

**Computing Orthogonal Rational
Functions
analytic outside the unit disc**

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Report TW 312, August 2000



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Abstract

The main accomplishment of this article is an algorithm that computes the reflection coefficients of the orthogonal rational functions, given the (finite number of) moments of the measure.

Therefore we give some results for the orthogonal rational functions (ORF) with respect to a given positive measure and with given poles inside the unit disc. These results are already known for ORF with poles outside the unit disc and the proofs are mainly a rewriting of the case where the poles lay outside the unit disc.

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Computing Orthogonal Rational Functions analytic outside the unit disc

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Abstract

The main accomplishment of this article is an algorithm that computes the reflection coefficients of the orthogonal rational functions, given the (finite number of) moments of the measure.

Therefore we give some results for the orthogonal rational functions (ORF) with respect to a given positive measure and with given poles inside the unit disc. These results are already known for ORF with poles outside the unit disc and the proofs are mainly a rewriting of the case where the poles lay outside the unit disc.

1 Introduction and motivation

In [3] the theory of orthogonal rational functions (ORF) with respect to a general measure and with prescribed poles, lying outside the unit disc, is well developed. These rational functions are a direct generalization of the well-known Szegő polynomials [8], in the sense that if all poles are placed at ∞ the rational functions coincide with the Szegő polynomials.

In the research area of system theory we see recently an increasing interest in the use of orthogonal rational functions for identification purposes (see [1, 2, 5, 6, 7, 9, 10]). Here it is however much more conventional to have poles inside the unit disc for stability reasons.

This is our motivation for a slight adaptation of the theory found in [3] to the case where all the poles lay inside the unit disc.

The main accomplishment of this article is however dealing with the computational problems that occur when some poles coincide. The Nevanlinna-Pick-type interpolation scheme to compute the reflection coefficients in the recurrence relation of the ORF then stops because of the indefinite form $0/0$. We solve this problem in the case where the number of moments is finite.

This paper is built up as follows: in the next section some definitions and notations are introduced. The third section handles the recurrence relation of the orthogonal rational functions. In the fourth section, the so-called *functions of the second kind* are introduced and the

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fifth section handles some interpolation theorems. Section six contains the algorithm to compute the reflection coefficients if the moments are known and section seven contains a numerical example as illustration. Some conclusions are made in section eight. The MATLAB source is given in the appendix.

2 Preliminaries

The field of real and complex numbers is denoted as \mathbb{R} and \mathbb{C} respectively. The integers are denoted by \mathbb{Z} and the non-negative integers by \mathbb{N} .

The unit circle, its exterior and interior are denoted by

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}, \quad \mathbb{E} = \{z \in \mathbb{C} : |z| > 1\} \text{ and } \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\},$$

respectively. The closed unit disc is defined as $\overline{\mathbb{D}} = \mathbb{T} \cup \mathbb{D}$.

Let μ be a positive measure on \mathbb{T} . Define the inner product

$$\langle f, g \rangle_\mu = \int_{\mathbb{T}} f(t) \overline{g(t)} d\mu(t), \quad f, g \in L_2(\mu).$$

We assume in the rest of this paper that the measure is normalized in the sense that $\int d\mu(t) = 1$. This does not affect the generality.

We define the moments as the Fourier coefficients

$$c_k = \int_{\mathbb{T}} e^{-ik\theta} d\mu(\theta), \quad k \in \mathbb{Z}. \quad (1)$$

The set of functions analytic outside the unit disc is denoted by $\mathcal{H}(\mathbb{E})$.

We define the Riesz-Herglotz kernel

$$D(t, z) = \frac{t+z}{t-z}.$$

We can associate a positive real function $\Omega(z)$ with the measure μ as follows

$$\Omega(z) = \int_{\mathbb{T}} D(t, z) d\mu(t), \quad z \in \mathbb{D}.$$

This function is analytic in \mathbb{D} and the relation between Ω and μ is one-to-one. If $\Omega \in H_1(\mathbb{D})$, the Hardy-space of all functions, that are absolutely integrable and analytic in \mathbb{D} , then we have

$$\Omega(z) = c_0 + 2 \sum_{k=1}^{\infty} c_k z^k, \quad (2)$$

which converges uniformly in \mathbb{D} .

By $\alpha = \{\alpha_1, \alpha_2, \dots\} \subset \mathbb{D}$, we denote a given sequence of points inside the unit disc. These points will be the poles of the orthogonal rational functions. We set $\alpha_0 = 0$ for notational reasons.

Now we can define the Blaschke factors ζ_k as

$$\zeta_k(z) = \frac{1 - \overline{\alpha_k} z}{z - \alpha_k}, \quad k \in \mathbb{N}, \quad (3)$$

and the Blaschke products B_k as

$$B_0(z) = 1, \quad B_k(z) = B_{k-1}(z)\zeta_k(z), \quad k = 1, 2, \dots \quad (4)$$

If we incorporate ζ_0 , we use the notation

$$\tilde{B}_{-1}(z) = 1, \quad \tilde{B}_k(z) = \zeta_0(z)B_k(z) = \frac{B_k(z)}{z}, \quad k \in \mathbb{N}. \quad (5)$$

The space of all rational functions with poles in $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is denoted by

$$\mathcal{L}_n = \text{span}\{B_0(z), B_1(z), \dots, B_n(z)\}. \quad (6)$$

The space \mathcal{L}_n can also be defined as

$$\mathcal{L}_n = \left\{ \frac{p_n(z)}{\pi_n(z)} : p_n \in \Pi_n, \pi_n(z) = \prod_{k=1}^n (z - \alpha_k) \right\},$$

where Π_n denotes the space of polynomials of degree at most n . The space of all rational functions with poles in α is denoted by

$$\mathcal{L} = \bigcup_{k=0}^{\infty} \mathcal{L}_k.$$

We also introduce the notation

$$B_{n/k}(z) = B_n(z)/B_k(z), \quad n, k \in \mathbb{N}.$$

If all $\alpha_k = 0$, we find that $B_n(z) = z^{-n} = 1/\pi_n(z)$ and that the space \mathcal{L}_n coincides with the space Π_{-n} of polynomials in z^{-1} with degree at most n .

Taking the complex conjugate of a function on the unit circle is extended to the whole complex plane by the substar transform

$$f_*(z) = \overline{f(1/\bar{z})}, \quad z \in \mathbb{C}.$$

The superstar transform is defined for functions $f \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ as

$$f^*(z) = B_n(z)f_*(z), \quad z \in \mathbb{C}.$$

Note that this operation depends on the degree n of the rational function. We do not incorporate this in the notation because it will be clear from the context. For polynomials of exact degree n the superstar transform is defined as above by replacing $B_n(z)$ by z^n .

3 Orthogonal Rational Functions

With the Gram-Schmidt procedure we can orthonormalize the basis $\{B_k\}_{k \in \mathbb{N}}$ to find a set of orthonormal rational functions $\phi_k \in \mathcal{L}_k$ with respect to the measure μ . This implies

$$\langle \phi_k, \phi_l \rangle_\mu = \delta_{kl}, \quad \forall k, l \in \mathbb{N},$$

where δ_{kl} denotes the Kronecker delta. This set is unique up to a unimodular constant factor for each ϕ_k . By choosing the leading coefficient¹ κ_k of ϕ_k to be positive, we have a unique set of orthonormal rational functions. It is easy to see that the leading coefficient is given by $\kappa_k = \phi_k^*(1/\overline{\alpha_k})$.

In this section we look at the recurrence relations for these orthonormal rational functions $\{\phi_k\}_{k=0}^\infty$.

Theorem 1 *With the notation above the following recursion for the orthonormal rational functions $\phi_n \in \mathcal{L}_n$ is valid:*

$$\begin{bmatrix} \phi_n(z) \\ \phi_n^*(z) \end{bmatrix} = \begin{bmatrix} d_n & 0 \\ 0 & \overline{d_n} \end{bmatrix} \frac{z - \alpha_{n-1}}{z - \overline{\alpha_{n-1}}} \begin{bmatrix} 1 & \overline{\lambda_n} \\ \lambda_n & 1 \end{bmatrix} \begin{bmatrix} \zeta_{n-1}(z) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_{n-1}(z) \\ \phi_{n-1}^*(z) \end{bmatrix},$$

where

$$\begin{aligned} \overline{d_n} &= \frac{1 - \overline{\alpha_{n-1}}\alpha_n}{1 - |\alpha_{n-1}|^2} \frac{\phi_n^*(1/\overline{\alpha_{n-1}})}{\kappa_{n-1}}; \\ \lambda_n &= \frac{\phi_n^*(\alpha_{n-1})}{\phi_n^*(1/\overline{\alpha_{n-1}})} \frac{1 - \overline{\alpha_n}\alpha_{n-1}}{1 - \overline{\alpha_{n-1}}\alpha_n}. \end{aligned}$$

Proof:

First we prove the existence of constants c_n and d_n such that

$$\frac{z - \alpha_n}{1 - \overline{\alpha_{n-1}}z} \phi_n(z) - d_n \phi_{n-1}(z) - c_n \frac{z - \alpha_{n-1}}{1 - \overline{\alpha_{n-1}}z} \phi_{n-1}^*(z) \in \mathcal{L}_{n-2}. \quad (7)$$

Define as before $\pi_n(z) = \prod_{k=1}^n (z - \alpha_k)$ and $p_n(z) = \phi_n(z)\pi_n(z)$. Note that $p_n^*(z) = \phi_n^*(z)\pi_n(z)$.

We can rewrite (7) as

$$\frac{p_n(z) - d_n(1 - \overline{\alpha_{n-1}}z)p_{n-1}(z) - c_n(z - \alpha_{n-1})p_{n-1}^*(z)}{(1 - \overline{\alpha_{n-1}}z)\pi_{n-1}(z)} = \frac{N(z)}{D(z)}.$$

If we want this to be in \mathcal{L}_{n-2} , we need to have $N(1/\overline{\alpha_{n-1}}) = 0 = N^*(1/\overline{\alpha_{n-1}})$. The first condition gives

$$c_n = \frac{\overline{\alpha_{n-1}}}{1 - |\alpha_{n-1}|^2} \frac{p_n(1/\overline{\alpha_{n-1}})}{p_{n-1}^*(1/\overline{\alpha_{n-1}})}. \quad (8)$$

From the second condition we get d_n :

$$\overline{d_n} = \frac{\overline{\alpha_{n-1}}}{1 - |\alpha_{n-1}|^2} \frac{p_n^*(1/\overline{\alpha_{n-1}})}{p_{n-1}^*(1/\overline{\alpha_{n-1}})}. \quad (9)$$

Since $p_n^*(z) = \phi_n^*(z)\pi_n(z)$, we have

$$\begin{aligned} p_{n-1}^*(1/\overline{\alpha_{n-1}}) &= \phi_{n-1}^*(1/\overline{\alpha_{n-1}})\pi_{n-1}(1/\overline{\alpha_{n-1}}) \\ &= \kappa_{n-1}\pi_{n-1}(1/\overline{\alpha_{n-1}}); \\ p_n(1/\overline{\alpha_{n-1}}) &= \phi_n(1/\overline{\alpha_{n-1}})\pi_n(1/\overline{\alpha_{n-1}}); \\ p_n^*(1/\overline{\alpha_{n-1}}) &= \phi_n^*(1/\overline{\alpha_{n-1}})\pi_n(1/\overline{\alpha_{n-1}}). \end{aligned}$$

¹The leading coefficient with respect to the Blaschke product basis is κ_k if $\phi_k = \kappa_k B_k + c_{k-1} B_{k-1} + c_{k-2} B_{k-2} + \dots + c_0 B_0$

This enables us to rewrite (8) and (9) as

$$c_n = \frac{1 - \overline{\alpha_{n-1}}\alpha_n}{1 - |\alpha_{n-1}|^2} \frac{\phi_n(1/\overline{\alpha_{n-1}})}{\kappa_{n-1}}, \quad (10)$$

$$\overline{d}_n = \frac{1 - \overline{\alpha_{n-1}}\alpha_n}{1 - |\alpha_{n-1}|^2} \frac{\phi_n^*(1/\overline{\alpha_{n-1}})}{\kappa_{n-1}}. \quad (11)$$

Thus with the choices (10) and (11) the expression (7) is in \mathcal{L}_{n-2} . However, at the same time, it is also orthogonal to \mathcal{L}_{n-2} . We check this by noting that for $k \leq n-2$, ϕ_k is orthogonal to every term in (7):

- $\left\langle \frac{z-\alpha_n}{1-\overline{\alpha_{n-1}}z} \phi_n, \phi_k \right\rangle_\mu = \left\langle \phi_n, \frac{1-\overline{\alpha_n}z}{z-\alpha_{n-1}} \phi_k \right\rangle_\mu = 0$, because the right factor is in \mathcal{L}_{n-1} .
- $\langle \phi_{n-1}, \phi_k \rangle_\mu = 0$, because $k \leq n-2$.
- $\left\langle \frac{z-\alpha_{n-1}}{1-\overline{\alpha_{n-1}}z} \phi_{n-1}^*, \phi_k \right\rangle_\mu = \left\langle B_{n-1} \phi_{k^*} \frac{z-\alpha_{n-1}}{1-\overline{\alpha_{n-1}}z}, \phi_{n-1} \right\rangle_\mu = \left\langle \phi_k^* \prod_{l=k+1}^{n-2} \zeta_l, \phi_{n-1} \right\rangle_\mu = 0$, because the left factor is in \mathcal{L}_{n-2} .

Thus we found that the rational function in (7) equals zero. This implies

$$\phi_n(z) = d_n \frac{z - \alpha_{n-1}}{z - \alpha_n} [\zeta_{n-1}(z) \phi_{n-1}(z) + \overline{\lambda}_n \phi_{n-1}^*(z)], \quad (12)$$

with

$$\lambda_n = \overline{c_n/d_n} = \frac{\phi_{n^*}(\alpha_{n-1})}{\phi_n^*(1/\overline{\alpha_{n-1}})} \frac{1 - \overline{\alpha_n}\alpha_{n-1}}{1 - \overline{\alpha_{n-1}}\alpha_n}. \quad (13)$$

Taking the superstar of (12) gives

$$\phi_n^*(z) = \overline{d}_n \frac{z - \alpha_{n-1}}{z - \alpha_n} [\lambda_n \zeta_{n-1}(z) \phi_{n-1}(z) + \phi_{n-1}^*(z)]. \quad (14)$$

The proof is complete if the initial conditions of the recurrence are correct, that is for $n = 1$. Now, since $\phi_0 = 1 = \phi_0^*$ and $\alpha_0 = 0$, we have

$$\phi_1(z) = d_1 \frac{1 + \overline{\lambda}_1 z}{z - \alpha_1} \text{ and } \phi_1^*(z) = \overline{d}_1 \frac{\lambda_1 + z}{z - \alpha_1}.$$

If we solve this for $z = 1/\overline{\alpha_0}$, we find (11) and (13). This completes the proof. ■

If we put $e_n = |d_n|$ and $\eta_n = d_n/|d_n|$, the recurrence relation of Theorem 1 becomes

$$\begin{bmatrix} \phi_n(z) \\ \phi_n^*(z) \end{bmatrix} = e_n \begin{bmatrix} \eta_n & 0 \\ 0 & \overline{\eta}_n \end{bmatrix} \frac{z - \alpha_{n-1}}{z - \alpha_n} \begin{bmatrix} 1 & \overline{\lambda}_n \\ \lambda_n & 1 \end{bmatrix} \begin{bmatrix} \zeta_{n-1}(z) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_{n-1}(z) \\ \phi_{n-1}^*(z) \end{bmatrix}. \quad (15)$$

Using (12) and the fact that $\langle \phi_k, \phi_n \rangle_\mu = 0$ for $k \leq n-1$, we find

$$\lambda_n = - \frac{\left\langle \phi_k, \frac{1-\overline{\alpha_{n-1}}z}{z-\alpha_n} \phi_{n-1} \right\rangle_\mu}{\left\langle \phi_k, \frac{z-\alpha_{n-1}}{z-\alpha_n} \phi_{n-1}^* \right\rangle_\mu}, \quad \forall k \leq n-1. \quad (16)$$

We can find an expression for e_n by using (11) and (13).

$$\begin{aligned}
e_n^2(1 - |\lambda_n|^2) &= |d_n|^2 \left(1 - \frac{|\phi_n(1/\overline{\alpha_{n-1}})|^2}{|\phi_n^*(1/\overline{\alpha_{n-1}})|^2} \right) \\
&= \frac{|1 - \overline{\alpha_{n-1}}\alpha_n|^2}{(1 - |\alpha_{n-1}^2|)^2} \frac{|\phi_n^*(1/\overline{\alpha_{n-1}})|^2 - |\phi_n(1/\overline{\alpha_{n-1}})|^2}{\kappa_{n-1}^2} \\
&= \frac{|1 - \overline{\alpha_{n-1}}\alpha_n|^2}{(1 - |\alpha_{n-1}^2|)^2} (1 - |\zeta_n(1/\overline{\alpha_{n-1}})|^2) \\
&= \frac{|1 - \overline{\alpha_{n-1}}\alpha_n|^2}{(1 - |\alpha_{n-1}^2|)^2} \frac{(1 - |\alpha_{n-1}|^2)(1 - |\alpha_n|^2)}{|1 - \overline{\alpha_n}\alpha_{n-1}|^2} \\
&= \frac{1 - |\alpha_n|^2}{1 - |\alpha_{n-1}|^2}. \tag{17}
\end{aligned}$$

The third line follows from the *Christoffel-Darboux* relation (Theorem 2). The case where all the poles are outside the unit disc and this can be found in [3, Theorem 3.1.3]. If we put $z = 1/\overline{\alpha_{n-1}} = w$ in this relation we find

$$\kappa_{n-1}^2 = \frac{|\phi_n^*(1/\overline{\alpha_{n-1}})|^2 - |\phi_n(1/\overline{\alpha_{n-1}})|^2}{1 - |\zeta_n(1/\overline{\alpha_{n-1}})|^2},$$

which was used above. We proved the following

Lemma 1 *The constant e_n can be calculated as the positive square root of*

$$e_n^2 = \frac{1 - |\alpha_n|^2}{1 - |\alpha_{n-1}|^2} \frac{1}{1 - |\lambda_n|^2}. \tag{18}$$

The Christoffel-Darboux relation is given next.

Theorem 2 *The following relation hold for the orthonormal basis functions ϕ_k*

$$\sum_{k=0}^{n-1} \overline{\phi_k(w)} \phi_k(z) = \frac{\phi_n^*(z) \overline{\phi_n^*(w)} - \phi_n(z) \overline{\phi_n(w)}}{1 - \zeta_n(z) \zeta_n(w)}.$$

We first have to prove some technical lemmas.

Lemma 2 *If $f \in \mathcal{L}_n$*

1. *If g and h are defined by the relations $f(z) - f(w) = (z - w)g(z) = (z - w)/(z - \alpha_n)h(z)$, then*
 - (a) $p_1(z)g(z) \in \mathcal{L}_n$, where p_1 is an arbitrary polynomial of maximal degree 1. Especially we have $g \in \mathcal{L}_n$.
 - (b) $h \in \mathcal{L}_{n-1}$.
2. *If $f(w) = 0$, then $(z - \alpha_n)/(z - w)f(z) \in \mathcal{L}_{n-1}$.*

Proof:

Clearly, we have $g(z) = (f(z) - f(w))/(z - w) = p_{n-1}(z)/\pi_n(z)$, with $\pi_n(z) = \prod_{k=1}^n (z - \alpha_k)$. This implies (1a).

From this and $h(z) = g(z)(z - \alpha_n)$, part (1b) is easily deduced.

(2) follows from (1b) if $f(w) = 0$. ■

Lemma 3 *Let $\{\phi_k\}_{k=0}^{n-1}$ denote the orthonormal basis functions for \mathcal{L}_{n-1} and ζ_k the Blaschke factor based on α_k . As a function of z , with w some parameter, we have*

$$l_n(z, w) = \frac{\phi_n^*(z)\overline{\phi_n^*(w)} - \phi_n(z)\overline{\phi_n(w)}}{1 - \zeta_n(z)\overline{\zeta_n(w)}} \in \mathcal{L}_{n-1}.$$

Proof:

A straightforward computation gives

$$\begin{aligned} 1 - \zeta_n(z)\overline{\zeta_n(w)} &= 1 - \frac{1 - \overline{\alpha_n}z}{z - \alpha_n} \frac{1 - \alpha_n\overline{w}}{\overline{w} - \overline{\alpha_n}} \\ &= \frac{(1 - |\alpha_n|^2)(z\overline{w} - 1)}{(z - \alpha_n)(\overline{w} - \overline{\alpha_n})}. \end{aligned} \quad (19)$$

If we take

$$f(z) = \frac{\overline{w} - \overline{\alpha_n}}{1 - |\alpha_n|^2} \left(\phi_n^*(z)\overline{\phi_n^*(w)} - \phi_n(z)\overline{\phi_n(w)} \right),$$

we have proven the lemma if $f(1/\overline{w}) = 0$, according to Lemma 2 (2). This is true if

$$\phi_n^*(1/\overline{w})\overline{\phi_n^*(w)} = \phi_n(1/\overline{w})\overline{\phi_n(w)},$$

which is easily checked. ■

Now we can prove the Christoffel-Darboux relation.

Proof of Theorem 2:

In Lemma 3 we proved that the right-hand side of Theorem 2 is in \mathcal{L}_{n-1} , so we only need to prove it reproduces.

Take some $f \in \mathcal{L}_{n-1}$. Then

$$\langle f, l_n(\cdot, w) \rangle_\mu = \langle f - f(w), l_n(\cdot, w) \rangle_\mu + f(w) \langle 1, l_n(\cdot, w) \rangle_\mu. \quad (20)$$

By Lemma 2, we find with $f(z) - f(w) = (z - w)g(z)$, that $g(z) \in \mathcal{L}_{n-1}$ and $p_1(z)g(z) \in \mathcal{L}_{n-1}$, for all $p_1 \in \Pi_1$. Thus if we call $N(z, w)$ the numerator of $l_n(z, w)$, we get, using (19),

$$\begin{aligned} \langle f - f(w), l_n \rangle_\mu &= \frac{w - \alpha_n}{1 - |\alpha_n|^2} \left\langle (z - w)g(z), \frac{z - \alpha_n}{z\overline{w} - 1} N(z, w) \right\rangle_\mu \\ &= \frac{w - \alpha_n}{1 - |\alpha_n|^2} \langle (1 - \overline{\alpha_n}z)g(z), N(z, w) \rangle_\mu. \end{aligned}$$

Because $(1 - \overline{\alpha_n}z)g(z) \in \mathcal{L}_{n-1}$, the inner product in the right-hand side gives

$$\begin{aligned} \langle (1 - \overline{\alpha_n}z)g(z), N(z, w) \rangle_\mu &= \langle (1 - \overline{\alpha_n}z)g(z), \phi_n^*(z) \rangle_\mu \phi_n^*(w) \\ &= \langle \zeta_n h, \phi_n^* \rangle_\mu \phi_n^*(w), \end{aligned}$$

with $h(z) = (z - \alpha_n)g(z) \in \mathcal{L}_{n-1}$. This is zero because $\phi_n^* \perp \zeta_n \mathcal{L}_{n-1}$.

It remains to show that $\langle 1, l_n(\cdot, w) \rangle_\mu = \eta(w) = 1$. Apply (20) to the function $f = l_n(\cdot, z)$. Then we get

$$\langle l_n(\cdot, z), l_n(\cdot, w) \rangle_\mu = l_n(w, z)\eta(w).$$

If we interchange z and w , we get

$$\langle l_n(\cdot, w), l_n(\cdot, z) \rangle_\mu = l_n(z, w)\eta(z).$$

Because the left-hand sides are each others conjugate and $l_n(z, w) = \overline{l_n(w, z)}$, we find $\eta(z) = \overline{\eta(w)}$. Thus η is a constant and we only have to prove it is one. For $z = w = 1/\overline{\alpha_n}$, we have

$$\begin{aligned} \sum_{k=0}^{n-1} \phi_k(1/\overline{\alpha_n}) \overline{\phi_k(1/\overline{\alpha_n})} &= \sum_{k=0}^n \phi_k(1/\overline{\alpha_n}) \overline{\phi_k(1/\overline{\alpha_n})} - \phi_n(1/\overline{\alpha_n}) \overline{\phi_n(1/\overline{\alpha_n})} \\ &= \kappa_n^2 - |\phi_n(1/\overline{\alpha_n})|^2. \end{aligned}$$

We also have

$$\begin{aligned} \sum_{k=0}^{n-1} \phi_k(1/\overline{\alpha_n}) \overline{\phi_k(1/\overline{\alpha_n})} &= l_n(1/\overline{\alpha_n}, 1/\overline{\alpha_n})/\eta \\ &= (\kappa_n^2 - |\phi_n(1/\overline{\alpha_n})|^2)/\eta. \end{aligned}$$

From which we find $\eta = 1$. ■

To get rid of the rotation η_n , we introduce the rotated orthonormal rational functions $\Phi_n = \varepsilon_n \phi_n$, where $\varepsilon_n \in \mathbb{T}$. Starting from the recurrence (15), we find

$$\begin{bmatrix} \Phi_n(z) \\ \Phi_n^*(z) \end{bmatrix} = e_n \frac{z - \alpha_{n-1}}{z - \alpha_n} \begin{bmatrix} \varepsilon_n \eta_n \overline{\varepsilon_{n-1}} & \varepsilon_n \eta_n \overline{\lambda_n} \varepsilon_{n-1} \\ \overline{\eta_n} \varepsilon_n \varepsilon_{n-1} \lambda_n & \overline{\eta_n} \varepsilon_n \varepsilon_{n-1} \end{bmatrix} \begin{bmatrix} \zeta_{n-1}(z) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Phi_{n-1}(z) \\ \Phi_{n-1}^*(z) \end{bmatrix}$$

Taking

$$\varepsilon_0 = 1, \quad \varepsilon_n = \varepsilon_{n-1} \overline{\eta_n}, \quad n = 1, 2, \dots, \quad (21)$$

and putting $L_n = \overline{\varepsilon_{n-1}^2} \lambda_n$, we find

$$\begin{bmatrix} \Phi_n(z) \\ \Phi_n^*(z) \end{bmatrix} = e_n \frac{z - \alpha_{n-1}}{z - \alpha_n} \begin{bmatrix} 1 & \overline{L_n} \\ L_n & 1 \end{bmatrix} \begin{bmatrix} \zeta_{n-1}(z) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Phi_{n-1}(z) \\ \Phi_{n-1}^*(z) \end{bmatrix}. \quad (22)$$

4 Functions of the second kind

Recall the definition of the Riesz-Herglotz kernel $D(t, z) = (t + z)/(t - z)$ and the Riesz-Herglotz representation $\Omega(z) = \int_{\mathbb{T}} D(t, z) d\mu(t)$, $z \in \mathbb{D}$. We define the kernel function $E(t, z) = 2t/(t - z) = 1 + D(t, z)$.

The functions of the second kind Ψ_n are defined as follows ($z \in \mathbb{E}$)

$$\begin{aligned} \Psi_n(z) &= \int_{\mathbb{T}} [E_*(t, z) \Phi_n(t) - D_*(t, z) \Phi_n(z)] d\mu(t) \\ &= \int_{\mathbb{T}} D_*(t, z) [\Phi_n(t) - \Phi_n(z)] d\mu(t) + \int_{\mathbb{T}} \Phi_n(t) d\mu(t) \\ &= \begin{cases} 1 & \text{if } n = 0, \\ \int_{\mathbb{T}} D_*(t, z) [\Phi_n(t) - \Phi_n(z)] d\mu(t) & \text{if } n \geq 1. \end{cases} \end{aligned} \quad (23)$$

Here the substar is with respect to z . This definition is for $z \in \mathbb{E}$, but, as we will show below in Lemma 4, these functions are rational and can therefore be defined in the whole complex plane.

Lemma 4 *The functions of the second kind Ψ_n belong to \mathcal{L}_n .*

Proof:

For $n = 0$ this is trivial, so we only consider the case where $n \geq 1$. The integrand in (23) has the form

$$[\Phi_n(t) - \Phi_n(z)] \frac{z+t}{z-t}.$$

The term between square brackets vanishes for $z = t$, so that the integral can be written as

$$\begin{aligned} \Psi_n(z) &= - \int_{\mathbb{T}} \frac{(t-z) \sum_{k=0}^n a_k(t) z^k}{(t-z) \pi_n(z)} d\mu(t) \\ &= - \frac{\sum_{k=0}^n [\int_{\mathbb{T}} a_k(t) d\mu(t)] z^k}{\pi_n(z)} \in \mathcal{L}_n. \end{aligned}$$

This completes the proof. ■

An equivalent definition for these functions of the second kind is given now.

Lemma 5 *We can replace (23) by the following definition for the functions of the second kind Ψ_n ($n > 0$).*

$$\begin{aligned} \frac{\Psi_n(z)}{B_k(z)} &= \int_{\mathbb{T}} D_*(t, z) \left[\frac{\Phi_n(t)}{B_k(t)} - \frac{\Phi_n(z)}{B_k(z)} \right] d\mu(t) \\ &= \int_{\mathbb{T}} \left[E_*(t, z) \frac{\Phi_n(t)}{B_k(t)} - D_*(t, z) \frac{\Phi_n(z)}{B_k(z)} \right] d\mu(t), \end{aligned}$$

for any $0 \leq k < n$. The second formula also holds for $n = 0$, if we take $B_k(z) = 1$.

Proof:

For $n = 0$ this is again trivial, so we only consider the case where $n > 0$. We will only prove the first relation, because the second can be proved following the same lines. We only need to prove

$$\int_{\mathbb{T}} D_*(t, z) \Phi_n(t) \left[1 - \frac{B_k(z)}{B_k(t)} \right] d\mu(t) = 0.$$

The term between square brackets vanishes for $z = t$, therefore we can rewrite the integral as

$$\int_{\mathbb{T}} \Phi_n(t) \frac{p_z(t)}{\pi_k^*(t)} d\mu(t),$$

with $p_z(t)$ a polynomial of degree at most k in t . This is of the form $\langle \Phi_n, f \rangle_\mu$, with $f \in \mathcal{L}_k$. Since $k < n$ and $\Phi_n \perp \mathcal{L}_k$, this is zero. ■

This can be further generalized by replacing $1/B_k$ by any function $f \in \mathcal{L}_{(n-1)*} = \{f | f_* \in \mathcal{L}_{n-1}\}$, but in this note we will only need the form we stated in Lemma 5.

Next we will derive an expression for the superstar conjugate function of the second kind Ψ_n^* , from the formula in Lemma 5. If we take the substar, we find

$$\frac{\Psi_{n^*}(z)}{B_{k^*}(z)} = \int_{\mathbb{T}} \left[E(t, z) \frac{\Phi_{n^*}(t)}{B_{k^*}(t)} - D(t, z) \frac{\Phi_{n^*}(z)}{B_{k^*}(z)} \right] d\mu(t).$$

This implies

$$\begin{aligned}
\frac{\Psi_n^*(z)}{B_{n/k}(z)} &= \int_{\mathbb{T}} \left[E(t, z) \frac{\Phi_n^*(t)}{B_{n/k}(t)} - D(t, z) \frac{\Phi_n^*(z)}{B_{n/k}(z)} \right] d\mu(t) \\
&= - \int_{\mathbb{T}} D_*(t, z) \left[\frac{\Phi_n^*(t)}{B_{n/k}(t)} - \frac{\Phi_n^*(z)}{B_{n/k}(z)} \right] d\mu(t) + \int_{\mathbb{T}} \frac{\Phi_n^*(t)}{B_{n/k}(t)} d\mu(t) \\
&= - \int_{\mathbb{T}} D_*(t, z) \left[\frac{\Phi_n^*(t)}{B_{n/k}(t)} - \frac{\Phi_n^*(z)}{B_{n/k}(z)} \right] d\mu(t) + \delta_{0n}.
\end{aligned} \tag{24}$$

For $k = 0$ we find the simple formula

$$\Psi_{n^*}(z) = - \int_{\mathbb{T}} D_*(t, z) [\Phi_{n^*}(t) - \Phi_{n^*}(z)] d\mu(t) + \delta_{0n}.$$

We show now that the functions of the second kind satisfy a recurrence relation, which has the same recurrence matrix as in (22) for the orthonormal rational functions.

Theorem 3 *For the functions of the second kind Ψ_n a recursion of the following form exists*

$$\begin{bmatrix} \Psi_n(z) \\ -\Psi_n^*(z) \end{bmatrix} = e_n \frac{z - \alpha_{n-1}}{z - \alpha_n} \begin{bmatrix} 1 & \overline{L_n} \\ L_n & 1 \end{bmatrix} \begin{bmatrix} \zeta_{n-1}(z) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Psi_{n-1}(z) \\ -\Psi_{n-1}^*(z) \end{bmatrix}. \tag{25}$$

Proof:

We only show the first relation, since the second one is easily derived by taking the superstar conjugate of the first.

For $n > 1$, we find from (23) and (24)

$$\begin{bmatrix} \Psi_{n-1}(z) \\ -\Psi_{n-1}^*(z) \end{bmatrix} = -\Omega_*(z) \begin{bmatrix} \Phi_{n-1}(z) \\ \Phi_{n-1}^*(z) \end{bmatrix} + \int_{\mathbb{T}} D_*(t, z) \begin{bmatrix} \Phi_{n-1}(t) \\ \frac{\zeta_{n-1}(z)}{\zeta_{n-1}(t)} \Phi_{n-1}^*(t) \end{bmatrix} d\mu(t).$$

Multiply from the left by

$$e_n \frac{z - \alpha_{n-1}}{z - \alpha_n} \begin{bmatrix} \zeta_{n-1}(z) & \overline{L_n} \end{bmatrix}.$$

Then the right-hand side becomes

$$-\Omega_*(z) \Phi_n(z) + \int_{\mathbb{T}} D_*(t, z) f(t, z) d\mu(t), \tag{26}$$

with

$$\begin{aligned}
f(t, z) &= e_n \frac{z - \alpha_{n-1}}{z - \alpha_n} \left[\zeta_{n-1}(z) \Phi_{n-1}(t) + \overline{L_n} \frac{\zeta_{n-1}(z)}{\zeta_{n-1}(t)} \Phi_{n-1}^*(t) \right] \\
&= \frac{\zeta_{n-1}(z)}{\zeta_{n-1}(t)} \frac{z - \alpha_{n-1}}{t - \alpha_{n-1}} \frac{t - \alpha_n}{z - \alpha_n} e_n \frac{t - \alpha_{n-1}}{t - \alpha_n} \left[\zeta_{n-1}(t) \Phi_{n-1}(t) + \overline{L_n} \Phi_{n-1}^*(t) \right] \\
&= \frac{1 - \overline{\alpha_{n-1}}z}{1 - \overline{\alpha_n}z} \frac{t - \alpha_n}{t - \alpha_{n-1}} \Phi_n(t),
\end{aligned}$$

where we used the recurrence (22) for Φ_n in the last line. We find that (26) becomes

$$-\Omega_*(z) \Phi_n(z) + \int_{\mathbb{T}} D_*(t, z) \frac{1 - \overline{\alpha_{n-1}}z}{1 - \overline{\alpha_n}z} \frac{t - \alpha_n}{t - \alpha_{n-1}} \Phi_n(t) d\mu(t).$$

This will equal $\Psi_n(z)$ if we may replace the latter integral by

$$\int_{\mathbb{T}} D_*(t, z) \Phi_n(t) d\mu(t).$$

This can indeed be done, since the difference integral equals

$$\int_{\mathbb{T}} D_*(t, z) \left[1 - \frac{1 - \overline{\alpha_{n-1}}z}{1 - \overline{\alpha_{n-1}}t} \frac{t - \alpha_n}{z - \alpha_n} \right] \Phi_n(t) d\mu(t) = \frac{1 - \alpha_n \overline{\alpha_{n-1}}}{z - \alpha_n} \int_{\mathbb{T}} \frac{t + z}{1 - \overline{\alpha_{n-1}}t} \Phi_n(t) d\mu(t) = 0,$$

because $\Phi_n \perp \mathcal{L}_{n-1}$.

We only have to check the case $n = 1$ to end the proof. We have to show that

$$\Psi_1(z) = e_1 \frac{1 - \overline{L_1}z}{z - \alpha_1}. \quad (27)$$

From the recurrence relation (22), we have

$$\Phi_1(z) = e_1 \frac{1 + \overline{L_1}z}{z - \alpha_1}.$$

Using this and definition (23), we find

$$\begin{aligned} \Psi_1(z) &= e_1 \int_{\mathbb{T}} \frac{t + z}{z - t} \left[\frac{1 + \overline{L_1}t}{t - \alpha_1} - \frac{1 + \overline{L_1}z}{z - \alpha_1} \right] d\mu(t) \\ &= e_1 \frac{1 + \overline{L_1}\alpha_1}{z - \alpha_1} \int_{\mathbb{T}} \frac{t + z}{t - \alpha_1} d\mu(t). \end{aligned}$$

We use orthonormality to find an expression for the latter integral.

$$0 = \langle \Phi_1, 1 \rangle_{\mu} = e_1 \int_{\mathbb{T}} \frac{1 + \overline{L_1}t}{t - \alpha_1} d\mu(t).$$

This gives

$$\int_{\mathbb{T}} \frac{d\mu(t)}{t - \alpha_1} = -\overline{L_1} \int_{\mathbb{T}} \frac{t}{t - \alpha_1} d\mu(t).$$

Taking this into account, we find

$$\Psi_1(z) = e_1 \frac{1 + \overline{L_1}\alpha_1}{z - \alpha_1} (1 - \overline{L_1}z) \int_{\mathbb{T}} \frac{t}{t - \alpha_1} d\mu(t).$$

The latter integral can be calculated as follows

$$\begin{aligned} \int_{\mathbb{T}} \frac{t}{t - \alpha_1} d\mu(t) &= 1 + \alpha_1 \int_{\mathbb{T}} \frac{d\mu(t)}{t - \alpha_1} \\ &= 1 - \alpha_1 \overline{L_1} \int_{\mathbb{T}} \frac{t}{t - \alpha_1} d\mu(t) \\ &= 1/(1 + \alpha_1 \overline{L_1}). \end{aligned}$$

Incorporating this in the expression for Ψ_1 , gives

$$\Psi_1(z) = e_1 \frac{1 - \overline{L_1}z}{z - \alpha_1},$$

and this is what we needed to prove. ■

Thus we proved the following recurrence relation for the orthogonal rational functions and the functions of the second kind.

Corollary 1 *The following coupled recurrence relation is valid for the orthogonal rational functions Φ_n and the functions of the second kind Ψ_n*

$$\begin{bmatrix} \Phi_n(z) & \Psi_n(z) \\ \Phi_n^*(z) & -\Psi_n^*(z) \end{bmatrix} = e_n \frac{z - \alpha_{n-1}}{z - \alpha_n} \begin{bmatrix} 1 & \overline{L_n} \\ L_n & 1 \end{bmatrix} \begin{bmatrix} \zeta_{n-1}(z) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Phi_{n-1}(z) & \Psi_{n-1}(z) \\ \Phi_{n-1}^*(z) & -\Psi_{n-1}^*(z) \end{bmatrix},$$

where e_n is defined in (18) and $L_n = \overline{\varepsilon_{n-1}^2} \lambda_n$, where ε_{n-1} is defined in (21) and λ_n is defined as in Theorem 1.

5 Interpolation

Recall the definition (5) of the Blaschke products \tilde{B}_k . The following interpolation properties are of great importance.

Theorem 4 *For the orthogonal rational functions Φ_n and the functions of the second kind Ψ_n , it holds that*

$$\frac{\Phi_n \Omega_* + \Psi_n}{\tilde{B}_{n-1}} = \begin{cases} \Omega_* + 1 \in \mathcal{H}(\mathbb{E}) & \text{if } n = 0, \\ g \in \mathcal{H}(\mathbb{E}) & \text{if } n > 0. \end{cases}$$

For their superstar conjugate, we find

$$\frac{\Phi_n^* \Omega_* - \Psi_n^*}{\tilde{B}_n} = \begin{cases} (\Omega_* - 1)/\zeta_0 \in \mathcal{H}(\mathbb{E}) & \text{if } n = 0, \\ g \in \mathcal{H}(\mathbb{E}) & \text{if } n > 0. \end{cases}$$

The proof is the same as in the case where the poles lay outside the unit disc (see [3, Theorem 6.1.1]).

We define the remainder functions R_{n1} and R_{n2} by

$$\begin{bmatrix} \tilde{B}_{n-1}(z)R_{n1}(z) \\ \tilde{B}_n(z)R_{n2}(z) \end{bmatrix} = \begin{bmatrix} \Phi_n(z) \\ \Phi_n^*(z) \end{bmatrix} \Omega_*(z) + \begin{bmatrix} \Psi_n(z) \\ -\Psi_n^*(z) \end{bmatrix}.$$

Using the recurrence relation for the orthogonal rational functions and the functions of the second kind, we become a recurrence relation for these remainder functions.

$$\begin{bmatrix} \tilde{B}_{n-1}(z)R_{n1}(z) \\ \tilde{B}_n(z)R_{n2}(z) \end{bmatrix} = e_n \frac{z - \alpha_{n-1}}{z - \alpha_n} \begin{bmatrix} 1 & \overline{L_n} \\ L_n & 1 \end{bmatrix} \begin{bmatrix} \zeta_{n-1}(z) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{B}_{n-2}(z)R_{n-1,1}(z) \\ \tilde{B}_{n-1}(z)R_{n-1,2}(z) \end{bmatrix}. \quad (28)$$

This can be rewritten as given in the next theorem.

Theorem 5 *The remainder functions as defined above satisfy the following recursion*

$$(z - \alpha_n) \begin{bmatrix} R_{n1}(z) \\ R_{n2}(z) \end{bmatrix} = e_n \begin{bmatrix} 1 & 0 \\ 0 & 1/\zeta_n(z) \end{bmatrix} \begin{bmatrix} 1 & \overline{L_n} \\ L_n & 1 \end{bmatrix} (z - \alpha_{n-1}) \begin{bmatrix} R_{n-1,1}(z) \\ R_{n-1,2}(z) \end{bmatrix},$$

where

$$L_n = - \lim_{z \rightarrow 1/\alpha_n} \frac{R_{n-1,2}(z)}{R_{n-1,1}(z)} \text{ and } e_n = \left[\frac{1 - |\alpha_n|^2}{1 - |\alpha_{n-1}|^2} \frac{1}{1 - |L_n|^2} \right]^{1/2}.$$

Proof:

Starting from (28), we find

$$\tilde{B}_{n-1}(z) \begin{bmatrix} R_{n1}(z) \\ \zeta_n(z)R_{n2}(z) \end{bmatrix} = e_n \frac{z - \alpha_{n-1}}{z - \alpha_n} \begin{bmatrix} 1 & \overline{L_n} \\ L_n & 1 \end{bmatrix} \tilde{B}_{n-1}(z) \begin{bmatrix} R_{n-1,1}(z) \\ R_{n-1,2}(z) \end{bmatrix},$$

which easily gives the required recurrence relation. To find the expression for L_n , we use the last line of the formula above for $z = 1/\overline{\alpha_n}$ to find

$$0 = L_n R_{n-1,1}(1/\overline{\alpha_n}) + R_{n-1,2}(1/\overline{\alpha_n}),$$

from which the result easily follows. The formula for e_n is already deduced in Lemma 1. ■

A nice corollary is the following.

Corollary 2 *Define the function $\Gamma_n(z)$ in terms of the remainder functions by*

$$\Gamma_n(z) = \frac{R_{n2}(z)}{R_{n1}(z)}.$$

Then

$$\Gamma_0(z) = \frac{1}{\zeta_0(z)} \frac{\Omega_*(z) - 1}{\Omega_*(z) + 1}$$

and for all $k \geq 0$ we have $\Gamma_{k*}(\mathbb{D}) \subseteq \mathbb{D}$, which implies that Γ_{k*} is a bounded analytic function and they can be recovered recursively by

$$\Gamma_n(z) = \frac{1}{\zeta_n(z)} \frac{L_n + \Gamma_{n-1}(z)}{1 + \overline{L_n} \Gamma_{n-1}(z)},$$

where $L_n = -\Gamma_{n-1}(1/\overline{\alpha_n})$.

This corollary gives an algorithm to compute the reflection coefficients L_n . However there can be some computational problems. Suppose $\alpha_n = \alpha_{n-1}$. Then $L_n = -\lim_{z \rightarrow 1/\overline{\alpha_n}} R_{n-1,2}(z)/R_{n-1,1}(z)$ and this is of the form $0/0$. So we cannot use the algorithm in the strict sense. In the following section an algorithm is developed to deal with this problem.

6 Computing the reflection coefficients

As mentioned at the end of the previous section, the computation of the reflection coefficients can give problems if some α_i are repeated. In this section we will derive an algorithm that deals with this problem.

Suppose that the sequence of moments c_k (1) is finite

$$c_{-N}, c_{-N+1}, \dots, c_0 = 1, \dots, c_{N-1}, c_N.$$

The expansion of the Riesz-Herglotz representation (2) becomes

$$\Omega(z) = c_0 + 2 \sum_{k=1}^N c_k z^k. \tag{29}$$

If we take the substar of the function Γ_n from Corollary 2, we find

$$\begin{aligned}\Gamma_{0*}(z) &= \zeta_0(z) \frac{\Omega(z) - c_0}{\Omega(z) + c_0}; \\ \Gamma_{n*}(z) &= \zeta_n(z) \frac{\overline{L}_n + \Gamma_{n-1*}(z)}{1 + L_n \Gamma_{n-1*}(z)},\end{aligned}$$

and

$$L_n = -\overline{\Gamma_{n-1*}(\alpha_n)}. \quad (30)$$

Note that these relations are also valid if $c_0 \neq 1$.

We claim that the numerator and denominator of Γ_{n*} are polynomials of degree not greater than N and we introduce the notation

$$\Gamma_{n*}(z) = \frac{\sum_{k=0}^N \gamma_{n,k}^{Num} z^k}{\sum_{k=0}^N \gamma_{n,k}^{Den} z^k}. \quad (31)$$

We will recursively calculate the coefficients $\gamma_{n,k}^{Num}$ and $\gamma_{n,k}^{Den}$.

By using (29), we find for $n = 0$ (recall $\alpha_0 = 0$)

$$\Gamma_{0*}(z) = \frac{1}{z} \frac{2 \sum_{k=1}^N c_k z^k}{2 \sum_{k=0}^N c_k z^k} = \frac{\sum_{k=0}^{N-1} c_{k+1} z^k}{\sum_{k=0}^N c_k z^k}.$$

Thus

$$\begin{aligned}\gamma_{0,k}^{Num} &= \begin{cases} c_{k+1}, & 0 \leq k \leq N-1, \\ 0, & k = N, \end{cases} \\ \gamma_{0,k}^{Den} &= c_k, \quad 0 \leq k \leq N.\end{aligned} \quad (32)$$

We can calculate L_1 by evaluating (30).

For $n = 1, 2, 3, \dots$, we have

$$\Gamma_{n*}(z) = \frac{1 - \overline{\alpha}_n z}{z - \alpha_n} \frac{\overline{L}_n + \Gamma_{n-1*}(z)}{1 + L_n \Gamma_{n-1*}(z)}.$$

Now α_n is a common zero of the numerator and denominator. Using (31), we find

$$\begin{aligned}\Gamma_{n*}(z) &= \frac{1 - \overline{\alpha}_n z}{z - \alpha_n} \frac{\sum_{k=0}^N (\overline{L}_n \gamma_{n-1,k}^{Den} + \gamma_{n-1,k}^{Num}) z^k}{\sum_{k=0}^N (\gamma_{n-1,k}^{Den} + L_n \gamma_{n-1,k}^{Num}) z^k} \\ &= \frac{1 - \overline{\alpha}_n z}{\sum_{k=0}^N (\gamma_{n-1,k}^{Den} + L_n \gamma_{n-1,k}^{Num}) z^k} \frac{\sum_{k=0}^N (\overline{L}_n \gamma_{n-1,k}^{Den} + \gamma_{n-1,k}^{Num}) z^k}{z - \alpha_n}.\end{aligned}$$

If we take a look at the last fraction, we have

$$\frac{\sum_{k=0}^N (\overline{L}_n \gamma_{n-1,k}^{Den} + \gamma_{n-1,k}^{Num}) z^k}{z - \alpha_n} = \sum_{k=0}^{N-1} \beta_{n,k} z^k,$$

because α_n is a zero of the numerator. The problem now is how to calculate these $\beta_{n,k}$. Multiply both sides with $z - \alpha_n$ to find

$$\sum_{k=0}^{N-1} \beta_{n,k} z^{k+1} - \alpha_n \sum_{k=0}^{N-1} \beta_{n,k} z^k = \sum_{k=0}^N (\overline{L}_n \gamma_{n-1,k}^{Den} + \gamma_{n-1,k}^{Num}) z^k.$$

Comparing the coefficients of z^k , gives us

$$\begin{aligned}\beta_{n,N-1} &= \overline{L_n} \gamma_{n-1,N}^{Den} + \gamma_{n-1,N}^{Num}, & k = N, \\ \beta_{n,k-1} &= \overline{L_n} \gamma_{n-1,k}^{Den} + \gamma_{n-1,k}^{Num} + \alpha_n \beta_{n,k}, & k = N-1, \dots, 1.\end{aligned}\tag{33}$$

Comparing the coefficients of z^0 gives:

$$\overline{L_n} \gamma_{n-1,0}^{Den} + \gamma_{n-1,0}^{Num} + \alpha_n \beta_{n,0} = 0.$$

Now we can calculate $\gamma_{n,k}^{Num}$ and $\gamma_{n,k}^{Den}$

$$\begin{aligned}\Gamma_{n^*}(z) &= \frac{\sum_{k=0}^N \gamma_{n,k}^{Num} z^k}{\sum_{k=0}^N \gamma_{n,k}^{Den} z^k} \\ &= \frac{\sum_{k=0}^{N-1} \beta_{n,k} z^k - \overline{\alpha_n} \sum_{k=0}^{N-1} \beta_{n,k} z^{k+1}}{\sum_{k=0}^N (\gamma_{n-1,k}^{Den} + L_n \gamma_{n-1,k}^{Num}) z^k}.\end{aligned}$$

Thus

$$\gamma_{n,k}^{Num} = \begin{cases} \beta_{n,0}, & k = 0, \\ \beta_{n,k} - \overline{\alpha_n} \beta_{n,k-1}, & 1 \leq k \leq N-1, \\ -\overline{\alpha_n} \beta_{n,N-1}, & k = N, \end{cases}\tag{34}$$

$$\gamma_{n,k}^{Den} = \gamma_{n-1,k}^{Den} + L_n \gamma_{n-1,k}^{Num}, \quad 0 \leq k \leq N.$$

We can now calculate L_{n+1} by evaluating (30).

If we combine all the previous results, we are able to calculate the orthogonal rational functions if we have some information. First of all we need to know which is the weight. This is given by its Fourier coefficients $C^N = \{c_0, \dots, c_N\}$, like in (1). Also the poles $\alpha^n = \{\alpha_0 = 0, \alpha_1, \dots, \alpha_n\}$ must be given. If we are interested in evaluating the ORF in some points, these are to be given too.

From above we know how to compute the reflection coefficients L_k . If we use Lemma 1 to calculate e_k and the recurrence (22) with the knowledge that $\Phi_0 = 1 = \Phi_0^*$, we have everything to compute the ORF $\{\Phi_k\}_{k=1}^n$ as presented in Algorithm 1. We make use of the following notation

- C^N = $[c_0, c_1, \dots, c_N]$, the vector containing the Fourier coefficients of the weight;
- α^n = $\{\alpha_0, \alpha_1, \dots, \alpha_n\}$, the poles of the ORF;
- \mathbf{z} = the vector containing the points where we want to evaluate the ORF;
- Φ = the matrix whose (i, j) -th element is $\Phi_{i-1}(\mathbf{z}(j))$.

We use MATLAB notation and the MATLAB source of Algorithm 1 can be found in the appendix. With $*$ and $/$, we mean the usual matrix multiplication and division. The notations $.*$ and $./$ denote elementwise multiplication and division. The transpose of a matrix A is denoted as A^T . The left- and right-shifted vector of $\mathbf{x} = [x_1 \dots x_m]$ is denoted as $\mathbf{x}| = [x_2 \dots x_m \ 0]$ and $|\mathbf{x} = [0 \ x_1 \dots x_{m-1}]$, respectively.

Algorithm 1

ORFcalcul $\langle IN: \alpha^n, C^N, \mathbf{z};$
 $OUT: \Phi \rangle$

1. $\Phi(1, :) = 1 = \Phi^*(1, :);$

$$2. \quad \begin{aligned} \gamma^{Num} &= C^N|; \\ \gamma^{Den} &= C^N; \end{aligned}$$

3. for $k = 1 : n$,

$$(a) \quad \mathbf{a} = [1 \ \alpha_k \ \alpha_k^2 \ \dots \ \alpha_k^{N-1}]^T;$$

$$(b) \quad L_k = -\overline{\gamma^{Num} * \mathbf{a} / \gamma^{Den} * \mathbf{a}};$$

$$(c) \quad e_k = \sqrt{(1 - |\alpha_k|^2) / ((1 - |\alpha_{k-1}|^2) * (1 - |L_k|^2))};$$

$$(d) \quad \Phi(k+1, :) = e_k * ((1 - \overline{\alpha_{k-1}} * \mathbf{z}) .* \Phi(k, :) + \overline{L_k} * (\mathbf{z} - \alpha_{k-1}) .* \Phi^*(k, :)) ./ (\mathbf{z} - \alpha_k);$$

$$(e) \quad \Phi^*(k+1, :) = e_k * (L_k * (1 - \overline{\alpha_{k-1}} * \mathbf{z}) .* \Phi(k, :) + (\mathbf{z} - \alpha_{k-1}) .* \Phi^*(k, :)) ./ (\mathbf{z} - \alpha_k);$$

(f) if $k < n$,

$$i. \quad \beta = \overline{L_k} * \gamma^{Den} + \gamma^{Num};$$

ii. for $l = N : -1 : 1$, $\beta(l) = \beta(l) + \alpha_k \beta(l+1)$; end for;

$$iii. \quad \gamma^{Den} = \gamma^{Den} + L_k * \gamma^{Num};$$

$$iv. \quad \gamma^{Num} = \beta - \overline{\alpha_k} * \beta;$$

end if;

end for;

7 Numerical illustration

We will consider three different weights w and two different sets α^n .

In order to illustrate Algorithm 1, we will give two examples.

(1) In Theorem 4 we showed that Ψ_n^* / Φ_n^* is an approximation of Ω_* in \mathbb{E} , since it interpolates in the points $\{1/\overline{\alpha_0}, 1/\overline{\alpha_1}, \dots, 1/\overline{\alpha_n}\}$. The second example is about orthogonality.

(2) If we compute the ORF $\{\Phi_k\}_{k=0}^n$ in N equidistant points \mathbf{z} on the unit circle \mathbb{T} , then we have with the notation introduced above

$$\lim_{N \rightarrow \infty} \Phi \text{diag}(w(\mathbf{z})) \Phi^H / N = I_{n+1},$$

where I_n denotes the unit matrix of dimension n and $\text{diag}(w(\mathbf{z}))$ denotes the diagonal matrix whose diagonal is $w(\mathbf{z})$, the weight evaluated in the points \mathbf{z} .

The moments and the poles we use are

$$C_1^N = [1 \ 0], \text{ the Lebesgue measure, corresponding to weight } w_1(\theta) = 1;$$

$$C_2^N = [1 \ (1-i)/(2\sqrt{2})], \text{ corresponding to weight } w_2(\theta) = 1 + (\cos(\theta) + \sin(\theta))/\sqrt{2};$$

$$C_3^N = [1 \ -.2 \ .4], \text{ corresponding to weight } w_3(\theta) = .2 - .4 * \cos(\theta) + 1.6 \cos^2(\theta);$$

$$\alpha_1^n = [.2 + .3i \ .2 - .3i \ -.3 + .4i \ -.3 - .4i \ .7 + .1i \ -.7 + .3i \ -.8 \ 0 \ .9 \ -.6],$$

repeated 20 times;

$$\alpha_2^n = \text{zeros}(1, 200) \text{ (the polynomial case).}$$

In figure 1-3, the weights $w_i(\theta)$, $0 \leq \theta \leq 2\pi$, $i = 1, 2, 3$ are given.

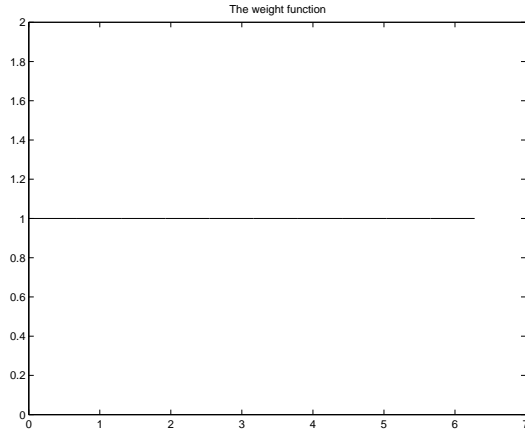


Figure 1: Weight w_1 : The Lebesgue weight

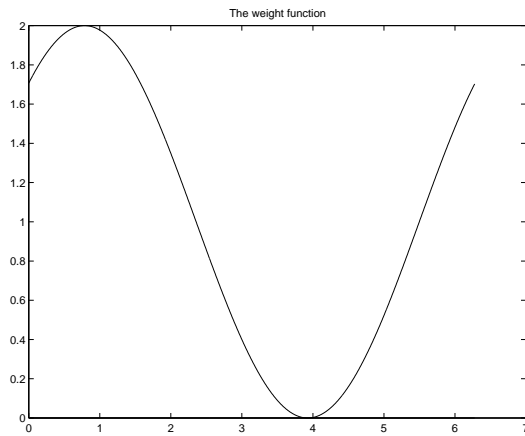


Figure 2: Weight w_2 with moments $[1(1 - i)/2\sqrt{2}]$: This weight is zero for $\theta = 5\pi/4$.

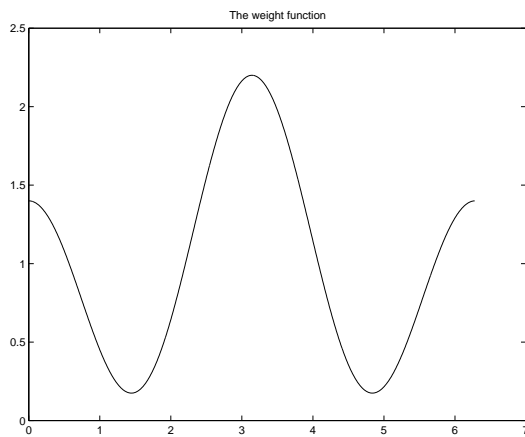


Figure 3: Weight w_3 with moments $[1 - .2 .4]$

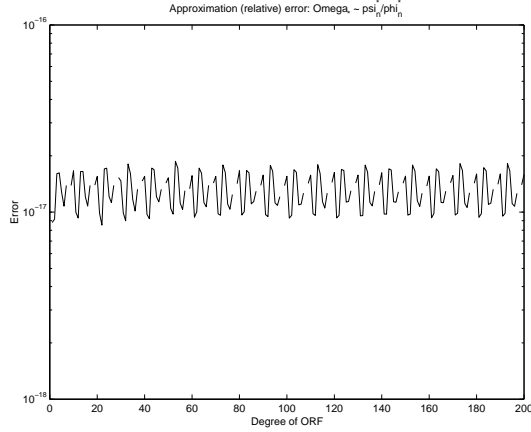


Figure 4: The weight: w_1 , with poles α_1^n and evaluated at the points on radius $r = 1.01$.

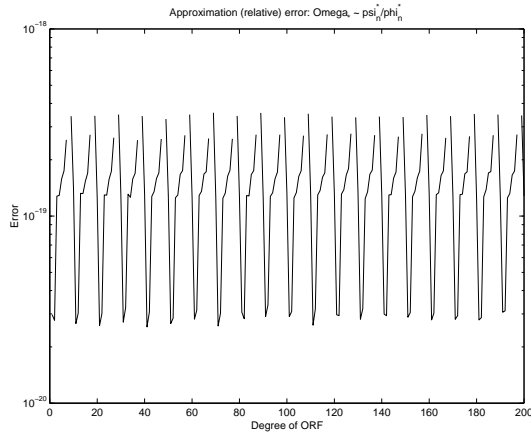


Figure 5: The weight: w_1 , with poles α_1^n and evaluated at the points on radius $r = 100$.

7.1 Example 1

We used $N = 1000$ equidistant points on the circle with radius $R = 1.1$ and $R = 100$. The figures 4-13 show the relative approximation error for every degree n for the different weights, poles and radii

$$error1(n) = \|\Omega_*(\mathbf{z}) - \Psi_n(\mathbf{z}) / \Phi_n(\mathbf{z})\| / \|\Omega_*(\mathbf{z})\|.$$

If we take the Lebesgue measure and all $\alpha_k = 0$ we find the monomial basis $\{1, z^{-1}, z^{-2}, \dots\}$. This implies $\Phi_n^* \equiv 1, \forall n \in \mathbb{N}$. It should be obvious that the function $\Omega_* \equiv 1$ and that the approximation is correct. This is the reason why we did not have figures with this combination.

The explanation for the *slow* convergence in figures 6 and 8 is the fact that $\log w \notin L^1(\mathbb{T})$ here. The known convergence results on the unit circle [4] are not valid anymore and the points \mathbf{z} are close to \mathbb{T} in those figures.

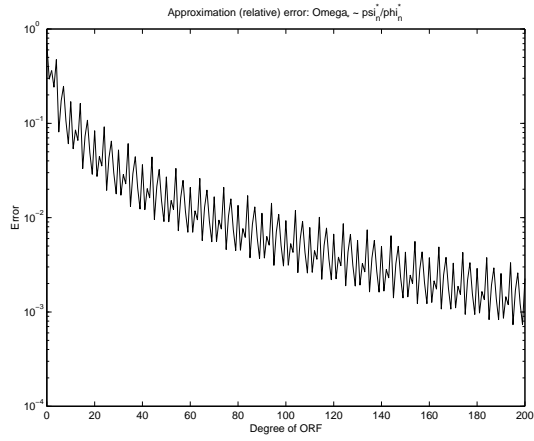


Figure 6: The weight: w_2 , with poles α_1^n and evaluated at the points on radius $r = 1.01$.

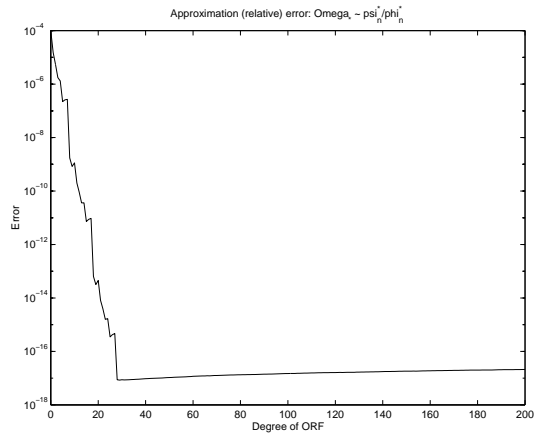


Figure 7: The weight: w_2 , with poles α_1^n and evaluated at the points on radius $r = 100$.

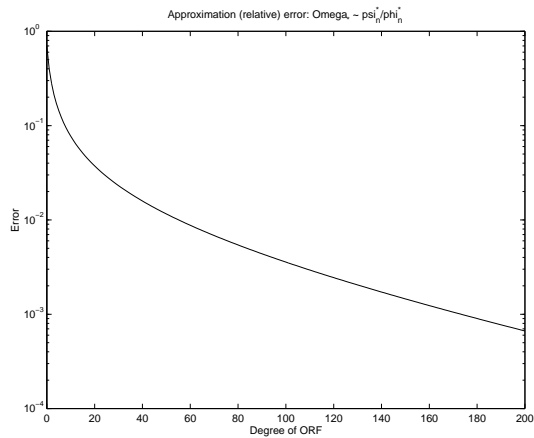


Figure 8: The weight: w_2 , with poles α_2^n and evaluated at the points on radius $r = 1.01$.

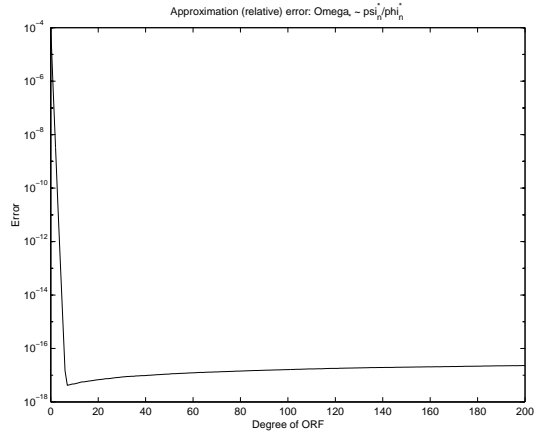


Figure 9: The weight: w_2 , with poles α_2^n and evaluated at the points on radius $r = 100$.

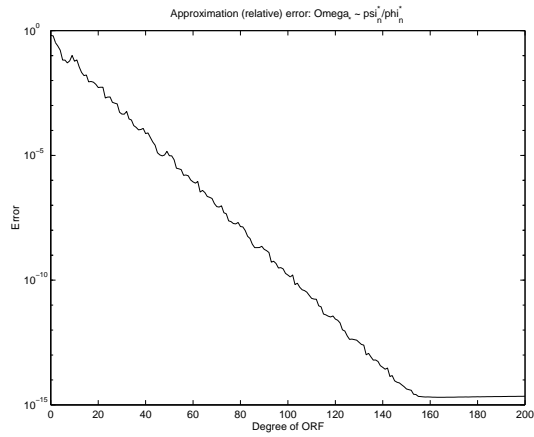


Figure 10: The weight: w_3 , with poles α_1^n and evaluated at the points on radius $r = 1.01$.

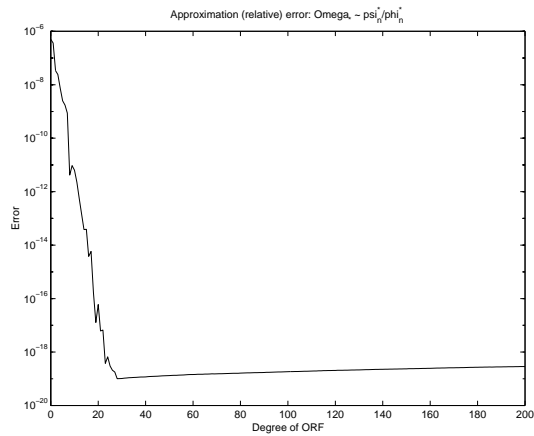


Figure 11: The weight: w_3 , with poles α_1^n and evaluated at the points on radius $r = 100$.

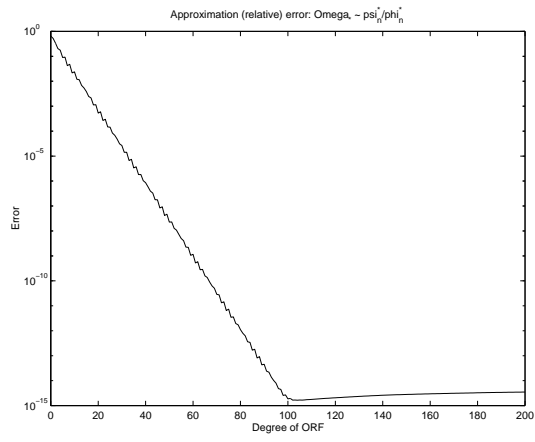


Figure 12: The weight: w_3 , with poles α_2^n and evaluated at the points on radius $r = 1.01$.

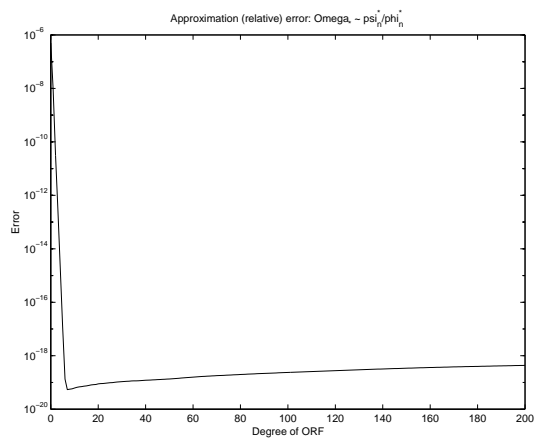


Figure 13: The weight: w_3 , with poles α_2^n and evaluated at the points on radius $r = 100$.

Table 1: The measure of orthogonality

error2	α_1^n	α_2^n
C_1^N	3.239177506284802e-13	4.314966222807296e-14
C_2^N	3.495867132064386e-13	5.764743833825424e-13
C_3^N	3.572096362350430e-13	3.992485708745933e-14

7.2 Example 2

Here we used $N = 1000$ equidistant points on the unit circle. Table 1 contains the measure of orthogonality

$$error2 = \|\Phi \text{diag}(w(\mathbf{z})) \Phi^H / N - I_{n+1}\|,$$

for the different weights and poles. We see that the orthogonality is very good. The computation is also very fast ($O(nN)$, can be calculated from Algorithm 1).

8 Conclusions

What we have written down in section 3, 4 and 5 is not a big surprise. All these results are known for orthogonal rational functions with poles outside the unit disc. However, the results stated here, are of interest e.g. for the use of these functions in the area of system identification, where the poles of a stable system lay inside the unit disc.

Computational problems that occur when not all the α_i are distinct, are solved for the case where the number of moments is finite and known. Another solution would imply derivatives, but there is (yet) not much known about derivatives of orthogonal rational functions.

As we have illustrated in section 7, our algorithm is accurate and fast.

References

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- [2] A. Bultheel and B. De Moor. Rational approximation in linear systems and control. *J. Comput. Appl. Math.*, 121:355–378, 2000.
- [3] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad. *Orthogonal rational functions*, volume 5 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, 1999.
- [4] A. Bultheel and P. Van gucht. Boundary asymptotics for orthogonal rational functions on the unit circle. *Acta Applicandae Mathematicae*, 61:333–349, 2000.
- [5] P.S.C. Heuberger, P.M.J. Van den Hof, and O. Bosgra. A generalized orthogonal basis for linear dynamical systems. *IEEE Trans. Automat. Control*, 40:451–465, 1995.
- [6] B. Ninness and F. Gustafsson. A unified construction of orthogonal bases for system identification. *IEEE Trans. Automat. Control*, 42:515–522, 1997.


```

%           if 2, compute L, phi(star) and psi(star)
%
% See also: comPhi, comPsi

% Patrick Van gucht and Adhemar Bultheel, august, 29, 2000.

%initializations
n = size(alpha,2); % the highest degree
N = size(points,2); % the number of points
m = size(coef,2); % the number of Fouriercoefficients

if (option~=0),
    phi = [ones(1,N)/sqrt(coef(1));zeros(n,N)];
    phistar = phi; % superstar
    if (option==2),
        psi = [ones(1,N)*sqrt(coef(1));zeros(n,N)]; % second kind
        psistar = psi; % superstar
    end
end

gamma_num = [coef(2:m) 0];
gamma_den = coef;
beta = zeros(1,m-1);
L = zeros(n,1);
alpha = [0 alpha]; % alpha_0=0

% Recurrence
for k=1:n,
    z = alpha(k+1).^transpose(0:m-1);
    L(k) = -conj((gamma_num*z)/(gamma_den*z)); % reflectioncoefficient
    if (option~=0),
        e_k = sqrt(((1-abs(alpha(k+1))^2)/((1-abs(alpha(k))^2)*(1-abs(L(k))^2)))
            ./ (points-alpha(k+1)));
        help = (1-alpha(k)'.*points).*phi(k,:);
        helpstar = (points-alpha(k)).*phistar(k,:);
        phi(k+1,:) = e_k.*(help + L(k)'.*helpstar);
        phistar(k+1,:) = e_k.*(L(k)*help + helpstar);
        if (option==2),
            help = (1-alpha(k)'.*points).*psi(k,:);
            helpstar = (points-alpha(k)).*psistar(k,:);
            psi(k+1,:) = e_k.*(help - L(k)'.*helpstar);
            psistar(k+1,:) = e_k.*(-L(k)*help + helpstar);
        end
    end
end
if k<n,
    beta(m-1) = L(k)'.*gamma_den(m) + gamma_num(m);
    for l=m-2:-1:1,

```

```

        beta(1) = L(k)'*gamma_den(l+1) + gamma_num(l+1) + alpha(k+1)*beta(l+1);
    end
    gamma_den = gamma_den + L(k)*gamma_num;
    gamma_num(1) = beta(1);
    gamma_num(2:m-1) = beta(2:m-1)-alpha(k+1)'*beta(1:m-2);
    gamma_num(m) = -alpha(k+1)'*beta(m-1);
end
end

```

Two other m-files `comPhi.m` and `comPsi.m` compute the Φ_n and the Ψ_n , respectively, given the reflection coefficients and the poles, in given points.

```

function [phi,phistar] = comPhi(L,alpha,points),

% COMPHI = Compute the orthogonal rational functions phi and their
%          superstar conjugate in given points if the reflection
%          coefficients L and the poles alpha are given.
%
% USE: [phi,phistar] = comPhi(L,alpha,points)
%
% output:
%   phi(star): matrix whose i-th row equals phi(star)_{i-1}(points)
% input:
%   L          : the reflection coefficients
%   alpha      : the poles of the rational functions
%   points     : rowvector of points wherein the phi are computed
%
% See also: ORFcalcul, comPsi

% Patrick Van gucht and Adhemar Bultheel, august, 29, 2000

%initialization
n = min(max(size(L)),max(size(alpha))); % the highest degree
N = size(points,2);                    % the number of points

phi = [ones(1,N);zeros(n,N)];
phistar = phi; % superstar

alpha = [0 alpha]; % alpha_0=0

% Recurrence
for k=1:n,
    e_k = sqrt(((1-abs(alpha(k+1)))^2)/((1-abs(alpha(k)))^2)*(1-abs(L(k))^2))
              ./ (points-alpha(k+1)));
    help = (1-alpha(k)')*points).*phi(k,:);
    helpstar = (points-alpha(k)).*phistar(k,:);
    phi(k+1,:) = e_k.*(help + L(k)')*helpstar);
end

```

```

    phistar(k+1,:) = e_k.*(L(k)*help + helpstar);
end

function [psi,psistar] = comPsi(L,alpha,points),

% COMPSI = Compute the functions of the second kind psi and their
%          superstar conjugate in given points if the reflection
%          coefficients L and the poles alpha are given.
%
% USE: [psi,psistar] = comPsi(L,alpha,points)
%
% output:
%   psi(star): matrix whose i-th row equals psi(star)_{i-1}(points)
% input:
%   L          : the reflection coefficients
%   alpha      : the poles of the rational functions
%   points     : rowvector of points wherein the psi are computed
%
% See also: ORFcalcul, comPhi

% Patrick Van gucht and Adhemar Bultheel, august, 29, 2000

%initialization
n = min(max(size(L)),max(size(alpha))); % the highest degree
N = size(points,2);                    % the number of points

psi = [ones(1,N);zeros(n,N)];
psistar = psi; % superstar

alpha = [0 alpha]; % alpha_0=0

% Recurrence
for k=1:n,
    e_k = sqrt((1-abs(alpha(k+1)))^2)/((1-abs(alpha(k))^2)*(1-abs(L(k))^2))
          ./(points-alpha(k+1)));
    help = (1-alpha(k)'.*points).*psi(k,:);
    helpstar = (points-alpha(k)).*psistar(k,:);
    psi(k+1,:) = e_k.*(help - L(k)'.*helpstar);
    psistar(k+1,:) = e_k.*(-L(k)*help + helpstar);
end

```