

On the Stability of Wavelet Bases in the Lifting Scheme

Jo Simoens
Stefan Vandewalle

Report TW 306, April 2000



Katholieke Universiteit Leuven
Department of Computer Science
Celestijnenlaan 200A – B-3001 Heverlee (Belgium)

On the Stability of Wavelet Bases in the Lifting Scheme

Jo Simoens
Stefan Vandewalle

Report TW 306, April 2000

Department of Computer Science, K.U.Leuven

Abstract

The lifting scheme provides an easy way to construct wavelet bases on irregular meshes in one or more dimensions. However, analysis shows that the standard implementation where a primal lifting step is used to increase the number of vanishing moments of the wavelets can compromise the stability of the resulting wavelet basis.

We propose a modification which solves this issue. With this modification, uniform stability can be guaranteed for a standard two-step lifted wavelet transform comprising one dual (prediction) and one primal (update) lifting step.

Keywords : multiscale decompositions, irregular meshes, nonstationary or second-generation wavelet bases, lifting, stability, conditioning.

AMS(MOS) Classification : 42C40, 65T60, 65F35.

On the Stability of Wavelet Bases in the Lifting Scheme

Jo Simoens ^{*†} Stefan Vandewalle [†]

April 2000

Abstract

The lifting scheme provides an easy way to construct wavelet bases on irregular meshes in one or more dimensions. However, analysis shows that the standard implementation where a primal lifting step is used to increase the number of vanishing moments of the wavelets can compromise the stability of the resulting wavelet basis.

We propose a modification which solves this issue. With this modification, uniform stability can be guaranteed for a standard two-step lifted wavelet transform comprising one dual (prediction) and one primal (update) lifting step.

1 Introduction

We are concerned with nonstationary biorthogonal wavelet bases. The basic structure is a strictly increasing sequence $\mathcal{V} := \{V_j\}_{j \geq 0}$ of closed subspaces of a Hilbert space \mathcal{H} whose union is dense in \mathcal{H} ,

$$V_j \subset V_{j+1}, \quad j \geq 0, \quad \text{and} \quad \text{clos} \bigcup_{j=0}^{\infty} V_j = \mathcal{H} .$$

This is called a *multiresolution analysis (MRA)*. Between every two successive spaces in \mathcal{V} , construct algebraic complements W_j so that

$$V_{j+1} = V_j \oplus W_j .$$

The symbol ‘ \oplus ’ denotes the inner sum of disjoint linear spaces. The complement space W_j is not necessarily orthogonal to V_j . With the notational convention that $W_{-1} := V_0$, we call the sequence $\mathcal{W} := \{W_j\}_{j \geq -1}$ a *multiscale decomposition (MSD)* of \mathcal{H} .

Our terminology is liberal, in that any basis $\Phi_j := \{\varphi_{jk} \mid k \in \mathcal{K}_j\}$ for the space V_j is referred to as a set of *scaling functions*, and any basis $\Psi_j := \{\psi_{jm} \mid m \in \mathcal{M}_j\}$ for any type of complement space W_j , $j \geq 0$, is called a set of *wavelets* at level j . This should be contrasted to the stationary setting, where all scaling functions and all wavelets are translates and dilates of a single father c.q. mother function. We consider the general, *nonstationary* case. Also, more restrictive definitions of the term ‘wavelet’ are found in the literature. In which sense the basis property has to be understood is made clear below.

^{*}Research Assistant of the Fund for Scientific Research – Flanders, Belgium (F.W.O.)

[†]Department of Computer Science, Katholieke Universiteit Leuven, Leuven, Belgium

A MSD \mathcal{W} with bases Φ_j and Ψ_j and a second MSD $\tilde{\mathcal{W}}$ with bases $\tilde{\Phi}_j$ and $\tilde{\Psi}_j$ are *biorthogonal* if

$$\begin{aligned} \langle \varphi_{jk}, \tilde{\varphi}_{jk'} \rangle &= \delta_{kk'} , & \langle \psi_{jm}, \tilde{\varphi}_{jk'} \rangle &= 0 , \\ \langle \varphi_{jk}, \tilde{\psi}_{jm'} \rangle &= 0 , & \langle \psi_{jm}, \tilde{\psi}_{jm'} \rangle &= \delta_{mm'} , \end{aligned}$$

where $\delta_{kk'}$ is the Kronecker symbol. By convention, \mathcal{W} is called the *primal* MSD and $\tilde{\mathcal{W}}$ the *dual* MSD, but their roles can obviously be interchanged.

The *lifting scheme*, proposed by Sweldens [13], starts from an initial pair of biorthogonal MSDs and builds a new pair, leaving the MRA on one side unchanged. In *primal lifting* the primal scaling functions are kept and the primal wavelets are modified to meet some chosen requirements by adding a linear combination of scaling functions:

$$\hat{\varphi}_{jk} := \varphi_{jk} , \quad \hat{\psi}_{jm} := \psi_{jm} - \sum_k u_{jkm} \varphi_{jk} . \quad (1.1)$$

This operation changes the complement spaces W_j . To maintain biorthogonality, the dual scaling functions and wavelets are modified accordingly. In *dual lifting* the analogous operation is performed on the dual side. The most common realization of the lifting scheme comprises two consecutive steps: a dual lifting steps from a trivial initial pair of MSDs, followed by a primal lifting step.

The lifting scheme can be a tool in the construction of wavelet bases: sometimes it is more convenient to define the lifting steps rather than the resulting basis functions. This is the point of view which is adopted in this paper. It has been shown [6] that lifting is rather powerful, in that any stationary biorthogonal wavelet transform with finite filters can be factored into lifting steps. The lifting scheme is easy to implement, gives efficient algorithms, but its main attraction is perhaps its flexibility, owing to the explicit character of (1.1). This is an important advantage when dealing with meshes of irregular geometry and connectivity.

Signal processing applications such as compression and denoising in a wavelet basis assume equivalence of the \mathbf{L}_2 -norm of a signal f and the ℓ_2 -norm of the sequence of its wavelet coefficients c ,

$$m \|c\|_{\ell_2} \leq \|f\|_{\mathbf{L}_2} \leq M \|c\|_{\ell_2} , \quad (1.2)$$

where m and M depend only on the wavelet basis and should be close to each other. Thus the error made in the signal by discarding one coefficient is bounded from above and from below by the magnitude of the coefficient up to the constants M and m . The property (1.2) will define \mathbf{L}_2 -*stability* of the wavelet basis.

Another class of applications where stability is essential are variational problems

$$\text{find } u \in \mathbf{H}^r \text{ such that } a(u, v) = f(v) \quad \forall v \in \mathbf{H}^r \quad (1.3)$$

in a Sobolev space \mathbf{H}^r , with $f \in \mathbf{H}^{-r}$ and $a(u, v)$ symmetric and \mathbf{H}^r -elliptic, i.e.

$$m_a^2 \|v\|_{\mathbf{H}^r}^2 \leq a(v, v) \leq M_a^2 \|v\|_{\mathbf{H}^r}^2 , \quad 0 < m_a, M_a < \infty .$$

The variational formulation of a second-order elliptic partial differential equation (where $r = 1$) is an example. If the MRA defined above is in \mathbf{H}^r , Galerkin discretization of (1.3) in any of the spaces V_j gives the discrete problem

$$\text{find } u_j \in V_j \text{ such that } a(u_j, v_j) = f(v_j) \quad \forall v_j \in V_j . \quad (1.4)$$

When expressed in a basis $\Xi_j = \{\xi_{jk}\}_k$ for V_j , (1.4) becomes the linear system

$$A_j c_j = f_j \quad (1.5)$$

with $A_j = [a(\xi_{jk}, \xi_{jl})]_{kl}$, $\sum_k c_{jk} \xi_{jk} = u_j$ and $f_{jk} = f(\xi_{jk})$. Here and in the sequel, we denote by c_{jk} the elements of a vector c_j . The number of unknowns grows exponentially as j is increased. In view of the numerical solution of (1.5), we would like the spectral condition number of the matrix A_j to remain bounded for increasing j . Using symmetry and \mathbf{H}^r -ellipticity of $a(\cdot, \cdot)$, one easily checks that this is equivalent to the basis Ξ_j being \mathbf{H}^r -stable uniformly in j , i.e.

$$m \|c_j\|_{\ell_2} \leq \left\| \sum_k c_{jk} \xi_{jk} \right\|_{\mathbf{H}^r} \leq M \|c_j\|_{\ell_2} ,$$

with m and M independent of j .

The standard implementations in [13] and subsequent papers do not consider stability issues. Although developed independently, lifting fits in the stability theory of [1], where a characterization is given of all stable wavelet bases starting from a given MRA. The objectives in the present paper are double. Firstly, we aim to explain some observed stability problems in the classical use of the two-step lifting scheme. In a recent report, Jansen [8] describes stability problems with lifted wavelet bases in the context of signal denoising. Our analysis locates the origin of the problem in the primal lifting step. Secondly, we want to propose a modification which is simple and yields wavelet bases and complement spaces that are uniformly stable. It should be noted that we do not show multiscale stability.

The remainder of this paper is structured as follows. Section 2 recalls some of the existing theory on the stability of multiscale bases and multiscale transforms. These concepts are next applied to the lifting scheme. Section 3 places the principle of lifting and the two-step construction in the context of the preceding stability theory, and in Section 4, we investigate stability issues in the so-called update step and develop a modified update method.

2 Multiscale Bases and Stability

2.1 Multiscale Decompositions

In a MSD \mathcal{W} as defined in Section 1, define the *two-scale transform* \mathcal{A}_j as

$$\mathcal{A}_j : V_j \boxplus W_j \rightarrow V_{j+1} : (v, w) \mapsto v + w . \quad (2.1)$$

To avoid notational confusion, we use the symbol ‘ \boxplus ’ instead of ‘ \oplus ’ to denote the direct sum of inner product spaces. The direct sum of V and W is the product space $V \times W$ endowed with the inner product $\langle (v, w), (v', w') \rangle_{V \boxplus W} := \langle v, v' \rangle_V + \langle w, w' \rangle_W$. Note that \mathcal{A}_j is invertible because of the assumption that W_j is an algebraic complement.

Definition 2.1. *The subspace W_j is a stable complement of V_j in V_{j+1} if \mathcal{A}_j in (2.1) is bounded and boundedly invertible. The complements $\{W_j\}$ are said to be uniformly stable if the condition number of \mathcal{A}_j is bounded uniformly in j .*

It is easy to find a more explicit condition. We introduce a number which characterizes the *degree of complement stability* in an intuitive way. It can be thought of intuitively as the cosine of the angle between two spaces.

Proposition 2.2. Given $V_{j+1} = V_j \oplus W_j$, define

$$\alpha_j := \sup_{\substack{v \in V_j, w \in W_j \\ v, w \neq 0}} \left| \left\langle \frac{v}{\|v\|}, \frac{w}{\|w\|} \right\rangle_{\mathcal{H}} \right|. \quad (2.2)$$

Then W_j is a stable complement of V_j if and only if $\alpha_j < 1$. The condition number $\kappa(\mathcal{A}_j)$ equals $(1 + \alpha_j)^{1/2}(1 - \alpha_j)^{-1/2}$.

Proof. The two-scale transform \mathcal{A}_j itself is always bounded, since

$$\begin{aligned} \|\mathcal{A}_j\|^2 &:= \sup_{v, w \neq 0} \frac{\|v + w\|_{\mathcal{H}}^2}{\|(v, w)\|_{V_j \boxplus W_j}^2} = 1 + \sup_{v, w \neq 0} \frac{2|\langle v, w \rangle|}{\|v\|_{\mathcal{H}}^2 + \|w\|_{\mathcal{H}}^2} \\ &= 1 + \sup_{\|v\|=\|w\|=1} |\langle v, w \rangle| = 1 + \alpha_j. \end{aligned}$$

The operator norm of \mathcal{A}_j^{-1} is given by

$$\|\mathcal{A}_j^{-1}\|^2 := \sup_{v, w \neq 0} \left(1 + \frac{2 \operatorname{Re} \langle v, w \rangle}{\|v\|_{\mathcal{H}}^2 + \|w\|_{\mathcal{H}}^2} \right)^{-1} = \frac{1}{1 - \alpha_j} \quad (2.3)$$

if $\alpha_j < 1$, and otherwise \mathcal{A}_j^{-1} is unbounded. \square

A fine-resolution space V_J can be written as a telescopic decomposition into a coarser resolution space and complement spaces,

$$V_J = V_j \oplus \bigoplus_{i=j}^{J-1} W_i. \quad (2.4)$$

Thus any element $v_J = v_j + \sum_{i=j}^{J-1} w_i$ of V_J can be written in *multiscale form* as a vector $(v_j, w_j, \dots, w_{J-1})$ in the direct sum space $V_j \boxplus W_j \boxplus \dots \boxplus W_{J-1}$.

The conversion from multiscale form to single-scale form is termed a *multiscale transform*

$$\mathcal{T}_J^j : V_j \boxplus \bigoplus_{i=j}^{J-1} W_i \rightarrow V_J : (v_j, w_j, \dots, w_{J-1}) \mapsto v_J = v_j + \sum_{i=j}^{J-1} w_i,$$

or, written in terms of the two-scale transform,

$$\begin{aligned} \mathcal{T}_J^j(v_j, w_j, \dots, w_{J-1}) &= \mathcal{A}_{J-1}(\mathcal{T}_{J-1}^j(v_j, w_j, \dots, w_{J-2}), w_{J-1}), \\ \mathcal{T}_{j+1}^j(v_j, w_j) &= \mathcal{A}_j(v_j, w_j). \end{aligned}$$

In a similar way as complement stability was based upon two-scale transforms, multiscale transforms define a notion of stability for MSDs.

Definition 2.3. A MSD of \mathcal{H} is stable if $\mathcal{T}_{\infty}^0 : \bigoplus_{i=-1}^{\infty} W_i \rightarrow \mathcal{H} : (w_{-1}, w_0, \dots) \mapsto \sum_{i=-1}^{\infty} w_i$ is bounded and boundedly invertible.

A stable MSD in the sense of the above definition is an instance of what is called a *stable subspace splitting* [16, 12] in the theory of Schwarz methods. Stability of a MSD is not equivalent to uniform complement stability; in $\mathbf{L}_2(\Omega)$ with $\Omega \subseteq \mathbb{R}^2$, the standard hierarchical basis [17] on two-dimensional triangulations is a known counterexample. However, we will see that the former implies the latter. First we show that the multiscale transform over all levels is connected to intermediate multiscale transforms over a finite number of levels in a way one would expect.

Proposition 2.4. *Let \mathcal{W} be a MSD. Then the following are equivalent:*

(i) \mathcal{T}_J^0 are uniformly well-conditioned, i.e. $\exists 0 < m, M < \infty$ independent of J such that

$$\|\mathcal{T}_J^0\| \leq M, \quad \|(\mathcal{T}_J^0)^{-1}\| \leq m^{-1}$$

(ii) \mathcal{T}_∞^0 is bounded and boundedly invertible.

Furthermore, the sequences $\{\|\mathcal{T}_J^0\|\}$ and $\{\|(\mathcal{T}_J^0)^{-1}\|\}$ (and hence the sequence of condition numbers $\{\kappa(\mathcal{T}_J^0)\}$) are monotone nondecreasing.

Proof. Monotonicity follows immediately from the embedding $V_J \subset V_{J+1}$. We show that if the \mathcal{T}_J^0 are uniformly well-conditioned, then \mathcal{T}_∞^0 and its inverse are both bounded; the converse is clear. Pick $v \in \mathcal{H}$, $v = \mathcal{T}_\infty^0(w_{-1}, w_0, \dots)$, and let $\mathcal{T}_J^0(w_{-1}, w_0, \dots, w_{J-1}) =: v_J \in V_J$. Then

$$\|v_J\| \leq M \|(w_{-1}, w_0, \dots, w_{J-1})\| \leq M \|(w_{-1}, w_0, \dots)\| ,$$

and $\lim_{J \rightarrow \infty} \|v_J\| = \|v\|$, so that $\|\mathcal{T}_\infty^0\| \leq M$. As to the norm of the inverse, for any J it holds that

$$\|(w_{-1}, w_0, \dots, w_{J-1})\| \leq m^{-1} \|v_J\| .$$

Since the right-hand side converges to $\|v\|$ and the left-hand side is strictly increasing for $J \rightarrow \infty$, it follows that $\|(\mathcal{T}_\infty^0)^{-1}\| \leq m^{-1}$. \square

Proposition 2.5. *If the multiscale transforms \mathcal{T}_J^0 are uniformly well-conditioned, then so are the two-scale transforms \mathcal{A}_j . More specifically, one has*

$$\kappa(\mathcal{T}_J^0) \geq \max_{0 \leq j < J} (1 - \alpha_j)^{-1/2}$$

with α_j from (2.2).

Proof. Let $\mathcal{T}_J^0(w_{-1}, w_0, \dots, w_{J-1}) = v_J \in V_J$ and $w_J \in W_J$. Then

$$\begin{aligned} \|v_J\|^2 + \|w_J\|^2 &\leq \|\mathcal{T}_J^0\|^2 \left(\sum_{i=-1}^{J-1} \|w_i\|^2 \right) + \|w_J\|^2 \\ &\leq \|\mathcal{T}_J^0\|^2 \|(w_{-1}, w_0, \dots, w_{J-1}, w_J)\|^2 \\ &\leq \|\mathcal{T}_J^0\|^2 \|(\mathcal{T}_{J+1}^0)^{-1}\|^2 \|v_J + w_J\|^2 . \end{aligned}$$

Since $\|\mathcal{T}_J^0\| \leq \|\mathcal{T}_{J+1}^0\|$, the result follows from (2.3) and the monotonicity of $\{\kappa(\mathcal{T}_J^0)\}$. \square

Hence a necessary condition for stability of the MSD is that the complement spaces be uniformly stable.

Sufficient conditions for stability of a MSD have been established by Dahmen [3, 4]. The main result is included here for completeness. Let \mathcal{W} be a MSD in \mathcal{H} and define the projectors

$$P_j : \mathcal{H} \rightarrow V_j : v = \mathcal{T}_\infty^0(w_{-1}, w_0, \dots) \mapsto v_j = \sum_{i=-1}^{j-1} w_i . \quad (2.5)$$

Then the spaces defined as the range of the adjoint projectors $\tilde{V}_j := \text{range } P_j^*$ are nested and dense in \mathcal{H} [3], and thus form a dual MRA $\tilde{\mathcal{V}}$. The following is shown.

Theorem 2.6 ([4, Theorem 5.6]). *Let \mathcal{V} be a MRA in \mathcal{H} , and let P_j be projectors with ranges V_j such that the spaces $\tilde{V}_j := \text{range } P_j^*$ are also nested. If the projectors P_j are uniformly bounded in \mathcal{H} , and if there is a family of uniformly bounded subadditive functionals*

$$\omega(\cdot, t) : \mathcal{H} \rightarrow \mathbb{R}^+, \quad t > 0 \quad \text{with} \quad \lim_{t \rightarrow 0^+} \omega(v, t) = 0 \quad \forall v \in \mathcal{H}$$

such that the pair of direct and inverse estimates

$$\begin{aligned} \inf_{v_j \in V_j} \|v - v_j\|_{\mathcal{H}} &\lesssim \omega(v, 2^{-j}) \quad \forall v \in \mathcal{H}, \\ \omega(v_j, t) &\lesssim (\min\{1, 2^j t\})^\gamma \|v_j\|_{\mathcal{H}} \quad \forall v_j \in V_j \end{aligned} \quad (2.6)$$

holds for \mathcal{V} and similarly for $\tilde{\mathcal{V}}$ with some $\gamma, \tilde{\gamma} > 0$, respectively, then

$$\|v\|_{\mathcal{H}}^2 \sim \sum_{j=0}^{\infty} \|(P_j - P_{j-1})v\|_{\mathcal{H}}^2,$$

with the convention that $P_{-1} := 0$.

Under the conditions of this theorem, the MSD corresponding to the projectors P_j through (2.5) is stable. The results in [3, 4] are formulated in terms of the primal and dual MRAs without the complement spaces intervening directly, which differs from our approach in this section. The rationale behind it is that it is often relatively easy to find simple uniformly stable bases that span a pair of biorthogonal MRAs.

2.2 Bases in Multiscale Decompositions

In the above, we have been dealing only with linear spaces without reference to bases for these spaces. We now introduce bases in the subspaces of a MSD. We briefly recall the definition of a Riesz basis that is used in this paper. For more details the reader is referred to [5, Ch.3].

Definition 2.7. *A countable set $\{\xi_k\}$ in $U \subseteq \mathcal{H}$ is a frame if there are positive constants m, M such that for any $u \in U$*

$$m^2 \|u\|_{\mathcal{H}}^2 \leq \sum_k |\langle u, \xi_k \rangle_{\mathcal{H}}|^2 \leq M^2 \|u\|_{\mathcal{H}}^2 . \quad (2.7)$$

The best constants m^2, M^2 in (2.7) are the frame bounds.

Definition 2.8. A frame in U which is minimal in the sense that it loses the frame property if any of its elements is removed, is a Riesz basis for U . The best m and M in (2.7) are called the Riesz constants of the Riesz basis. The ratio M/m is then the Riesz condition number.

If a frame $\{\xi_k\}$ is a Riesz basis, the representation $u = \sum_k c_k \xi_k$ is unique, and there is a unique dual Riesz basis $\{\tilde{\xi}_k\}$ with Riesz constants M^{-1}, m^{-1} , and with the property that $u = \sum_k \langle u, \tilde{\xi}_k \rangle \xi_k = \sum_k \langle u, \xi_k \rangle \tilde{\xi}_k$.

One has that $\{\xi_k\}$ is a Riesz basis for the closure of its linear span if and only if for any sequence of coefficients $c \in \ell_2$

$$m \|c\|_{\ell_2} \leq \left\| \sum_k c_k \xi_k \right\|_{\mathcal{H}} \leq M \|c\|_{\ell_2} . \quad (2.8)$$

The best constants in this formulation are equal to the Riesz constants.

Definition 2.9. A sequence of Riesz bases $\{\xi_{jk}\}$ for spaces $U_j \subseteq \mathcal{H}$ with Riesz constants m_j, M_j is uniformly stable if m_j^{-1} and M_j are bounded uniformly in j .

For convenience of notation, an expression like $\forall j, \forall u_j = \sum_k c_{jk} \xi_{jk} : \|u_j\|_{\mathcal{H}} \leq M \|c_j\|_{\ell_2}$ with $M < \infty$ independent of j , will be written more concisely as $\|u_j\|_{\mathcal{H}} \lesssim \|c_j\|_{\ell_2}$. The notation $\|u_j\|_{\mathcal{H}} \sim \|c_j\|_{\ell_2}$ means that $\|c_j\|_{\ell_2} \lesssim \|u_j\|_{\mathcal{H}} \lesssim \|c_j\|_{\ell_2}$.

In the sequel, we will assume a Riesz basis Φ_j is given for each V_j , and a Riesz basis Ψ_j for each W_j ,

$$\begin{aligned} V_j &= \text{clos}_{\mathcal{H}} \text{span } \Phi_j, & \Phi_j &:= \{ \varphi_{jk} \mid k \in \mathcal{K}_j \} \\ W_j &= \text{clos}_{\mathcal{H}} \text{span } \Psi_j, & \Psi_j &:= \{ \psi_{jm} \mid m \in \mathcal{M}_j \} , \end{aligned}$$

with \mathcal{K}_j and \mathcal{M}_j unspecified index sets. In many practical applications, this is actually the way the spaces V_j are defined: as the closed span of a set Φ_j .

The multiresolution structure of V_j and W_j is equivalent to refinability of the bases: there exist unique refinement masks $\{h_{jlk}\}$ and $\{g_{jlm}\}$ such that

$$\varphi_{jk} = \sum_{l \in \mathcal{K}_{j+1}} h_{jlk} \varphi_{j+1,l} , \quad \psi_{jm} = \sum_{l \in \mathcal{K}_{j+1}} g_{jlm} \varphi_{j+1,l} .$$

Letting the sets Φ_j and Ψ_j double as (possibly infinite) row vectors, the refinement relations above can be written as matrix expressions

$$\Phi_j = \Phi_{j+1} H_j \quad \text{and} \quad \Psi_j = \Phi_{j+1} G_j .$$

Unconditional convergence of these expressions follows from the Riesz basis property of Φ_{j+1} .

2.3 Multiscale Bases and Multiscale Transforms

We can now reformulate the two-scale transform in terms of the given bases, as

$$A_j : \ell_2(\mathcal{K}_j) \boxplus \ell_2(\mathcal{M}_j) \rightarrow \ell_2(\mathcal{K}_{j+1}) : (a_j, b_j) \mapsto H_j a_j + G_j b_j \quad (2.9)$$

or in matrix notation

$$A_j := [H_j \quad G_j], \quad A_j \begin{bmatrix} a_j \\ b_j \end{bmatrix} = H_j a_j + G_j b_j . \quad (2.10)$$

Complement stability, being defined as bounded invertibility of the basis-free two-scale transform \mathcal{A}_j , is immediately related to the two-scale transform matrix A_j . The following result is a uniform version of a remark from [2].

Proposition 2.10. *Assume that Φ_j are uniformly stable Riesz bases in a MRA $\{V_j\}$, with refinement matrices H_j , Then the following are equivalent:*

- (i) $W_j := \text{clos}_{\mathcal{H}} \text{span } \Psi_j$, with $\Psi_j := \Phi_{j+1}G_j$, are uniformly stable complements of V_j in V_{j+1} , and Ψ_j are uniformly stable Riesz bases for W_j
- (ii) $A_j := [H_j \ G_j]$ and its inverse are both uniformly bounded.

Proof. The implication (i) \Leftrightarrow (ii) is clear. To prove the forward implication, note that if A_j are uniformly bounded, then so are G_j . Let $w_j = \Phi_{j+1}G_j b_j$. Since Φ_{j+1} are uniformly stable, uniform boundedness of G_j implies $\|w_j\| \sim \|G_j b_j\| \lesssim \|b_j\|$. On the other hand, denote the inverse of A_j as $[\tilde{H}_j \ \tilde{G}_j]^*$. Then \tilde{G}_j^* are also uniformly bounded and $b_j = \tilde{G}_j^* G_j b_j$, so that $\|b_j\| \lesssim \|G_j b_j\| \sim \|w_j\|$. Hence $\Phi_{j+1}G_j$ are uniformly stable bases for their closed spans. \square

Because of the decomposition (2.4), an element of V_j can be expressed in any of two ways,

$$\Phi_J a_J = \Phi_j a_j + \sum_{i=j}^{J-1} \Psi_i b_i ,$$

in single-scale form by its coordinates a_J in the single-scale basis Φ_J , and in multiscale form by the vector of coefficients $[a_j^T \ b_j^T \ \dots \ b_{J-1}^T]^T$ in the multiscale basis $\Xi_J^j := \Phi_j \cup \bigcup_{i=j}^{J-1} \Psi_i$. It is clear that if the complement spaces W_j are orthogonal to the respective V_j and if Ψ_j are uniformly stable bases for W_j , then by the Pythagorean theorem Ξ_J^0 are uniformly stable bases for V_J . As the next result states, this also holds for MSDs that are not necessarily orthogonal, as long as they are stable.

Proposition 2.11. *A necessary and sufficient condition for Ξ_J^0 being uniformly stable bases for V_J is that*

- (i) \mathcal{T}_J^0 are uniformly well-conditioned, and
- (ii) Ψ_j are uniformly stable bases for W_j .

Furthermore, the Riesz condition number of Ξ_J^0 is bounded by $\kappa(\mathcal{T}_J^0) \max_{j < J} \{\kappa(\Psi_j)\}$.

Proof. Let $v_J = \sum_{j=-1}^{J-1} w_j$ with $w_j = \sum_{m \in \mathcal{M}_j} b_{jm} \psi_{jm}$. If Ψ_j are uniformly stable, then $\sum_{j=-1}^{J-1} \|w_j\|_{\mathcal{H}}^2 \sim \sum_{j=-1}^{J-1} \|b_j\|_{\ell_2(\mathcal{M}_j)}^2$. Uniform well-conditionedness of \mathcal{T}_J^0 means that $\|v_J\|_{\mathcal{H}}^2 \sim \sum_{j=-1}^{J-1} \|w_j\|_{\mathcal{H}}^2$, while Ξ_J^0 being uniformly stable is equivalent to $\|v_J\|_{\mathcal{H}}^2 \sim \sum_{j=-1}^{J-1} \|b_j\|_{\ell_2(\mathcal{M}_j)}^2$. Hence sufficiency is trivial. Necessity follows from the fact that if Ξ_J^0 are uniformly stable, then so are their subsets Ψ_j . \square

The multiscale transform from multiscale to single-scale form is efficiently performed using Mallat's multiresolution algorithm. The operation can be written in matrix form as the application of

$$T_J^j := \begin{bmatrix} A_{J-1} & 0 \\ 0 & I \end{bmatrix} T_{J-1}^j , \quad T_{j+1}^j := A_j .$$

If the spaces V_j are finite-dimensional, we may expand T_J^j into an $n_J \times n_J$ matrix, where n_J is the dimension of V_J .

Efficiency of the multiscale transform written in this form depends strongly on the matrices A_j being sparse. This is a property of the bases involved, and we will always strive to find bases that possess this property.

Definition 2.12. *The basis Φ_j is local relative to Φ_{j+1} if the number of nonzero entries in a column of the refinement matrix H_j is bounded. An analogous definition is made for Ψ_j and G_j .*

The hierarchy of bases $\{\Phi_j\}$ is local if for every j , Φ_j is local relative to Φ_{j+1} and the number of nonzero entries per column of H_j is bounded uniformly in j . The collection of wavelet bases $\{\Psi_j\}$ is local if for every j , both Φ_j and Ψ_j are local relative to Φ_{j+1} and the number of nonzero entries per column of H_j and G_j are bounded uniformly in j .

If the bases are local, the application of T_J^j requires $\mathcal{O}(n_J)$ operations.

2.4 A Riesz Basis for \mathcal{H}

One can prove a similar fact as in Proposition 2.4 in terms of the bases in the MSD. The next result is obtained in [9] for a slightly different definition of a frame. The modifications required in the proof for our definition are straightforward. We include a reworded proof for the convenience of the reader.

Theorem 2.13. *Let \mathcal{W} be a MSD of \mathcal{H} , and let Ψ_j be a Riesz basis for W_j . Then the following are equivalent:*

- (i) *the systems Ξ_j^0 are uniformly stable bases for V_J*
- (ii) *$\Xi_\infty^0 := \cup_{j=-1}^\infty \Psi_j$ is a Riesz basis for \mathcal{H} .*

Moreover, the sequences of Riesz constants m_J, M_J of Ξ_J^0 are monotone and

$$0 < m^* = \lim_{J \rightarrow \infty} m_J \leq m_J \leq M_J \leq \lim_{J \rightarrow \infty} M_J = M^* < \infty$$

for all J , where m^, M^* are the Riesz constants of Ξ_∞^0 .*

Proof. To prove monotonicity, take $v_J = \sum_{j=-1}^{J-1} \sum_m c_{jm} \psi_{jm} \in V_J \subset V_{J+1}$. By (2.8) one has that

$$M_J^{-2} \|v_J\|_{\mathcal{H}}^2 \leq \sum_{j=-1}^{J-1} \sum_m |c_{jm}|^2 \leq m_J^{-2} \|v_J\|_{\mathcal{H}}^2, \quad (2.11)$$

and also, because of nestedness,

$$M_{J+1}^{-2} \|v_J\|_{\mathcal{H}}^2 \leq \sum_{j=-1}^J \sum_m |c_{jm}|^2 \leq m_{J+1}^{-2} \|v_J\|_{\mathcal{H}}^2.$$

The coefficients c are unique, so $c_{Jm} = 0$. Since m_J, M_J are the best constants, one must have $m_{J+1} \leq m_J$ and $M_J \leq M_{J+1}$.

We proceed to show that if $m_\infty := \lim_{J \rightarrow \infty} m_J > 0$ and $M_\infty := \lim_{J \rightarrow \infty} M_J < \infty$, then Ξ_∞^0 is a Riesz basis for \mathcal{H} . For any $v \in \mathcal{H}$, let $\{v_n\}$ with $v_n = \sum_{j=-1}^{J_n-1} \sum_m c_{jm}^n \psi_{jm} \in V_{J_n}$ be a sequence converging to v . Since the m_J^{-2} are bounded, $\{c^n\}$ is a Cauchy sequence in ℓ_2 . Let c^* be its limit. Then $\sum_{j=-1}^\infty \sum_m c_{jm}^* \psi_{jm}$ converges strongly to v . Using convergence of $\{c^n\}$ and $\{v_n\}$ to c^* and v , and the inequalities (2.11) relating both sequences, finally gives

$$M_\infty^{-2} \|v\|_{\mathcal{H}}^2 \leq \|c^*\|_{\ell_2}^2 \leq m_\infty^{-2} \|v\|_{\mathcal{H}}^2.$$

By density, m_∞ and M_∞ are the best constants.

Conversely, let Ξ_∞^0 be a Riesz basis for \mathcal{H} with constants m^*, M^* . Any subset of Ξ_∞^0 is a Riesz basis for its linear span, with constants bounded by m^* and M^* . Again, density of \mathcal{V} implies that $\lim_{J \rightarrow \infty} m_J = m^*$ and $\lim_{J \rightarrow \infty} M_J = M^*$. \square

Corollary 2.14. *Assume that Φ_J are uniformly stable bases for V_J . Then the following are equivalent:*

- (i) *the multiscale transforms T_J^0 are uniformly well-conditioned*
- (ii) *Ξ_∞^0 is a Riesz basis for \mathcal{H} .*

Proof. This is an immediate consequence of Theorem 2.13. It suffices to see that if the Φ_J are uniformly stable, T_J^0 having uniformly bounded condition numbers is equivalent to uniform stability of Ξ_J^0 . \square

The last corollary states that in order to have uniformly well-conditioned multiscale transforms, one needs a multiscale Riesz basis for \mathcal{H} . As the next section recalls, this implies a biorthogonal MSD. Thus biorthogonality appears as a necessary condition for stability.

2.5 Biorthogonal Multiscale Decompositions

Let us first assume for a moment that Ξ_∞^0 is a Riesz basis for \mathcal{H} . Then there is a unique dual Riesz basis $\tilde{\Xi}_\infty^0 = \bigcup_{j=-1}^\infty \tilde{\Psi}_j$, which is biorthogonal to Ξ_∞^0 and which defines a dual MSD by

$$\tilde{W}_j := \text{clos}_{\mathcal{H}} \text{span } \tilde{\Psi}_j, \quad \tilde{V}_J := \bigoplus_{j=-1}^{J-1} \tilde{W}_j.$$

Moreover, if Φ_j are uniformly stable bases for V_j , then there are uniformly stable bases $\tilde{\Phi}_j$ for \tilde{V}_j which are biorthogonal to Φ_j . The bases Φ_j, Ψ_j and their duals $\tilde{\Phi}_j, \tilde{\Psi}_j$ define a pair of biorthogonal MSDs.

We now drop the assumption that Ξ_∞^0 is a Riesz basis. A pair of MSDs \mathcal{W} and $\tilde{\mathcal{W}}$ equipped with the single-level Riesz bases Φ_j, Ψ_j and $\tilde{\Phi}_j, \tilde{\Psi}_j$ is called *biorthogonal* if $\langle \varphi_{jk}, \tilde{\varphi}_{jk'} \rangle = \delta_{kk'}$ and $\langle \psi_{jm}, \tilde{\psi}_{jm'} \rangle = \delta_{jj'} \delta_{mm'}$, or

$$\begin{aligned} \langle \varphi_{jk}, \tilde{\varphi}_{jk'} \rangle &= \delta_{kk'}, & \langle \psi_{jm}, \tilde{\varphi}_{jk'} \rangle &= 0, \\ \langle \varphi_{jk}, \tilde{\psi}_{jm'} \rangle &= 0, & \langle \psi_{jm}, \tilde{\psi}_{jm'} \rangle &= \delta_{mm'}. \end{aligned}$$

One then has that $W_j \perp \tilde{V}_j$ and $V_j \perp \tilde{W}_j$. Also, the matrices in the refinement relations

$$\begin{aligned} \Phi_j &= \Phi_{j+1} H_j, & \Psi_j &= \Phi_{j+1} G_j, \\ \tilde{\Phi}_j &= \tilde{\Phi}_{j+1} \tilde{H}_j, & \tilde{\Psi}_j &= \tilde{\Phi}_{j+1} \tilde{G}_j, \end{aligned}$$

have the form

$$\begin{aligned} H_j &= [\langle \varphi_{jk}, \tilde{\varphi}_{j+1,l} \rangle]_{kl}^T, & G_j &= [\langle \psi_{jm}, \tilde{\varphi}_{j+1,l} \rangle]_{ml}^T, \\ \tilde{H}_j &= [\langle \tilde{\varphi}_{jk}, \varphi_{j+1,l} \rangle]_{kl}^T, & \tilde{G}_j &= [\langle \tilde{\psi}_{jm}, \varphi_{j+1,l} \rangle]_{ml}^T. \end{aligned}$$

These refinement matrices satisfy

$$\begin{aligned} \tilde{H}_j^* H_j &= I, & \tilde{H}_j^* G_j &= 0, \\ \tilde{G}_j^* H_j &= 0, & \tilde{G}_j^* G_j &= I, \end{aligned}$$

which can be rewritten very succinctly using the two-scale transform matrix A_j from (2.10) and its dual counterpart, as $\tilde{A}_j^* A_j = A_j \tilde{A}_j^* = I$. A pair of matrices with this property are said to be *biorthogonal matrices*. Hence the inverse of A_j is given by \tilde{A}_j^* , and the forward and inverse two-scale transforms read $a_{j+1} = H_j a_j + G_j b_j$ and $a_j = \tilde{H}_j^* a_{j+1}$, $b_j = \tilde{G}_j^* a_{j+1}$.

Note that if both the primal and the dual bases are local, then both the forward and the inverse multiscale transforms T_j^0 and $(T_j^0)^{-1}$ have linear complexity.

2.6 Order and Vanishing Moment Conditions

When dealing with MRAs on possibly bounded domains, we find it convenient to use a slightly modified definition of the order of a MRA, which coincides with the usual definition in the stationary case.

Definition 2.15. *The (polynomial) order of a nonstationary MRA on a domain in \mathbb{R}^d is given by*

$$\tilde{N} := \max\{n \mid \exists j_0 \text{ such that } \forall j \geq j_0 : \Pi_n \text{ is locally contained in } V_j\},$$

where Π_n is the space of polynomials of total degree at most n .

Here local inclusion is understood as an inclusion which holds on every bounded subdomain of the possibly unbounded domain in \mathbb{R}^d . The modification is needed to allow for finite-dimensional spaces V_j , with possibly $\dim V_j < N$ at the coarsest levels.

More generally, the spaces Π_n need not consist of polynomials, as long as they are appropriately nested, i.e. $\Pi_n \subseteq \Pi_{n'}$ when $n \leq n'$, and have increasing local approximation power: the sole objective of imposing some polynomial order is usually to guarantee the existence of a direct estimate (2.6) through a Bramble–Hilbert type argument.

Definition 2.16. *In a nonstationary MRA on a domain in \mathbb{R}^d , the primal wavelets are said to have N vanishing moments if W_j is orthogonal to Π_N for $j \geq j_0$.*

In the biorthogonal setting, the number of vanishing moments of the dual wavelets is of course defined analogously. One has that $S \subset \mathcal{H}$ is orthogonal to W_j for all $j \geq j_0$ if and only if $S \subseteq \tilde{V}_{j_0}$. Hence the quantities N and \tilde{N} are dual to each other in that the number of primal vanishing moments N equals the order of the dual MRA, and the order \tilde{N} of the primal MRA equals the number of dual vanishing moments.

3 The Lifting Scheme

3.1 Principle and Stability Properties

We may now apply the concepts from the previous section to the lifting scheme. The lifting scheme exploits the following observation [13, Theorem 8.1].

Theorem 3.1. *Let $A_j = [H_j \ G_j]$ and $\tilde{A}_j = [\tilde{H}_j \ \tilde{G}_j]$ be a pair of bounded biorthogonal two-scale transform matrices. If U_j is bounded, then the new two-scale transform matrices*

$$\hat{A}_j = [H_j \ \hat{G}_j] := [H_j \ G_j] \begin{bmatrix} I & -U_j \\ 0 & I \end{bmatrix} \quad \text{and} \quad \hat{\tilde{A}}_j := [\hat{\tilde{H}}_j \ \hat{\tilde{G}}_j] = [\tilde{H}_j \ \tilde{G}_j] \begin{bmatrix} I & 0 \\ U_j^* & I \end{bmatrix} \quad (3.1)$$

are also bounded and biorthogonal.

Remark 3.2. If the two-scale transform matrices A_j are uniformly well-conditioned, then the new matrices \hat{A}_j defined by (3.1) for all j are also uniformly well-conditioned if and only if the matrices U_j are uniformly bounded.

Theorem 3.1 can be obtained as an instance of a more general result.

Theorem 3.3 ([1]). *Given H_j , any G_j such that $[H_j \ G_j]$ is bounded and boundedly invertible, is called a stable completion of H_j . Starting with an initial stable completion G_j , any \hat{G}_j obtained as*

$$[H_j \ \hat{G}_j] := [H_j \ G_j] \begin{bmatrix} I & L \\ 0 & K \end{bmatrix}$$

with L bounded and K bounded and boundedly invertible, is also a stable completion of H_j . Moreover, all stable completions of H_j are of this form.

The lifting scheme uses $K = I$. It has been shown [6] that in the stationary setting any pair of local biorthogonal refinement operators $[H \ G]$ can be obtained by lifting from a trivial initial choice.

By Proposition 2.10 and the next result, the new biorthogonal two-scale transform matrices define new sets of biorthogonal basis functions, at least when the conditions on the new dual MRA are satisfied.

Proposition 3.4 ([1, Remark 2.5]). *Let $\{\hat{\Phi}_j\}$, $\{\hat{\tilde{\Phi}}_j\}$ be uniformly stable biorthogonal systems of refinable functions with refinement matrices \hat{H}_j c.q. $\hat{\tilde{H}}_j$, both spanning a MRA. Then the following are equivalent:*

- (i) $\hat{\Psi}_j := \hat{\Phi}_{j+1}\hat{G}_j$ and $\hat{\tilde{\Psi}}_j := \hat{\tilde{\Phi}}_{j+1}\hat{\tilde{G}}_j$ form biorthogonal sets of functions
- (ii) $\hat{A}_j := [\hat{H}_j \ \hat{G}_j]$ and $\hat{\tilde{A}}_j := [\hat{\tilde{H}}_j \ \hat{\tilde{G}}_j]$ are biorthogonal.

Finding the new primal wavelets is obvious: they are derived from the unchanged scaling functions by the new refinement matrices \hat{G}_j . The new dual scaling functions are defined as the limit functions of the nonstationary subdivision scheme determined by the dual refinement matrices $\hat{\tilde{H}}_j$. Convergence of this subdivision scheme is not guaranteed. If the new dual scaling functions exist, the new dual wavelets are easily found using the matrices $\hat{\tilde{G}}_j$.

As was already stressed in [13], while it does maintain biorthogonality, lifting in itself does not guarantee the new MSDs being stable. We can now synthesize much of preceding discussion into a statement on the stability of complement spaces and single-scale bases in a pair of biorthogonal MSDs after lifting.

Theorem 3.5. *Let \mathcal{W} and $\tilde{\mathcal{W}}$, with the bases Φ_j, Ψ_j and $\tilde{\Phi}_j, \tilde{\Psi}_j$, be a pair of biorthogonal MSDs. Assume that the bases Φ_j and Ψ_j are uniformly stable and that W_j are uniformly stable complements, and use the same assumptions on the dual MSD $\tilde{\mathcal{W}}$.*

With H_j, G_j denoting the refinement matrices in \mathcal{W} and \tilde{H}_j, \tilde{G}_j the refinement matrices in $\tilde{\mathcal{W}}$, the new refinement matrices after lifting, $\hat{H}_j = H_j$ and \hat{G}_j as defined in (3.1), determine a new MSD $\hat{\mathcal{W}}$.

If the matrices U_j in (3.1) are uniformly bounded, then the following holds:

- (i) *The lifted bases $\hat{\Phi}_j := \Phi_j$ and $\hat{\Psi}_j := \Phi_{j+1}\hat{G}_j$ are uniformly stable, and the spaces $\hat{W}_j := \text{clos}_{\mathcal{H}} \text{span } \hat{\Psi}_j$ are uniformly stable complements.*
- (ii) *If, in addition, the lifted dual refinement matrices $\hat{\tilde{H}}_j$, as defined in (3.1), determine sets of dual scaling functions $\hat{\tilde{\Phi}}_j$ that are uniformly stable then also $\hat{\tilde{\Psi}}_j := \hat{\tilde{\Phi}}_{j+1}\hat{G}_j$ are uniformly stable, and $\hat{W}_j := \text{clos}_{\mathcal{H}} \text{span } \hat{\tilde{\Psi}}_j$ are uniformly stable complements.*

Proof. If the single-scale bases Φ_j and Ψ_j are uniformly stable and the spaces W_j are uniformly stable complements, then by Proposition 2.10, the two-scale transform matrices A_j are uniformly well-conditioned, and by biorthogonality so are \tilde{A}_j . Uniform boundedness of the update matrices U_j implies by Theorem 3.1 and Remark 3.2 that the new two-scale transform matrices \hat{A}_j and their biorthogonal matrices $\hat{\tilde{A}}_j$ are still uniformly well-conditioned, so that simply using Proposition 2.10 in the other sense gives uniform complement stability of \hat{W}_j and uniform stability of the new single-scale bases $\hat{\Psi}_j$.

As to the second assertion, assume that nonstationary subdivision with the sequence $\{\hat{\tilde{H}}_j\}$ converges, defining the dual scaling functions $\hat{\tilde{\Phi}}_j$. Because of the relation $\hat{\tilde{H}}_j^* H_j = I$, corresponding piecewise constant iterates of Φ_j and $\hat{\tilde{\Phi}}_j$ in the nonstationary cascade algorithm are biorthogonal. Hence if the limit functions $\hat{\tilde{\Phi}}_j$ exist, they are still biorthogonal to Φ_j . If furthermore the bases $\hat{\tilde{\Phi}}_j$ are uniformly stable, then by Proposition 3.4 also $\hat{\tilde{\Psi}}_j$ are biorthogonal to $\hat{\tilde{\Phi}}_j$, and by Proposition 2.10 the spaces \hat{W}_j are uniformly stable complements and $\hat{\tilde{\Psi}}_j$ are uniformly stable bases for \hat{W}_j . \square

Of course, the roles of both biorthogonal MSDs in Theorem 3.1 can be interchanged. This yields the matrices

$$\hat{A}_j = [\hat{H}_j \quad \hat{G}_j] := [H_j \quad G_j] \begin{bmatrix} I & 0 \\ P_j & I \end{bmatrix} \quad \text{and} \quad \hat{\tilde{A}}_j := [\hat{\tilde{H}}_j \quad \hat{\tilde{G}}_j] = [\tilde{H}_j \quad \tilde{G}_j] \begin{bmatrix} I & -P_j^* \\ 0 & I \end{bmatrix}. \quad (3.2)$$

In order to distinguish between the two, (3.1) is referred to as *primal lifting*, while (3.2) is known as *dual lifting*. A result analogous to Theorem 3.5 can be formulated.

Theorem 3.6. *Let \mathcal{W} and $\tilde{\mathcal{W}}$, with the bases Φ_j, Ψ_j and $\tilde{\Phi}_j, \tilde{\Psi}_j$, be a pair of biorthogonal MSDs. Assume that the bases Φ_j and Ψ_j are uniformly stable and that W_j are uniformly stable complements, and use the same assumptions on the dual MSD $\tilde{\mathcal{W}}$.*

With H_j, G_j denoting the refinement matrices in \mathcal{W} and \tilde{H}_j, \tilde{G}_j the refinement matrices in $\tilde{\mathcal{W}}$, the new dual refinement matrices after dual lifting, $\hat{H}_j = \tilde{H}_j$ and \hat{G}_j as defined in (3.2), determine a new MSD $\hat{\mathcal{W}}$.

If the matrices P_j in (3.2) are uniformly bounded, then the following holds:

- (i) The lifted bases $\hat{\Phi}_j := \tilde{\Phi}_j$ and $\hat{\Psi}_j := \tilde{\Phi}_{j+1}\hat{G}_j$ are uniformly stable, and the spaces $\hat{W}_j := \text{clos}_{\mathcal{H}} \text{span} \hat{\Psi}_j$ are uniformly stable complements.
- (ii) If, in addition, the lifted primal refinement matrices \hat{H}_j , as defined in (3.2), determine sets of scaling functions $\hat{\Phi}_j$ that are uniformly stable then also $\hat{\Psi}_j := \tilde{\Phi}_{j+1}\hat{G}_j$ are uniformly stable, and $\hat{W}_j := \text{clos}_{\mathcal{H}} \text{span} \hat{\Psi}_j$ are uniformly stable complements.

3.2 Two-Step Construction

If A_j and \tilde{A}_j correspond to a pair of biorthogonal MSDs, then after primal lifting the primal MRA \mathcal{V} has been left unchanged. The new bases $\hat{\Phi}_j$ and $\hat{\Psi}_j$ are related to the old ones by $\hat{\Phi}_j = \Phi_j$ and $\hat{\Psi}_j = \Psi_j - \Phi_j U_j$. This allows to easily build a new $\hat{\Psi}_j$ out of Ψ_j and Φ_j . One may start with a simple biorthogonal MSD and then adapt it so as to obtain certain desirable properties. Not only is it often easier to find U_j and implement the multiscale transform without explicitly finding A_j and \tilde{A}_j^* , it often also yields a faster algorithm.

The standard realization of the lifting scheme uses two consecutive lifting steps. The starting point is a simple orthogonal MSD with an orthogonal two-scale transform matrix $[\check{H}_j \ \check{G}_j]$. A dual lifting step — or *prediction* step — yields a primal MSD of the desired order, and a primal lifting step — or *update* step — increases the number of vanishing moments of the primal wavelets. The second step has no effect on the order. The terms *update* and *prediction* are due to an interpretation [6] of the effect of both operations.

Both consecutive steps are reflected in the multiscale transform. The two-scale transform reads

$$a_{j+1} = [\check{H}_j \ \check{G}_j] \begin{bmatrix} I & 0 \\ \check{P}_j & I \end{bmatrix} \begin{bmatrix} I & -U_j \\ 0 & I \end{bmatrix} \begin{bmatrix} a_j \\ b_j \end{bmatrix}.$$

Direct implementation in factored form leads to easy and efficient algorithms (see Figure 3). Often it takes significantly less operations to apply the multiscale transform in factored form than to assemble the lifted refinement matrices and multiply these with the vectors of coefficients. In [6] a comparison is made for some stationary multiscale transforms.

Finally, since both the inverse of the original multiscale transform and of the individual lifting steps are trivial, the inverse transform has a simple expression,

$$\begin{bmatrix} a_j \\ b_j \end{bmatrix} = \begin{bmatrix} I & U_j \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ -\check{P}_j & I \end{bmatrix} \begin{bmatrix} \check{H}_j^* \\ \check{G}_j^* \end{bmatrix} a_{j+1},$$

and is of the same complexity as the forward transform. If the orthogonal bases before lifting are local and the prediction and update matrices are uniformly banded, then both the new primal bases and the new dual bases after lifting are still local, and the forward and inverse multiscale transforms have linear complexity.

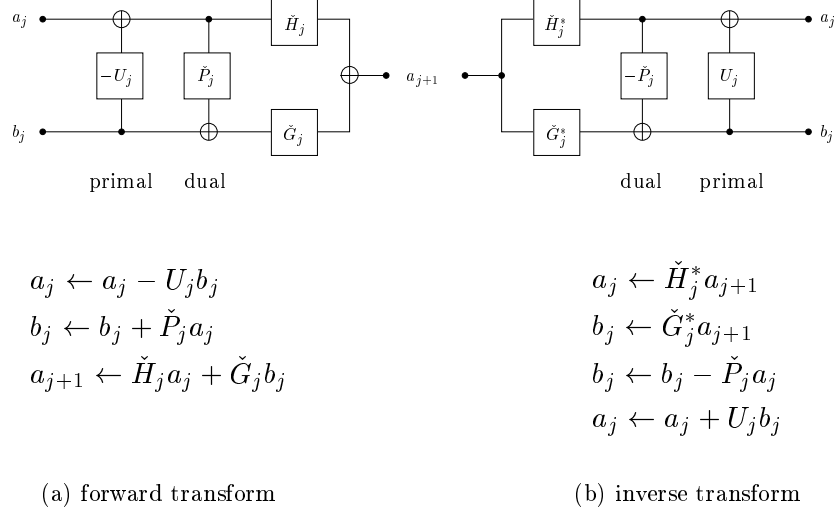


Figure 3: Factored implementation of lifting steps in the forward and inverse two-scale transforms.

Section 3.1 has made clear that uniform boundedness in norm of the matrices P_j and U_j is essential to the stability of the new MSDs after lifting. The prediction step usually amounts to polynomial interpolation, and it can be shown that in common cases, the prediction matrices P_j are indeed uniformly bounded in norm, so that Theorem 3.5 can be invoked. The update step is more delicate.

4 Update Methods

In view of Theorem 3.5, we introduce one last stability concept and define stability for algorithms that choose update matrices.

Definition 4.1. *An update method is called stable if starting from a MSD with uniformly stable bases and uniformly stable complements, it yields update matrices U_j that are uniformly bounded in norm.*

4.1 Classical Update

The requirement that the complement space \hat{W}_j be orthogonal to Π_N does not in itself uniquely characterize \hat{W}_j . For each wavelet ψ_{jm} , the nonzero entries in the m th column U_{jm} of the update matrix U_j determine a linear combination of a subset $\Phi_{jm} \subset \Phi_j$,

$$\hat{\psi}_{jm} = \psi_{jm} - \Phi_{jm} u_{jm}.$$

The subset Φ_{jm} is usually chosen a priori. The (possibly) nonzero entries u_{jm} , each having its associated index in U_{jm} , will be called the *update stencil* for ψ_{jm} . The number of entries in the stencil is the stencil width.

The standard implementation of the primal lifting step in the two-step construction of Section 3.2 — to which we will refer as the *classical update method* — goes as follows. A stencil of width N is used, involving the N scaling functions supported closest to the support of the wavelet to be lifted. The choice of Φ_{jm} also has an effect on \hat{W}_j . The lifting coefficients are found as the solution to an $N \times N$ system,

$$u_{jm} = \begin{bmatrix} \langle \Phi_{jm}, p_0 \rangle \\ \vdots \\ \langle \Phi_{jm}, p_{N-1} \rangle \end{bmatrix}^{-1} \begin{bmatrix} \langle \psi_{jm}, p_0 \rangle \\ \vdots \\ \langle \psi_{jm}, p_{N-1} \rangle \end{bmatrix}, \quad (4.1)$$

where the functions p_i , $i = 0, \dots, N - 1$ form a basis for Π_N .

Whether the system is always nonsingular under the conditions at hand is an open question. In our experience in the one-dimensional case the above system is indeed nonsingular but nevertheless prone to bad conditioning. A fixed monomial basis for Π_N is unsatisfactory. Even when a translated monomial basis is used (giving *local moments*), its condition number tends to grow larger at coarser levels and for increasing N . Using local orthogonal polynomials might help, but this would present an increase in computational cost and would also further complicate the implementation.

The magnitude of the lifting coefficients can be very sensitive to the choice of Φ_{jm} . The lifting coefficients appear to be unbounded, and at least not sufficiently well-behaved for practical purposes. This can compromise complement stability and single-scale condition numbers for the new wavelet bases. Figure 4 gives a numerical example for an average-interpolating MRA [7] of order 5 on a mildly irregular mesh on the interval. The top row shows the situation before the update step. The bases Φ_4 and Ψ_4 have condition numbers 1.96 and 1.72, respectively, with a complement stability constant $\alpha_4 \approx .833$. Symmetry around the center of the interval is only approximate. A classical update step is performed to obtain a new wavelet basis $\hat{\Psi}_4$ with five vanishing moments. Using the five nearest scaling functions as is done in (c) does not give acceptable results for the wavelet $\hat{\psi}_{4,4}$ marked in bold: the lifting coefficients for this wavelet are several orders of magnitude larger than for the surrounding wavelets. The condition number $\kappa(\hat{\Psi}_4)$ of the new basis is approximately 24 and the complement stability constant is close to .998. In (d) another set of scaling functions was selected for the update of $\psi_{4,4}$ only, namely $\hat{\Phi}_{4,4} = \{\varphi_{4,1}, \varphi_{4,2}, \varphi_{4,3}, \varphi_{4,4}, \varphi_{4,6}\}$ instead of $\{\varphi_{4,2}, \dots, \varphi_{4,6}\}$. This alternative choice results in $\kappa(\hat{\Psi}_4) \approx 2.35$ and $\alpha_4 \approx .473$.

4.2 Semiorthogonal Lifting

In contrast, there is a simple update method that is provably stable, which makes it interesting at least from a theoretical point of view. Choosing the update operator U_j as the orthogonal projection from W_j into V_j yields an orthogonal MSD, i.e. a MSD in which all complement spaces W_j are orthogonal complements of V_j in V_{j+1} and hence are orthogonal to W_i for $i < j$.

We refer to this operation as *semiorthogonal lifting*. Basis functions for these particular complement spaces are sometimes called *prewavelets*, when the name *wavelets* is reserved for the case of an orthogonal basis. Since we call any basis function in a complement space a wavelet, we will use the terms *semiorthogonal wavelets* and *orthogonal wavelets*, respectively.

Semiorthogonal lifting from the hierarchical basis of [17] is considered in [14]. In general, one has the following.

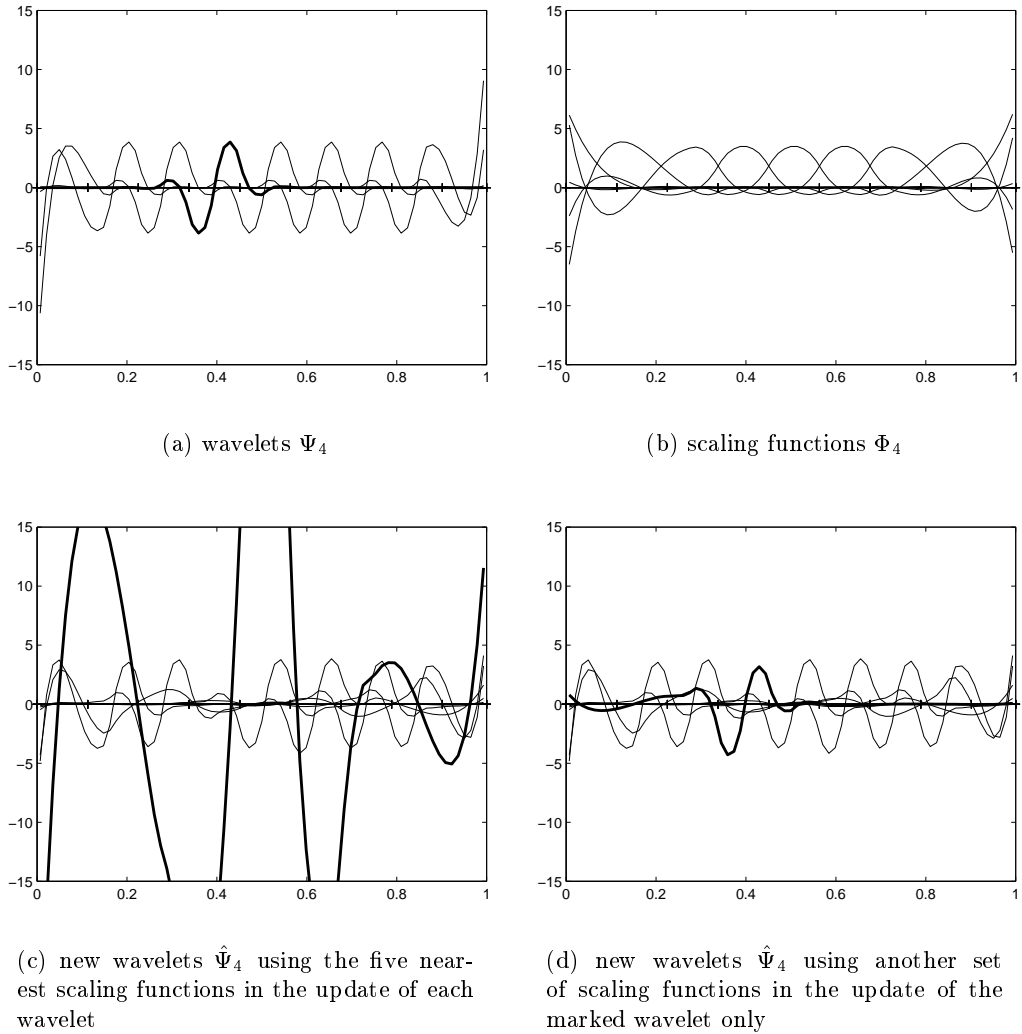


Figure 4: An example showing that the magnitude of the lifting coefficients can be very sensitive to the choice of Φ_{jm} in (4.1), with adverse effects on complement stability and stability of the lifted wavelet basis.

Proposition 4.2. *Let \mathcal{W} be a MSD of \mathcal{H} . Assume that W_j are uniformly stable complements of V_j in V_{j+1} , and that Ψ_j are uniformly stable bases for W_j . Then the new multiscale basis $\hat{\Xi}_\infty^0 := \cup_{j=-1}^\infty \hat{\Psi}_j$ obtained by semiorthogonal lifting is a Riesz basis for \mathcal{H} .*

Proof. Since \mathcal{V} is dense in \mathcal{H} and the new complement spaces \hat{W}_j are mutually orthogonal, stability of the new MSD follows from the Pythagorean theorem.

Let P_j be the orthogonal projection of V_{j+1} onto V_j , and let $\hat{\psi}_{jm} := \psi_{jm} - \Phi_j U_{jm} = (I - P_j)\psi_{jm}$. For any $w \in W_j$,

$$\|w\|^2 \geq \|(I - P_j)w\|^2 = \|w\|^2 - \|P_j w\|^2 \geq (1 - \alpha_j^2) \|w\|^2 .$$

If W_j is a stable complement, the operator $(I - P_j)$ with domain restricted to W_j is a topological isomorphism between W_j and \hat{W}_j . If Ψ_j is a Riesz basis for W_j with constants m and M , then

$$m \|(I - P_j)^{-1}\|^{-1} \|d\| \leq \left\| \sum_m d_m \hat{\psi}_{jm} \right\| \leq M \|I - P_j\| \|d\| ,$$

so that $\hat{\Psi}_j$ is a Riesz basis for \hat{W}_j . □

The lifted basis $\hat{\Psi}_j$ will be better conditioned if the complement stability constant α_j before semiorthogonal lifting is better. In any case, if W_j is a stable complement and if both bases Φ_j and Ψ_j are stable, then the matrix Q_j corresponding to orthogonal projection from W_j onto V_j is bounded, and this holds uniformly in j .

Hence we may conclude stability of semiorthogonal lifting as an update method. In general the matrix Q_j is full, so that the lifted wavelets are not local. Local bases for the same complement spaces may exist, but this is not our point of interest here. Even when Q_j is full, it has a decay property which will be important in the next sections.

Lemma 4.3. *If the bases Φ_j and Ψ_j are local and if Φ_j are uniformly stable, then the matrices of lifting coefficients Q_j in semiorthogonal lifting have exponential off-diagonal decay of a rate that is essentially independent of j .*

The notions of bandedness and off-diagonal decay are used with the convention that the distance between matrix entries is counted as the distance between their associated nodes in the mesh, rather than their relative position in the matrix for some ordering of the index sets.

Proof. The matrix Q_j is found as $Q_j = \langle \Phi_j, \Phi_j \rangle^{-1} \langle \Psi_j, \Phi_j \rangle$. In terms of the *frame operator* F_j of the Riesz basis Φ_j and its adjoint F_j^* ,

$$\begin{aligned} F_j &: V_j \rightarrow \ell_2(\mathcal{K}_j) : v \mapsto \{ \langle v, \varphi_{jk} \rangle \}_{k \in \mathcal{K}_j} \\ F_j^* &: \ell_2(\mathcal{K}_j) \rightarrow V_j : a \mapsto \sum_{k \in \mathcal{K}_j} a_k \varphi_{jk} , \end{aligned}$$

the hermitian matrix $R_j := \langle \Phi_j, \Phi_j \rangle$ represents the operator $F_j F_j^*$. Since

$$\langle F_j F_j^* a, a \rangle_{\ell_2} = \langle F_j^* a, F_j^* a \rangle_{\mathcal{H}} = \left\| \sum_{k \in \mathcal{K}_j} a_k \varphi_{jk} \right\|_{\mathcal{H}}^2 ,$$

this operator satisfies $m_j^2 \|a\|_{\ell_2}^2 \leq \langle F_j F_j^* a, a \rangle_{\ell_2} \leq M_j^2 \|a\|_{\ell_2}^2$, or shorter $m_j^2 \mathbf{1} \leq F_j F_j^* \leq M_j^2 \mathbf{1}$, where m_j, M_j are the Riesz constants of Φ_j , and $\mathbf{1}$ is the identity operator on $\ell_2(\mathcal{K}_j)$. Rewriting yields

$$0 \leq \mathbf{1} - \frac{1}{M_j^2} F_j F_j^* \leq \left(1 - \frac{m_j^2}{M_j^2}\right) \mathbf{1} < \mathbf{1} .$$

This implies that $\|(I - M_j^{-2} R_j)\|_2$ is bounded by $1 - m_j^2 M_j^{-2}$, which in turn is uniformly bounded away from unity if the bases Φ_j are uniformly stable. As a result, one has that

$$R_j^{-1} = M_j^{-2} \sum_{n=0}^{\infty} (I - M_j^{-2} R_j)^n .$$

By locality, the matrices R_j are banded with uniformly bounded bandwidth. Hence the matrix powers of $(I - M_j^{-2} R_j)$, having linearly increasing bandwidth and elements of exponentially decreasing magnitude, sum to a matrix with exponential off-diagonal decay uniformly in j . Since also the matrices $\langle \Psi_j, \Phi_j \rangle$ are uniformly banded, the final matrix multiplication giving Q_j preserves the asserted property. \square

4.3 Local Semiorthogonalization

Wavelets from semiorthogonal lifting are not automatically local any more. Yet exact semiorthogonality is not necessary for stability. Hence the idea of an approximation that gives local refinement matrices [10, 11, 14, 15].

Let Q_j be the update matrix produced by semiorthogonal lifting, and denote its m th column by Q_{jm} . An easy approximation strategy would be to truncate the smallest entries in the update matrix. Note that whatever the approximation to the update matrix Q_j , the forward and inverse lifted wavelet transforms are still exact inverses to one another as long as both make the same approximation to Q_j . Because of the form of Q_j known from Lemma 4.3, omitting the smallest entries will cause the truncated matrix to be banded. Local semiorthogonalization exploits this by choosing the support of the update stencil in advance: only a subset of Φ_j enters into the semiorthogonalization.

One way to view this strategy is to think of it as an exact semiorthogonalization of $\hat{\psi}_{jm}$ with respect to a subspace of V_j spanned by scaling functions supported around the support of ψ_{jm} [10, 11]. If this subset is sufficiently large, the lifted wavelet will be approximately orthogonal to V_j .

There is another way to think of local semiorthogonalization. Suppose a subset Φ_{jm} of Φ_j is fixed in advance, and determine the vector u_{jm} so as to minimize the quantity

$$\max_{v \in V_j} \frac{|\langle \psi_{jm} - \Phi_{jm} u_{jm}, v \rangle|}{\|\psi_{jm} - \Phi_{jm} u_{jm}\| \|v\|} . \quad (4.2)$$

The choice of u_{jm} minimizing (4.2) is $\langle \Phi_{jm}, \Phi_{jm} \rangle^{-1} \langle \psi_{jm}, \Phi_{jm} \rangle$, which corresponds precisely to $\hat{\psi}_{jm}$ being orthogonal to $\text{span } \Phi_{jm}$. This alternative formulation shows how to maximize the angle with V_j when lifting with an arbitrary subspace V_{jm} of V_j : $\Phi_{jm} u_{jm}$ should be the orthogonal projection of ψ_{jm} on V_{jm} .

Since Π_N is contained in V_j , the new wavelets could also be expected to be close to having N primal vanishing moments. But while the allowable error in orthogonalizing with respect to V_j is rather large, in order to benefit from the side effect of sufficiently small primal moments, the precision needs to be high. In effect, all vanishing moments are lost in the approximation. Without at least one primal vanishing moment, the new pair of biorthogonal MSDs cannot be stable.

One can, however, guarantee that the update matrix is still uniformly bounded in norm. For each wavelet, one column U_{jm} of the matrix U_j is calculated. It suffices to bound the relative deviation of U_{jm} from Q_{jm} by testing whether

$$\|Q_{jm} - U_{jm}\|_\infty \leq \delta \|Q_{jm}\|_\infty \quad (4.3)$$

for some $\delta > 0$, and enlarging the update stencil until this is true. Since Q_{jm} itself has exponentially decaying elements, it is reasonable to assume that enlarging the update stencil eventually decreases the norm of the error exponentially in the stencil width. Consequently, the support of the update stencil is bounded uniformly in j . A first consequence of this is that the lifted wavelets will still be local, if the wavelets and scaling functions before lifting were local. For the matrix U_j as a whole, (4.3) implies that

$$\|Q_j - U_j\|_1 \leq \delta \|Q_j\|_1 .$$

For band matrices with uniformly bounded bandwidth, the 1-norm and the 2-norm are equivalent uniformly in the dimensions of the matrix. An $m \times n$ matrix A with off-diagonal decay satisfies the same norm equivalence $b^{-1} \|A\|_1 \leq \|A\|_2 \leq \sqrt{b} \|A\|_1$ as a band matrix with bandwidth b if $\|A\|_1, \|A\|_\infty \leq b \max_{1 \leq i \leq m, 1 \leq j \leq n} |a_{ij}|$. This is clearly satisfied if the decay is exponential. Hence (4.3) ensures that

$$\|U_j\|_2 \leq b^{3/2} (1 + \delta) \|Q_j\|_2 .$$

The parameter δ determines the trade-off between smaller update support and better complement stability.

With the above device, the update method inherits its stability from exact semiorthogonal lifting. The resulting wavelets are local.

An alternative approach in approximating semiorthogonal lifting is used in [14, 15], where the action of $Q_j = \langle \Phi_j, \Phi_j \rangle^{-1} \langle \Psi_j, \Phi_j \rangle$ is approximated by a few iterations of a conjugate gradient solver. It is shown experimentally that a consistently small number of iterations is sufficient for the applications considered. Our approach has the advantage of being localized: it does not require a global system to be solved (even approximately). Also, it only extends the update stencil where this is needed.

4.4 A Combined Method

When it comes to enhancing complement stability, approximate semiorthogonalization is an obvious choice. However, the previous section has made clear that this implies losing vanishing moments, which is undesirable. We therefore propose a combined method. While enforcing exact orthogonality of the lifted complement space \tilde{W}_j with respect to a subspace Π_N of its corresponding coarse scale space V_j , we will use additional degrees of freedom to maximize the angle of the lifted wavelets with V_j . This effect could be obtained in a second update step after a classical update, by lifting \tilde{W}_j with the subspace \tilde{W}_{j-1} of V_j instead of with V_j

itself, but, the bases $\hat{\Psi}_j$ from the unstabilized update being arbitrarily ill-conditioned, this is something we wish to avoid.

Again, assume that a finite subset $\Phi_{jm} \subset \Phi_j$ is given, with $\#\Phi_{jm} \geq N$, and let V_{jm} be its linear span. Use the orthogonal decomposition

$$V_{jm} = (P_{jm}\Pi_N) \oplus (V_{jm} \cap \Pi_N^\perp) =: V_{jm}^- \oplus V_{jm}^+, \quad (4.4)$$

where P_{jm} denotes the orthogonal projection of V_{j+1} onto V_{jm} . To verify that the orthogonal complement of $P_{jm}\Pi_N$ in V_{jm} is indeed $V_{jm} \cap \Pi_N^\perp$, it suffices to see that

$$(P_{jm}\Pi_N)_{V_{jm}}^\perp := \{v \in V_{jm} \mid \langle v, P_{jm}p \rangle = \langle v, p \rangle = 0 \ \forall p \in \Pi_N\}.$$

Let $\Phi_{jm}B_{jm}^-$ and $\Phi_{jm}B_{jm}^+$ be bases for the respective subspaces V_{jm}^- and V_{jm}^+ . Such matrices are easily obtained; a matrix B_{jm}^- can be found from the projection into V_{jm} of the basis $[p_0 \dots p_{N-1}]$ for Π_N , whence a suitable B_{jm}^+ results from a partial Gram–Schmidt orthogonalization. We consider the general case in which $\dim P_{jm}\Pi_N = N$.

With the decomposition (4.4), any vector of lifting coefficients u_{jm} is correspondingly decomposed as $u_{jm} = B_{jm}^- u_{jm}^- + B_{jm}^+ u_{jm}^+$. The first component moves the unlifted wavelet ψ_{jm} into the orthogonal complement of Π_N if

$$u_{jm}^- = \left(\begin{array}{c} \langle \Phi_{jm}, p_0 \rangle \\ \vdots \\ \langle \Phi_{jm}, p_{N-1} \rangle \end{array} B_{jm}^- \right)^{-1} \begin{array}{c} \langle \psi_{jm}, p_0 \rangle \\ \vdots \\ \langle \psi_{jm}, p_{N-1} \rangle \end{array},$$

and the second component cannot make it leave $\Pi_N^\perp \cap V_{j+1}$. The vector u_{jm} minimizing (4.2) under the constraint that $\psi_{jm} - \Phi_{jm}u_{jm}$ is orthogonal to Π_N is then found by choosing $\Phi_{jm}B_{jm}^+ u_{jm}^+$ as the orthogonal projection of the partially lifted wavelet $\psi_{jm} - \Phi_{jm}B_{jm}^- u_{jm}^-$ into V_{jm}^+ . By orthogonality of the decomposition (4.4), this equals the orthogonal projection of ψ_{jm} into V_{jm}^+ , so that

$$u_{jm}^+ = \left(B_{jm}^{+*} \langle \Phi_{jm}, \Phi_{jm} \rangle B_{jm}^+ \right)^{-1} B_{jm}^{+*} \langle \psi_{jm}, \Phi_{jm} \rangle.$$

Not too surprisingly, the set of data needed is a combination of those for the classical update and those for local semiorthogonalization.

The same device that was presented for ensuring stability of local semiorthogonalization can be used here. This yields an update method which is stable, produces wavelets that are local, and gives any desired number of primal vanishing moments ranging from 1 to \tilde{N} .

5 Conclusion

The standard two-step construction of wavelet bases on irregular meshes by means of the lifting scheme does not take into account stability considerations, and this causes noticeable problems. We have presented a modification to the update step that proves to be stable. The scaling functions and wavelets on both the primal and the dual side are still local.

Acknowledgments

This paper presents research results of the Belgian Program on Interuniversity Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture. The scientific responsibility rests with the authors.

The first author was funded as a Research Assistant of the Fund for Scientific Research – Flanders, Belgium (F.W.O.).

References

- [1] J. Carnicer, W. Dahmen, and J. Peña. Local decomposition of refinable spaces and wavelets. *Appl. Comput. Harmon. Anal.*, 3:127–153, 1996.
- [2] W. Dahmen. Some remarks on multiscale transformations, stability and orthogonality. In P. J. Laurent, A. L. Méhauté, and L. L. Schumaker, editors, *Curves and Surfaces II*, pages 1–32. A. K. Peters, Boston, 1991.
- [3] W. Dahmen. Stability of multiscale transformations. *J. Fourier Anal. Appl.*, 4:341–362, 1996.
- [4] W. Dahmen. Wavelet and multiscale methods for operator equations. In *Acta Numerica*, volume 6, pages 55–228. Cambridge Univ. Press, Cambridge, 1997.
- [5] I. Daubechies. *Ten Lectures on Wavelets*. Number 61 in CBMS-NSF Regional Conference Series in Applied Mathematics. SIAM, Philadelphia, 1992.
- [6] I. Daubechies and W. Sweldens. Factoring wavelet transforms into lifting steps. *J. Fourier Anal. Appl.*, 4(3), 1999.
- [7] D. L. Donoho. Smooth wavelet decompositions with blocky coefficient kernels. In L. L. Schumaker and G. Webb, editors, *Recent Advances in Wavelet Analysis*, pages 1–43. Academic Press, 1993.
- [8] M. Jansen and A. Bultheel. Smoothing irregularly sampled signals using wavelets and cross validation. Report TW289, Katholieke Universiteit Leuven, Department of Computer Science, Leuven, Belgium, Apr. 1999.
- [9] R. Lorentz and P. Oswald. Criteria for hierarchical bases in Sobolev spaces. *Appl. Comput. Harmon. Anal.*, 6:219–251, 1999.
- [10] M. Lounsbery. *Multiresolution Analysis for Surfaces of Arbitrary Topological Type*. Ph.D. thesis, University of Washington, Department of Computer Science and Engineering, 1994.
- [11] M. Lounsbery, T. D. DeRose, and J. Warren. Multiresolution analysis for surfaces of arbitrary topological type. *ACM Trans. on Graphics*, 16(1):34–73, 1997.
- [12] P. Oswald. Stable subspace splittings for Sobolev spaces and domain decomposition algorithms. In D. E. Keyes and J. Xu, editors, *Domain Decomposition Methods in Scientific and Engineering Computing: Proc. Seventh Int. Conf. on Domain Decomposition*, Oct.

27–30, 1993, *Pennsylvania State University*, volume 180 of *Contemporary Mathematics*, pages 87–98, Providence, 1994. Amer. Math. Soc.

- [13] W. Sweldens. The lifting scheme: A construction of second generation wavelets. *SIAM J. Math. Anal.*, 29(2):511–546, 1997.
- [14] P. S. Vassilevski and J. Wang. Stabilizing the hierarchical basis by approximate wavelets, I: Theory. *Num. Lin. Alg. Appl.*, 4(2):103–126, 1997.
- [15] P. S. Vassilevski and J. Wang. Stabilizing the hierarchical basis by approximate wavelets, II: Implementation and numerical results. *SIAM J. Sci. Comput.*, 20:490–514, 1999.
- [16] J. Xu. Iterative methods by space decomposition and subspace correction. *SIAM Review*, 34(4):581–613, 1992.
- [17] H. Yserentant. On the multi-level splitting of finite element spaces. *Numer. Math.*, 49:379–412, 1986.

Jo Simoens
Katholieke Universiteit Leuven
Department of Computer Science
Celestijnenlaan 200A
B-3001 Leuven, Belgium
jo.simoens@cs.kuleuven.ac.be
<http://www.cs.kuleuven.ac.be/~jo/>