1)n. \tag{26}$$

The solutions of this system determine the remaining nodes of the cubature formula.
The nodes of the cubature formula in Theorem 4 are unknown. Hence we shall proceed as in the algebraic case. By $V_k$ we denote the set of the common zeroes of all polynomials of degree $k$ in the orthonormal system $F_j(x),\ j = (t(n, k-1)+1)(1)t(n, k)$. As $a^{(1)}$ we pick any point which does not belong to $V_k$. The point $a^{(1)}$ defines a trigonometric polynomial $K_k(a^{(1)}, x)$ of degree $k$ since $a^{(1)} \notin V_k$. As $a^{(2)}$ we will take a zero of the equation $K_k(a^{(1)}, x) = 0$ which does not belong to $V_k$. Let us assume that the points $a^{(j)},\ j = 1(1)r-1,\ 2 \le r \le n$, have been chosen. As $a^{(r)}$ we will pick a solution of the system

$$K_k(a^{(i)}, x) = 0,\ i = 1(1)r-1\ ,$$

which does not belong to $V_k$. We get the points $a^{(j)},\ j = 1(1)n$, which define $n$ trigonometric polynomials $K_k(a^{(j)}, x),\ j = 1(1)n$, of degree $k$, which satisfy the following conditions

$$K_k(x^{(r)}, x^{(\rho)}) = b_r \delta_{r\rho},\ r, \rho = 1(1)n.$$

We consider the system of equations (26), which is composed by these polynomials. By $x^{(j)},\ j = 1(1)s$, we denote all solutions of system (26). If $n + s = \kappa$, then the points $a^{(i)},\ i = 1(1)n,\ x^{(j)},\ j = 1(1)s$ are taken as nodes of the cubature formula

$$I(f) \cong \sum_{i=1}^{n} A_i f(a^{(i)}) + \sum_{j=1}^{s} B_j f(x^{(j)}).$$

The coefficients are defined by virtue of (25) by

$$A_i^{-1} = K_k(a^{(i)}, a^{(i)}),\ i = 1(1)n,\ B_j^{-1} = K_k(x^{(j)}, x^{(j)}),\ j = 1(1)s.$$

Suppose that the integral (1) satisfies condition (18), i.e. it vanishes for any odd trigonometric monomial. In [2] the following conjecture has been stated. If a cubature formula for the evaluation of the integral (1) exists which is exact for trigonometric polynomials of degree not greater than $2k+1$ and with a number of $2v = \tau(k+1, n)$ nodes, then it is of the form

$$I(f) \cong \sum_{j=1}^{v} A_j \left[ f(x^{(j)}) + f(x^{(j)} + \Pi) \right],\ \Pi = (\pi, \pi, \dots, \pi). \tag{27}$$

We assume that $f(x) = f(x_1, x_2, \dots, x_n)$ is a $2\pi$-periodic function with respect to all variables. N. N. Osipov proved this conjecture in [7].

**Theorem 5.** For the cubature formula (27) to be exact for trigonometric polynomials of degree not greater than $2k+1$ it is necessary and sufficient that the following conditions be fulfilled:

$$\tilde{K}_k(x^{(r)}, x^{(s)}) = d_r \delta_{rs},\ d_r \neq 0,\ r, s = 1(1)v. \tag{28}$$

If conditions (28) are satisfied, the coefficients in formula (27) are defined by the equalities

$$A_r^{-1} = 2d_r = 2\tilde{K}_k(x^{(r)}, x^{(r)}),\ r = 1(1)v.$$

The proof of Theorem 5 is given in [2]. This theorem permits to extend the modified method of reproducing kernel for the construction of cubature formulae exact for algebraic polynomials, given in [4] to the trigonometric case.

If $a^{(1)}$ is a node of cubature formula (27) exact for trigonometric polynomials of degree not greater than $2k+1$, then the remaining nodes by virtue of (28) satisfy the equation $\tilde{K}_k(a^{(1)}, x) = 0$. Using the above described method we find points $a^{(i)}$, $i = 1(1)n$ satisfying the following conditions

$$\tilde{K}_k(a^{(p)}, a^{(q)}) = d_p\, \delta_{pq},\ d_p \neq 0,\ p, q = 1(1)n.$$

We consider the system of equations $\tilde{K}_k(a^{(l)}, x) = 0,\ l = 1(1)n$ and denote all solutions of this system by $x^{(j)}, j = 1(1)s$. The points $a^{(p)}, a^{(p)} + \Pi,\ p = 1(1)n$, and $x^{(j)},\ j = 1(1)s$, if $2n + s = 2v = \tau(k+1, n)$, are the nodes of the cubature formula

$$I(f) \cong \sum_{i=1}^{n} A_i \left[ f(a^{(i)}) + f(a^{(i)} + \Pi) \right] + \sum_{j=1}^{v-n} B_j \left[ f(x^{(j)}) + f(x^{(j)} + \Pi) \right],$$

where

$$A_i^{-1} = 2d_i = 2\tilde{K}_k(a^{(i)}, a^{(i)}),\ B_j^{-1} = 2d_j = 2\tilde{K}_k(x^{(j)}, x^{(j)}).$$

The papers [8,9] contain a survey of results on the evaluation of integral (1) in the case $\Omega = [0,1]^n$ and $w = 1$ by means of cubature formulae that are exact for trigonometric polynomials. These results are published in the Russian mathematical literature. These papers contain also new results, namely new cubature formulae for the evaluation of two-dimensional integrals with a minimal number of nodes. The following are the main new results in [9]. A closed form is obtained for expressing the reproducing kernels for the spaces of trigonometric polynomials in two dimensions, and a definition of shift symmetric cubature formulae is given. Theorems are proved which are analogous to Theorems 4 and 5 in this paper.

In what follows we shall consider the special case of the integral (1),

$$I_n(f) = \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} f(x)dx. \tag{29}$$

The trigonometric monomials $\varphi_j(x),\ j = 1, 2, \ldots$ coincide with the orthonormal system with respect to the scalar product (20), which is determined by integral (29). This permits to re-write the reproducing kernels (21) and (23). The kernel (21) is of the form

$$K_k(u, x) = \sum_{l=0}^{k} \sum_{||\alpha||_1 = l} e^{i(\alpha, x-u)}, \tag{30}$$

where the summation in the inner sum is taken over all integer vectors $\alpha$, which satisfy the equality $||\alpha||_1 = l$. We re-write the kernel (23) as well

$$\tilde{K}_k(u, x) = {\sum_{l=\sigma}^{k}}' \sum_{||\alpha||_1 = l} e^{i(\alpha, u-x)},\ \sigma = k - 2\left\lfloor \frac{2}{k} \right\rfloor, \tag{31}$$

where the prime on the first summation sign indicates that the summation is taken over all $l$ with the same parity as that of $k$.

With the help of Theorem 5 and the representation of the reproducing kernel in (31) we can construct cubature formulae for the evaluation of the integral $I_2(f)$ that are exact for trigonometric polynomials of degree not greater than $2k + 1$ and have the minimal number of nodes $\tau(k + 1, 2) = 2(k + 1)^2$. This is done in [6] for $k = 1(1)4$. These cubature formulae have been obtained by other methods for any $k$ in [3].

The cubature formula for the integral (29) is called a lattice cubature formula if it is of the form

$$I_n(f) \cong \frac{1}{N} \sum_{j=1}^{N} f\left( \left\{ \frac{jp_1}{N} \right\} 2\pi, \left\{ \frac{jp_2}{N} \right\} 2\pi, \ldots, \left\{ \frac{jp_n}{N} \right\} 2\pi \right), \tag{32}$$

where $p_1, p_2, \ldots, p_n$ are positive integers and $\{h\}$ is the fractional part of $h$. In Western literature a cubature formula (32) is called a lattice rule of rank 1, see, e.g., [9].

The point $\Theta = (0, 0, 0, \ldots, 0)$ is a node of the cubature formula (32), which we obtain for $j = N$. If $N = 2s$ is even and the numbers $p_j,\ j = 1(1)N$, are odd, then the lattice cubature formula (32) is of the form (27).

We indicate the connection of the $r$-th coordinates of the nodes which we obtain for $j = l$ and $j = l + s$. We have

$$\left\{ \frac{(l+s)p_r}{N} \right\} 2\pi = \left\{ \frac{lp_r}{N} + \frac{1}{2} p_r \right\} 2\pi = \left\{ \frac{lp_r}{2} + \frac{1}{2} \right\} 2\pi. \tag{33}$$

If $\{p_r/N\} < 1/2$, then the right-hand side of equation (33) equals $\{lp_r/N\}2\pi + \pi$.
If $\{lp_r/N\} \geq 1/2$, then

$$\left\{ \frac{lp_r}{N} + \frac{1}{2} \right\} 2\pi = \left( \left\{ \frac{lp_r}{N} \right\} - \frac{1}{2} \right) 2\pi = \left\{ \frac{lp_r}{N} \right\} 2\pi - \pi \, .$$

Adding $2\pi$ we get that the right-hand side of (33) equals $\{lp_r/N\}\pi + \pi$. The node for $j = l+s$ does not belong to $[0, 2\pi]^n$, if $\{lp_r/N\} > 1/2$. This is of no significance, since we can extend the integrand $f(x)$ $2\pi$-periodically with respect to every variable.
The Theorems 4 and 5 and the reproducing kernels (30) and (31) may be used for the construction of lattice cubature formulae. Such cubature formulae for the evaluation of the integral $I_2(f)$ exact for trigonometric polynomials of degree not greater than $2k$ with a minimal number of $t(2,k)$ nodes have been constructed for $k = 1, 2, 3$ in [6]. These cubature formulae are obtained with other methods for all $k$ in [10]. In order to illustrate this method for the construction of cubature formulae we consider the case in which Theorem 4 is applied for $k = 2$, $n = 2$. The reproducing kernel (30) is of the form

$$K_2(u, x) = 1 + \sum_{||\alpha||_1 = 1} e^{i(\alpha, x-u)} + \sum_{||\alpha||_1 = 2} e^{i(\alpha, x-u)}.$$

The first sum consists of four summands for the indices

$$\alpha = (\pm 1, 0), \ (0, \pm 1).$$

The second sum consists of eight summands for the indices

$$\alpha = (\pm 2, 0), \ (0, \pm 2), \ (\pm 1, \pm 1).$$

Joining the summands

$$e^{i(\alpha, x-u)} + e^{i(-\alpha, x-u)} = 2\cos(\alpha, x-u)$$

and transforming it, we get

$$\begin{aligned} K_2(u, x) &= 1 + 2[\cos(x_1 - u_1) + \cos(x_2 - u_2)] \\ &\quad + 2[\cos 2x_1 + 2\cos x_1 \cos x_2 + \cos 2x_2]. \end{aligned}$$

Since the point $(0, 0)$ is a node of the lattice cubature formula, the remaining nodes due to Theorem 4 satisfy

$$\begin{aligned} K_2(0, 0; x_1, x_2) &= 1 + 2[\cos x_1 + \cos x_2] + \\ &\quad 2[\cos 2x_1 + 2\cos x_1 \cos x_2 + \cos 2x_2] = 0. \end{aligned}$$

If the lattice cubature formula exists and $(x_1, x_2)$ is the first node, then $(2x_1, 2x_2)$ is a node as well, consequently, $K_2(0, 0; 2x_1, 2x_2) = 0$. We obtain a system of two equations in $x_1, x_2$:

$$\begin{aligned} 0 &= 1 + 2[\cos x_1 + \cos x_2] + 2[\cos 2x_1 + 2\cos x_1 \cos x_2 + \cos 2x_2], \\ 0 &= 1 + 2[\cos 2x_1 + \cos 2x_2] + 2[\cos 4x_1 + 2\cos 2x_1 \cos 2x_2 + \cos 4x_2]. \end{aligned}$$

Introducing $a = \cos x_1$, $b = \cos x_2$ and applying the formulae $\cos 2\varphi = 2\cos^2\varphi - 1$, $\cos 4\varphi = 8\cos^4\varphi - 8\cos^2\varphi + 1$, the system is of the form

$$\begin{aligned} 0 &= 4(a^2 + ab + b^2) + 2(a + b) - 3 \\ 0 &= 16(a^4 + a^2b^2 + b^4) - 20(a^2 + b^2) + 5. \end{aligned}$$

The right-hand sides of this system are symmetric polynomials in $a$ and $b$, therefore we introduce the new unknowns $\sigma = a + b$ and $\tau = ab$, which are the elementary symmetric polynomials of $a$ and $b$. It is not difficult to see that $a^2 + b^2 = (a+b)^2 - 2ab = \sigma^2 - 2\tau$. Since $(a^2 + b^2)^2 = a^4 + b^4 + 2a^2b^2$, hence

$$a^4 + b^4 = (\sigma^2 - 2\tau)^2 - 2\tau^2 = \sigma^4 - 4\sigma^2\tau + 2\tau^2.$$

The system in $\sigma$ and $\tau$ may be written as

$$\begin{aligned} 0 &= 4(\sigma^2 - \tau) + 2\sigma - 3, \\ 0 &= 16(\sigma^4 - 4\sigma^2\tau + 3\tau^2) - 20(\sigma^2 - 2\tau) + 5 = 0. \end{aligned}$$

From the first equation we get $\tau = (4\sigma^2 + 2\sigma - 3)/4$. Substituting this into the second equation we obtain an algebraic equation of degree three, $8\sigma^3 + 4\sigma^2 - 8\sigma + 1 = 0$. The three roots are

$$\sigma_1 = 0.688601\ldots, \; \sigma_2 = 0.136945\ldots, \; \sigma_3 = -1.325546\ldots$$

With the help of $\sigma_1$ we find $\tau_1 = 0.0684726\ldots$. The values $a_1$ and $b_1$ are determined by the quadratic equation $z^2 - \sigma_1 z + \tau_1 = 0$ and they are calculated as

$$a_1 = 0.568064\ldots, \; b_1 = 0.120537\ldots.$$

From the equations $a_1 = \cos x_1^{(1)}$, $b_1 = \cos x_2^{(1)}$ we find $(x_1^{(1)}, x_2^{(2)}) = (\frac{2}{13}, \frac{3}{13})2\pi$, which is the first node of the lattice cubature formula

$$I_2(f) \cong \frac{1}{13} \sum_{j=1}^{13} f\left(\left\{j\,\frac{2}{13}\right\} 2\pi, \left\{j\,\frac{3}{13}\right\} 2\pi\right). \tag{34}$$

The coordinates of the nodes in the cubature sum may be rearranged. The roots $\sigma_2$ and $\sigma_3$ generate a cubature formula (34) as well, where the first node is different.
The indicated method of constructing a lattice cubature formulae may be applied for the integrals $I_n(f)$ for $n > 2$. For example, by means of Theorem 5 the two cubature formulae that are exact for trigonometric of polynomials of degree not greater than 5 with a minimal number of 38 nodes are constructed in [11].
One of these formulae has been derived before in [13] and [12]. With the help of Theorem 4 the following assertion is proved in [14]. In $\mathbb{R}^3$ a lattice cubature formula of rank 1 that is exact for trigonometric polynomials of degree not greater than 4 with a minimal number of 25 nodes does not exist. The reproducing kernel (30) in this case may be written in the form

$$\begin{aligned} K_2(u,x) \;=\; & 1 + 2[\cos(x_1 - u_1) + \cos(x_2 - u_2) + \cos(x_3 - u_3)] + \\ & +2\left[\cos 2(x_1 - u_1) + \cos 2(x_2 - u_2) + \cos 2(x_3 - u_3)\right. \\ & +2\cos(x_1 - u_1)\cos(x_2 - u_2) + 2\cos(x_1 - u_1)\cos(x_3 - u_3) \\ & \left.+2\cos(x_2 - u_2)\cos(x_3 - u_3)\right]. \end{aligned}$$

Consider the one-dimensional case where the integral is of the form

$$I(f) := \frac{1}{2\pi} \int_0^{2\pi} w(x) f(x) dx. \tag{35}$$

It is known that for $w(x) = 1$ the composite rectangular quadrature formula

$$I(f) \cong \frac{1}{m} \sum_{j=1}^{m} f\left(\alpha + (j-1)\frac{2\pi}{m}\right), \ \alpha \in \left[0, \frac{2\pi}{m}\right] \tag{36}$$

has the highest trigonometric degree of accuracy $m-1$. The quadrature sum in (36) contains the parameter $\alpha \in [0, \frac{2\pi}{m}]$; in other words, the quadrature formula (36) is not unique.
We assume that the weight function $w(x)$ is nonnegative on $[0, 2\pi]$ and that it is integrable. In [15] a simple description of the set of all quadrature formulae of highest trigonometric degree of accuracy for evaluation of the integral (35) is given and an algorithm for their calculation is presented.
In particular, it follows from these results that in the case $w(x) = 1$ there is no quadrature formula of highest trigonometric degree different from the composite rectangle quadrature formula (36). The kernel of Szegő, $S_k(u, x)$, cf. [16, p. 298], plays an important role in this investigation. The connection between the reproducing kernel for the space of trigonometric polynomials in one variable (21), $K(u, x)$ and the kernel $S_k(u, x)$ is studied in [14].

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