Cubature formulas of a nonalgebraic degree of precision

Ronald Cools and Juan Carlos Santos-León

Report TW 301, February 2000

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Abstract

In this paper, we deal with the construction of cubature formulas that are exact for polynomials and also for polynomials multiplied by r, where r is the Euclidean distance to the origin. A general lower bound for the number of nodes for a specified degree of precision is given. This bound is improved for centrally symmetric integrals. A set of constraints (consistency conditions) is introduced for the construction of fully symmetric formulas. For one dimension and arbitrary degree, it is shown that the lower bound is sharp for centrally symmetric integrals. For higher dimensions, this is only illustrated for low degrees.

Keywords : Cubature formulas, lower bounds, fully symmetric, consistency conditions
AMS(MOS) Classification : Primary : 65D32

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Abstract

In this paper, we deal with the construction of cubature formulas that are exact for polynomials and also for polynomials multiplied by r, where r is the Euclidean distance to the origin. A general lower bound for the number of nodes for a specified degree of precision is given. This bound is improved for centrally symmetric integrals. A set of constraints (consistency conditions) is introduced for the construction of fully symmetric formulas. For one dimension and arbitrary degree, it is shown that the lower bound is sharp for centrally symmetric integrals. For higher dimensions, this is only illustrated for low degrees.

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1 Introduction

Consider an integral

$$I[f] := \int_\Omega f(\mathbf{x})\omega(\mathbf{x})d\mathbf{x} \tag{1.1}$$

where $\Omega \subset I\!R^n$ and $\omega(\mathbf{x}) \geq 0$, $\mathbf{x} \in \Omega$. In this paper, we deal with the problem of constructing cubature formulas for the integral (1.1) of the form

$$Q[f] := \sum_{j=1}^{N} w_j f(\mathbf{y}^{(j)}), \;\; w_j \in I\!R,\; \mathbf{y}^{(j)} \in \Omega,\; j = 1, \ldots, N, \tag{1.2}$$

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such that $Q[f] = I[f]$ when f is a polynomial or a polynomial multiplied by r, where

$$r(\mathbf{x}) := ||\mathbf{x}||_2 = \sqrt{x_1^2 + \cdots + x_n^2}.$$

The need for such rules appears, e.g., in the solution via pseudospectral methods, in particular collocation, of atomic and molecular Schrödinger equations [4].

We will denote by P^n the vector space of algebraic polynomials in n variables and by P^n_k, the subspace of such polynomials of degree at most k. In order to be as general as possible in the construction of rules Q, we consider for a given $\ell \in I\!N$ the sequence of vector spaces $R^{n,\ell}_0 \subset R^{n,\ell}_1 \subset R^{n,\ell}_2 \subset \cdots$ where $R^{n,\ell}_k$ is defined through

$$\begin{array}{lll} R^{n,\ell}_k := P^n_k + rP^n_{k-\ell}, & \text{if } k \geq \ell & \\ & & k = 0, 1, \ldots \\ R^{n,\ell}_k := P^n_k, & \text{if } k < \ell. & \end{array} \qquad (1.3)$$

In the rest of this paper we will always assume that $n, \ell, k, s, N \in I\!N$.

For a recent survey on the construction of cubature formulas see [3]. There it was suggested that some results about cubature formulas that are exact for a vector space of polynomials, can be generalized to other vector spaces. In this paper we will consider cubature formulas that are exact on the vector space $R^{n,\ell}_k$. It is our aim to generalize some of the known theoretical results on lower bounds and consistency conditions, i.e., the basis of the two most popular approaches to construct cubature formulas. The actual construction of cubature formulas is beyond the scope of this paper. Some are included to illustrate the theory.

Our interest is, for a given ℓ, to construct cubature formulas that are exact for all elements of $R^{n,\ell}_k$, in particular for $k \geq \ell$. It will be shown that $R^{n,\ell}_k$ $(k \geq \ell)$ has different properties for ℓ odd or even. Since we consider $\mathbf{y}^{(j)} \in \Omega$, $j = 1, \ldots, N$, we will identify the functions in (1.3) which are equal in Ω. Furthermore, we only consider strictly positive integrals I, i.e., $I[f] > 0$ for all functions $f \geq 0$ in Ω, not equivalent to zero and $f \in R^{n,\ell}_k$ for all n, ℓ, $k \in I\!N$.

Definition 1.1 *For a given ℓ we will say that the cubature formula Q has degree d if*

$$Q[f] = I[f], \text{ for all } f \in R^{n,\ell}_d. \qquad (1.4)$$

Definitions 1.1 1. *The integral I is called centrally symmetric if $\mathbf{x} \in \Omega$ then $-\mathbf{x} \in \Omega$ and $\omega(\mathbf{x}) = \omega(-\mathbf{x})$. A cubature formula Q is called centrally symmetric if it can be written in the form*

$$Q[f] = \sum_{j=1}^{M} w_j[f(\mathbf{y}^{(j)}) + f(-\mathbf{y}^{(j)})].$$

2. *For $\mathbf{y} = (y_1, \ldots, y_n) \in I\!R^n$, let $G(\mathbf{y}) := \{(s_1 y_{p_1}, \ldots, s_n y_{p_n}) : s_i \in \{-1, 1\}, i = 1, \ldots, n$ and $p_1, \ldots, p_n$ is any permutation of $1, \ldots, n\}$.*

The integral I is called fully symmetric if $\mathbf{x} \in \Omega$ then $G(\mathbf{x}) \subset \Omega$ and $\omega(\mathbf{x}) = \omega(\mathbf{y})$ for all $\mathbf{y} \in G(\mathbf{x})$. A cubature formula Q is called fully symmetric if it can be written in the form

$$Q[f] = \sum_{j=1}^{M} w_j [\sum_{\mathbf{y} \in G(\mathbf{y}^{(j)})} f(\mathbf{y})]. \tag{1.5}$$

Note that if a basis of $R_k^{n,\ell}$ is fixed, the construction of a cubature formula directly from (1.4) requires the solution of a nonlinear (and if $k \geq \ell$ nonalgebraic) system, of dim $R_k^{n,\ell}$ equations with $N(n+1)$ unknowns: the weights and the coordinates of the nodes.

This paper is arranged in the following way. In Section 2, we generalize the known lower bounds (i.e., for $\ell > k$) for the number of nodes (see, e.g., [3]) to other values of ℓ. Most known cubature formulas (for $\ell > k$) were constructed using consistency conditions for the defining system of polynomial equations. Section 3 is devoted to generalize this approach and is based on [5]. Using consistency conditions derived for fully symmetric cubature formulas that are exact for polynomials, we derive a set of consistency conditions for (1.5). In Section 4 we illustrate the theoretical results with the practical construction of cubature formulas. As an illustration, we have fixed $\ell = 2$. For one dimension and arbitrary degree it is shown that minimal formulas, i.e. formulas that attain our lower bound, can be constructed. For higher dimensions, minimal formulas of degree 3 are considered.

In our examples we will use the weight functions $\omega_1(\mathbf{x}) = e^{-r}$, $\omega_2(\mathbf{x}) = e^{-r^2}$ and $\omega_3(\mathbf{x}) = e^{-r}/r$ where $\mathbf{x} \in \Omega = I\!R^n$. The moments that will be needed in the examples are given in Tables 1 and 2 (Γ denotes the Gamma-function). These are particular cases of the following general formula:

$$\int_{I\!R^n} x_1^{i_1} \cdots x_n^{i_n} r^k e^{-r^\alpha} dx_1 \ldots dx_n = \frac{2\Gamma(\frac{i_1+1}{2}) \cdots \Gamma(\frac{i_n+1}{2}) \Gamma(\frac{i_1+\cdots+i_n+n+k}{\alpha})}{\alpha \Gamma(\frac{i_1+\cdots+i_n+n}{2})},$$

where $\alpha \in I\!R$, $\alpha > 0$ and the $i_j, j = 1, \ldots, n$, are even positive integers. The above integral is equal to zero when an i_j is an odd positive integer.

2 Lower bounds

In [3] a survey is given of the known results on lower bounds for cubature formulas that are exact on P_k^n. In this section we will generalize some of these results to the space $R_k^{n,\ell}$. Some of these generalizations are straightforward, but others are not as they depend on ℓ in a particular way.

Theorem 2.1 *If the cubature formula Q given by (1.2) is exact for all functions of $R_{2k}^{n,\ell}$ with $\ell \geq 1$, then the number of nodes $N \geq$ dim $R_k^{n,\ell}$.*

Proof. Suppose a cubature formula Q of degree $2k$ with $N <$ dim $R_k^{n,\ell}$ exists. Then, we can find $f \in R_k^{n,\ell}$ not equivalent to zero, vanishing at all the nodes of

Table 1: Moments for 2-dimensional integrals

	$\omega_1(\mathbf{x}) = e^{-r}$	$\omega_2(\mathbf{x}) = e^{-r^2}$	$\omega_3(\mathbf{x}) = e^{-r}/r$
$I[1] =$	2π	π	2π
$I[r] =$	4π	$\frac{\pi\sqrt{\pi}}{2}$	2π
$I[x^2] = I[y^2] =$	6π	$\frac{\pi}{2}$	2π
$I[x^2 r] = I[y^2 r] =$	24π	$\frac{3\pi\sqrt{\pi}}{8}$	6π
$I[x^2 y^2] =$	30π	$\frac{\pi}{4}$	6π
$I[x^4] = I[y^4] =$	90π	$\frac{3\pi}{4}$	18π
$I[x^i y^j] = I[x^i y^j r] =$	0 if i or j is odd		

Table 2: Moments for n-dimensional integrals

	$\omega_1(\mathbf{x}) = e^{-r}$	$\omega_2(\mathbf{x}) = e^{-r^2}$	$\omega_3(\mathbf{x}) = e^{-r}/r$
$I[1] =$	$\frac{2\pi^{\frac{n}{2}}(n-1)!}{\Gamma(\frac{n}{2})}$	$\pi^{\frac{n}{2}}$	$\frac{2\pi^{\frac{n}{2}}(n-2)!}{\Gamma(\frac{n}{2})}$
$I[r] =$	$\frac{2\pi^{\frac{n}{2}}n!}{\Gamma(\frac{n}{2})}$	$\frac{\pi^{\frac{n}{2}}\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}$	$\frac{2\pi^{\frac{n}{2}}(n-1)!}{\Gamma(\frac{n}{2})}$
$I[x_i^2] =$	$\frac{2\pi^{\frac{n}{2}}(n+1)!}{n\Gamma(\frac{n}{2})}$	$\frac{\pi^{\frac{n}{2}}}{2}$	$\frac{2\pi^{\frac{n}{2}}(n-1)!}{\Gamma(\frac{n}{2})}$

this cubature formula by solving a system of linear equations. Since $\ell \geq 1$, $f^2 \in R_{2k}^{n,\ell}$ and hence $I[f^2] = Q[f^2] = 0$, in contradiction with our assumption on the integral I. $\square$

Theorem 2.2 *If the cubature formula Q given by (1.2) is exact for all functions of $R_d^{n,\ell}$ with $\ell \geq 1$ and $d \geq 1$, then it has at least* dim $R_k^{n,\ell}$ *positive weights, where $k = \lfloor \frac{d}{2} \rfloor$, the integer part of $\frac{d}{2}$.*

Proof. Let $d = 1$. Then $Q[1] = \sum_{j=1}^{N} w_j = I[1] > 0$. So, there must be at least one positive weight and the Theorem holds.

Let now $d \geq 2$. Suppose that there are at most $t <$ dim $R_k^{n,\ell}$ positive weights, $w_1, \ldots, w_t$, with corresponding nodes $\mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(t)}$. There exists an $f \in R_k^{n,\ell}$ not equivalent to zero and vanishing at these nodes. Taking into account that $\ell \geq 1$, $I[f^2] = Q[f^2] \leq 0$, and this is in contradiction with our assumption on the integral I. $\square$

Note that for $\ell = 0$, the proofs of Theorem 2.1 and Theorem 2.2 are not valid. Indeed, for a given $k \geq 0$, $\exists f \in R_k^{n,0} : f^2 \notin R_{2k}^{n,0}$. Combining Theorem 2.1 and Theorem 2.2 one obtains the following.

Corollary 2.1 *If the cubature formula (1.2) attains the lower bound of Theorem 2.1, then all its weights are positive.*

Theorem 2.1 gives a general lower bound for the number of nodes. This bound can be improved under additional assumptions on the integral and on the cubature formula. This is the case when central symmetry is assumed. In the rest of this section we deal with cubature formulas Q of the form (1.2) and integrals I which are centrally symmetric.

Let Q be of odd degree $d = 2s + 1$ and define

$$\left.\begin{array}{l} h_{2k}^{n,\ell} := \{f \in R_{2k}^{n,\ell} \ : \ f(\mathbf{x}) = f(-\mathbf{x})\} \\[2ex] h_{2k+1}^{n,\ell} := \{f \in R_{2k+1}^{n,\ell} \ : \ f(\mathbf{x}) = -f(-\mathbf{x})\}. \end{array}\right. \qquad k = 0, 1, 2, \ldots$$

$h_{2k}^{n,\ell}$ contains only even functions and $h_{2k+1}^{n,\ell}$ contains only odd functions. From here on, the value zero is considered an even number. Note that for all even values of ℓ the direct sum of these spaces

$$h_{s+1}^{n,\ell} \oplus h_s^{n,\ell} = R_{s+1}^{n,\ell}, \tag{2.1}$$

but for odd values of ℓ this is not true. Indeed, if we denote by $\langle f_1, f_2, \ldots \rangle$ the vector space generated by the functions $f_1, f_2, \ldots$ and if $n = 2,\ \ell = 1$ and $s = 1$ then $h_2^{2,1} \oplus h_1^{2,1} = \langle 1, x^2, xy, y^2, r \rangle \oplus \langle x, y \rangle \neq R_2^{2,1} = \langle 1, x, y, x^2, xy, y^2, r, rx, ry \rangle$.

Define

$$O_{s+1}^{n,\ell} := \{f \in R_{s+1}^{n,\ell} \ : \ g \in R_s^{n,\ell} \ \Rightarrow \ I[fg] = 0\}.$$

Lemma 2.1 *For even values of ℓ it holds that $O_{s+1}^{n,\ell} \subset h_{s+1}^{n,\ell}$.*

Proof. Let $f \in O_{s+1}^{n,\ell}$. The function f can be split in the form $f = f_e + f_o$ with f_e an even function and f_o odd.

Let $\overline{f}(\mathbf{x}) := f(-\mathbf{x})$, then by virtue of symmetry $\overline{f} \in O_{s+1}^{n,\ell}$. Since $f + \overline{f} = 2f_e$ and $f - \overline{f} = 2f_o$, one obtains $f_e,\ f_o \in O_{s+1}^{n,\ell}$.

Consider first $s = 2t$. Then $f \in R_{2t+1}^{n,\ell}$ and by virtue of (2.1), $f_e \in h_{2t}^{n,\ell}$ and $f_o \in h_{2t+1}^{n,\ell}$. The function f_e is equivalent to zero in Ω. Indeed, since $I[f_e g] = 0$ for all $g \in R_{2t}^{n,\ell}$, for $g = f_e$ is $I[f_e^2] = 0$ and hence $f_e \equiv 0$ in Ω. Thus, $f = f_o \in h_{2t+1}^{n,\ell} = h_{s+1}^{n,\ell}$.

For $s = 2t + 1$, one can proceed analogously. □

In the following example we consider ℓ to be an odd number. Its purpose is to show that an orthogonal function in $R_k^{n,\ell}$ may not be even or odd and thus Lemma 2.1 has no counterpart for odd values of ℓ. This is an important difference with the polynomial case.

Example 2.1 *Consider $R_2^{2,1}$ and let the integral I be given by (1.1) where $\Omega = \mathbb{R}^2$ and $\omega(\mathbf{x}) = \omega_3(x, y)$. After some calculations it is deduced*

$$O_2^{2,1} = \{f \ : \ f(x,y) = \alpha_1 x^2 + \alpha_2 xy + \alpha_3 y^2 - 3\alpha_4 x - 3\alpha_5 y + \alpha_1 +$$

$$\alpha_3 - 2(\alpha_1 + \alpha_3) r + \alpha_4 rx + \alpha_5 ry,\ \alpha_i \in \mathbb{R},\ i = 1, \ldots, 5\}.$$

A function in $O_2^{2,1}$ may not be even or odd. E.g., the orthogonal function $f(x, y) = x^2 - 3x + 1 - 2r + rx \in O_2^{2,1}$ is not even and is not odd. □

Recall that we have identified the functions of $R_k^{n,\ell}$ which are equal in Ω. Since $\mathbf{y}^{(j)} \in \Omega,\ j = 1, \ldots, N$, the linear functionals

$$\begin{array}{rcl} L_j : R_{s+1}^{n,\ell} & \rightarrow & I\!R \\ f & \mapsto & L_j[f] = f(\mathbf{y}^{(j)}) \end{array} \tag{2.2}$$

are well defined.

Let $\{F_1, F_2, \ldots\}$ be a set of functionals. With $\langle F_1, F_2, \ldots \rangle$ we denote the vector space generated by this set. With $F\,|_P$ we denote the restriction of the functional F to the vector space P.

We summarize some elementary results on linear functionals which we will need, in the following lemma.

Lemma 2.2 *[7, section 2.6] Let $\mathcal{V}$ be a vector space of finite dimension and $\mathcal{F}$ the vector space of linear functionals on $\mathcal{V}$. $\mathcal{V}$ and $\mathcal{F}$ are isomorphic. Let S be a subspace of $\mathcal{V}$ and S^o the subspace of $\mathcal{F}$ consisting of all those linear functionals in $\mathcal{F}$ which map every element of S to zero. Then* $\dim S + \dim S^o = \dim \mathcal{V} = \dim \mathcal{F}$*.*

Lemma 2.3 *For a set of N functionals $L_j, j = 1, \ldots, N$, defined as in (2.2)*

$$\begin{array}{c} \dim \left\langle L_1 \Big|_{h_{s+1}^{n,\ell}}, \ldots, L_N \Big|_{h_{s+1}^{n,\ell}} \right\rangle \geq \\ \dim\langle L_1 \Big|_{h_s^{n,\ell}}, \ldots, L_N \Big|_{h_s^{n,\ell}} \rangle + \left\{ \begin{array}{rl} -1, & if\ zero\ is\ a\ node\ and\ s\ even \\ 0, & otherwise. \end{array} \right. \end{array} \tag{2.3}$$

Proof. Let

$$\begin{array}{rcl} L : R_{s+1}^{n,\ell} & \rightarrow & I\!R \\ f & \mapsto & L[f] = f(\mathbf{0}). \end{array}$$

Take m functionals, say $L_1, \ldots, L_m$ in $\{L_1, \ldots, L_N\} \setminus \{L\}$, such that, when they are restricted to $h_s^{n,\ell}$ they form a basis of $\langle L_1 \Big|_{h_s^{n,\ell}}, \ldots, L_N \Big|_{h_s^{n,\ell}} \rangle$. If zero is a node then L is one of the functionals L_j which may be necessary to get a basis. If this is the case, add the functional L to a set $\{L_1, \ldots, L_m\}$ to get a basis. In any case, one can find a $q \in P_1^n$ such that $L[q] = 0,\ L_j[q] \neq 0,\ j = 1, \ldots, m$. Then by taking into account $L_i(qf) = L_i(q)L_i(f)$ and $L_i(q) \neq 0,\ i = 1, \ldots, m$, we obtain that $L_1, \ldots, L_m$ are linearly independent in $qh_s^{n,\ell}$. Since $L[q] = 0$ we know that $q \in P_1^n \setminus P_0^n$ and thus $qh_s^{n,\ell} \subset h_{s+1}^{n,\ell}$. Hence, $L_1, \ldots, L_m$ are linearly independent in $h_{s+1}^{n,\ell}$. We have obtained

$$\dim\langle L_1 \Big|_{h_{s+1}^{n,\ell}}, \ldots, L_N \Big|_{h_{s+1}^{n,\ell}} \rangle \ \geq$$

$$\dim\langle L_1 \Big|_{h_s^{n,\ell}}, \ldots, L_N \Big|_{h_s^{n,\ell}} \rangle + \left\{ \begin{array}{rl} -1, & if\ zero\ is\ a\ node \\ 0, & otherwise. \end{array} \right.$$

Now one has to take into account that even when zero is a node, L is not an element of the basis when s is odd. Indeed when L is restricted to $h_s^{n,\ell}$ it is the zero functional. This completes the proof. □

Theorem 2.3 *For even values of $\ell \geq 2$ a cubature formula of degree $2s+1$ requires*

$$N \geq 2 \dim\, h_s^{n,\ell} + \begin{cases} -1, & \textit{if zero is a node and } s \textit{ even} \\ 0, & \textit{otherwise.} \end{cases} \tag{2.4}$$

Proof. Because $\ell \geq 2$, if $f \in R_{s+1}^{n,\ell}$ vanishes in all nodes of the cubature formula then $f \in O_{s+1}^{n,\ell}$. By virtue of Lemma 2.1, $f \in h_{s+1}^{n,\ell}$. It is known, by virtue of Lemma 2.2, that

$$\dim\langle L_1 \Big|_{R_{s+1}^{n,\ell}}, \ldots, L_N \Big|_{R_{s+1}^{n,\ell}} \rangle \;=\; \dim R_{s+1}^{n,\ell} - \dim S \tag{2.5}$$

where

$$S = \{f \in R_{s+1}^{n,\ell} : f(\mathbf{y}^{(1)}) = \cdots = f(\mathbf{y}^{(N)}) = 0\}.$$

Since in our situation we also have

$$S = \{f \in h_{s+1}^{n,\ell} : f(\mathbf{y}^{(1)}) = \cdots = f(\mathbf{y}^{(N)}) = 0\},$$

we know that

$$\dim\langle L_1 \Big|_{h_{s+1}^{n,\ell}}, \ldots, L_N \Big|_{h_{s+1}^{n,\ell}} \rangle = \dim h_{s+1}^{n,\ell} - \dim S \tag{2.6}$$

and

$$\dim\langle L_1 \Big|_{h_s^{n,\ell}}, \ldots, L_N \Big|_{h_s^{n,\ell}} \rangle = \dim h_s^{n,\ell}. \tag{2.7}$$

Combining (2.1), (2.5) and (2.6) gives

$$\begin{aligned} \dim\langle L_1 \Big|_{R_{s+1}^{n,\ell}}, \ldots, L_N \Big|_{R_{s+1}^{n,\ell}} \rangle &= \dim h_s^{n,\ell} + \dim h_{s+1}^{n,\ell} - \dim S \\ &= \dim h_s^{n,\ell} + \dim\langle L_1 \Big|_{h_{s+1}^{n,\ell}}, \ldots, L_N \Big|_{h_{s+1}^{n,\ell}} \rangle. \end{aligned}$$

Combining this with

$$N \geq \dim\langle L_1 \Big|_{R_{s+1}^{n,\ell}}, \ldots, L_N \Big|_{R_{s+1}^{n,\ell}} \rangle$$

and Lemma 2.3 and (2.7) completes this proof. □

In Section 4 we will show that this bound is sharp for $n = \ell = 2$. For this case it cannot be improved without taking into account more information about the region and weight function. Observe that we had to exclude $\ell = 0$ again because the first sentence in the above proof does not hold in that case. This is illustrated in the following example.

Example 2.2 *Consider $n = 2$, $\ell = 0$, $d = 3$ and the integral I given by (1.1) where $\omega(\mathbf{x}) = \omega_1(x, y)$. After some calculations, the following cubature formula of degree $d = 3$ is obtained:*

$$\begin{aligned} Q[f] = w_1[f(a_1,0) + f(-a_1,0) + f(0,a_1) + f(0,-a_1)]+ \\ w_2[f(a_2,0) + f(-a_2,0) + f(0,a_2) + f(0,-a_2)] \end{aligned} \tag{2.8}$$

where

$$a_1 = 3 + \sqrt{3},\ w_1 = \frac{3-\sqrt{3}}{12}\pi,\ a_2 = 3 - \sqrt{3},\ w_2 = \frac{3+\sqrt{3}}{12}\pi.$$

The function $f(x,y) = (x^2 + y^2 - a_1^2)(r - a_2) \in R_2^{2,0}$ *vanishes in all nodes of cubature formula (2.8), nevertheless* $f \notin O_2^{2,0}$. *Indeed, the function* $r \in R_1^{2,0}$ *and* $I[fr] = 24\pi \neq 0$. $\square$

For 1 dimension and even $\ell \geq 2$, Theorem 2.3 can be improved. Indeed, let $s = 2t + 1$. Hence, the quadrature formula is exact in $R_{2s+1}^{1,\ell} = R_{4t+3}^{1,\ell}$. One can write

$$\dim\langle L_1 \Big|_{h_{s+1}^{1,\ell}}, \ldots, L_N \Big|_{h_{s+1}^{1,\ell}} \rangle = \dim\langle \Big\{ L_j \Big|_{\langle 1, x^2, \ldots, x^{2(t+1)}, |x|, |x|x^2, \ldots, |x|x^{2(t+1)-\ell} \rangle} \Big\}_{j=1}^{N} \rangle \geq$$

$$\dim\langle \Big\{ L_j \Big|_{\langle 1, x^2, \ldots, x^{2t}, |x|, \ldots, |x|x^{2(t+1)-\ell} \rangle} \Big\}_{j=1}^{N} \rangle = \begin{cases} 2t + 3 - \frac{\ell}{2}, & \text{if } \ell \leq s+1 \\ t + 1, & \text{if } \ell > s+1. \end{cases}$$

To obtain the last equality, take into account that the only function f in $\langle 1, x^2, \ldots, x^{2t},\ |x|, |x|x^2, \ldots, |x|x^{2(t+1)-\ell} \rangle$ which vanishes in all nodes of the quadrature formula is $f \equiv 0$. Otherwise, $Q[f^2] = I[f^2] = 0$ and one has a contradiction with our assumption on the integral I. Thus

$$\dim\langle L_1 \Big|_{h_{s+1}^{1,\ell}}, \ldots, L_N \Big|_{h_{s+1}^{1,\ell}} \rangle \geq \begin{cases} 2t + 3 - \frac{\ell}{2} = \dim\ h_s^{1,\ell} + 1, & \text{if } \ell \leq s+1 \\ t + 1 = \dim\ h_s^{1,\ell}, & \text{if } \ell > s+1. \end{cases} \tag{2.9}$$

Following the proof of Theorem 2.3 and taking into account equation (2.9) one obtains

$$\begin{aligned} N \geq \dim\langle L_1 \Big|_{R_{s+1}^{1,\ell}}, \ldots, L_N \Big|_{R_{s+1}^{1,\ell}} \rangle = \dim\langle L_1 \Big|_{h_{s+1}^{1,\ell}}, \ldots, L_N \Big|_{h_{s+1}^{1,\ell}} \rangle + \\ \dim\langle L_1 \Big|_{h_s^{1,\ell}}, \ldots, L_N \Big|_{h_s^{1,\ell}} \rangle \geq \begin{cases} 2\dim\ h_s^{1,\ell} + 1, & \text{if } \ell \leq s+1 \\ 2(t+1) = 2\dim\ h_s^{1,\ell}, & \text{if } \ell > s+1. \end{cases} \end{aligned}$$

Therefore, we have proved

Theorem 2.4 *For 1 dimension and* $\ell \geq 2$ *even, the number of nodes* N *in a quadrature formula of degree* $d = 2s + 1$ *for a centrally symmetric integral is bounded by*

$$N \geq 2\dim\ h_s^{1,\ell} + \begin{cases} -1, & \textit{if zero is a node and } s \textit{ even} \\ 1, & \textit{if } s \textit{ is odd and } \ell \leq s+1 \\ 0, & \textit{otherwise.} \end{cases}$$

In Section 4 we will show that this bound is sharp for $\ell = 2$: it cannot be improved without further information about the region and the weight function.

3 Consistency conditions for fully symmetric cubature formulas

In this section we consider fully symmetric cubature formulas (1.5) for a fully symmetric integral I. Note that for even values of ℓ, if a fully symmetric formula has degree $d = 2s$ then it also has degree $d = 2s + 1$. This is not true if ℓ is odd.

Example 3.1 *Consider* $n = 2$, $\ell = 1$, $d = 2$ *and the integral* I *given by (1.1) where* $\omega(\mathbf{x}) = \omega_2(x, y)$. *The cubature formula*

$$Q[f] = w_1 f(0,0) + w_2[f(\tau,0) + f(-\tau,0) + f(0,\tau) + f(0,-\tau)]$$

where

$$w_1 = \pi(4-\pi)/4,\ w_2 = \pi^2/16,\ \tau = 2/\sqrt{\pi},$$

has degree $d = 2$ *but not degree* $d = 3$. *Indeed,* $Q[x^2 r] = \sqrt{\pi} \neq I[x^2 r] = 3\pi\sqrt{\pi}/8$.
□

If f is a monomial and $\delta \in \{0,1\}$ then $I[r^\delta f] = Q[r^\delta f] = 0$ if there is an odd power in a variable of f. If there are only even powers in the variables of f, the integral and the cubature formula depend on such powers, and not on the ordering of the variables. Thus, one obtains

Theorem 3.1 *A fully symmetric cubature formula* Q *has degree* d *if* $Q[f] = I[f]$ *for all functions* f *of the form*

$$f(\mathbf{x}) = x_1^{2j_1} \cdots x_n^{2j_n},\ j_1 \geq \cdots \geq j_n \geq 0,\ \sum_{i=1}^{n} j_i \leq \lfloor \frac{d}{2} \rfloor \tag{3.1}$$

and

$$f(\mathbf{x}) = r x_1^{2j_1} \cdots x_n^{2j_n},\ j_1 \geq \cdots \geq j_n \geq 0,\ \sum_{i=1}^{n} j_i \leq \lfloor \frac{d-\ell}{2} \rfloor,\ \textit{if } d \geq \ell. \tag{3.2}$$

The rest of this section builds upon the work of Patrick Keast and James Lyness [5], who extended and formalized the work started by Francis Mantel and Philip Rabinowitz [6].

Definition 3.1 *[5] A basic rule* $\mathcal{R}$ *is defined by*

$$\mathcal{R}(\alpha_1, \ldots, \alpha_n)[f] := \frac{1}{2^n n!} \sum_{s_1} \cdots \sum_{s_n} \sum_{P} f(s_1 \alpha_{p_1}, \ldots, s_n \alpha_{p_n}) \tag{3.3}$$

where $s_i \in \{-1, 1\}$ *and* $\sum_P$ *indicates a sum over the permutations* $p_1, \ldots, p_n$ *of* $1, \ldots, n$. *The set of numbers* $\alpha_1, \ldots, \alpha_n$ *are called generators of* $\mathcal{R}$ *and their ordering is immaterial.*

Definition 3.2 *[5] The class* $[\mathbf{n}] = [n_1, \ldots, n_k]$ *of a basic rule includes all basic rules which may be expressed in the form*

$$\mathcal{R}(\alpha_1, \ldots, \alpha_1, \ldots, \alpha_k, \ldots, \alpha_k, 0, \ldots, 0)$$

where $\alpha_1, \ldots, \alpha_k$ *are distinct nonzero generators and* α_i *appears* n_i *times,* $i = 1, \ldots, k$. *The rest of the* n *components are filled in, if this is the case, with zeros. The class* $[.]$ *is the one in which all generators are equal to zero.*

Note that a fully symmetric cubature formula (1.5) can be written in terms of basic rules (3.3). The number of distinct basic rules of class $[\mathbf{n}]$ in a given cubature formula is denoted by $K[\mathbf{n}]$. Such numbers are called the *rule structure parameters.* One considers $K[.] = 0$ or 1. The general expression for a fully symmetric cubature formula Q is then given by

$$Q = \sum_{\mu=0}^{D} \sum_{i=1}^{K_\mu} w_{i,\mu} \mathcal{R}_{i,\mu} \tag{3.4}$$

where $\mathcal{R}_{i,\mu}$ is a basic rule of class $[\mathbf{n}_\mu] = [n_1, \ldots, n_k]$, i.e.,

$$\mathcal{R}_{i,\mu} = \mathcal{R}(\underbrace{\alpha_1^{(i,\mu)}, \ldots, \alpha_1^{(i,\mu)}}_{n_1}, \ldots, \underbrace{\alpha_k^{(i,\mu)}, \ldots, \alpha_k^{(i,\mu)}}_{n_k}, 0, \ldots, 0),$$

where $\alpha_j^{(i,\mu)}$ appear n_j times, $j = 1, \ldots, k$. $K_\mu = K[\mathbf{n}_\mu]$ is the rule structure parameter, indicating the number of basic rules of class $[\mathbf{n}_\mu]$. $D+1$ is the number of distinct classes possible and $w_{i,\mu}$ are the weights.

Each class introduces a certain number of unknowns and nodes in the cubature formula. This is summarized in Tables 3 and 4 for one and two dimensions respectively.

Table 3: Rule structure parameters in 1 dimension

rule structure parameters	generator	number of unknowns	number of nodes introduced by class	unknowns
$K_0 := K[.]$	0	1	1	weight
$K_1 := K[1]$	α	2	2	α, weight

Table 4: Rule structure parameters in 2 dimensions

rule structure parameters	generator	number of unknowns	number of nodes introduced by class	unknowns
$K_0 := K[.]$	(0,0)	1	1	weight
$K_1 := K[1]$	$(\alpha_1,0)$	2	4	α_1, weight
$K_2 := K[2]$	(α_1,α_1)	2	4	α_1, weight
$K_3 := K[1,1]$	(α_1,α_2)	3	8	α_1, α_2, weight

Before one can use Theorem 3.1 to calculate the nodes and weights in (3.4), one has to decide its form, that is, choose the different classes $[\mathbf{n}]$ and the rule structure parameters $K[\mathbf{n}]$. This is an important point. Depending on it, the resulting system may be easier to solve or the cubature formula has less nodes. In this sense, certain inequalities for the $K[\mathbf{n}]$, which are called *consistency conditions*, are used. They guarantee to obtain a system where the number of unknowns is greater than or equal to the number of equations in each subsystem. Such a system is in general nonlinear. Furthermore, as the number of dimensions n and the degree of precision d increase, it is also nonalgebraic. Even if the proposed form for Q satisfies the consistency conditions, it is not assured that the system has a real solution. Rather closely to the polynomial case, see [5], we have derived the set of such consistency conditions for the cubature formulas we are interested in.

We introduce the space $\overline{G}_d^{n,\ell}$ as the space spanned by the functions in (3.1) and (3.2), i.e.,

$$\begin{aligned} \overline{G}_d^{n,\ell} &:= \overline{S}^n_{\lfloor \frac{d}{2} \rfloor} \oplus r\overline{S}^n_{\lfloor \frac{d-\ell}{2} \rfloor} \quad && \text{if } d \geq \ell \\ \overline{G}_d^{n,\ell} &:= \overline{S}^n_{\lfloor \frac{d}{2} \rfloor} && \text{if } d < \ell \end{aligned} \tag{3.5}$$

where

$$\overline{S}^n_q := \langle x_1^{2j_1} \cdots x_n^{2j_n},\ j_1 \geq \cdots \geq j_n \geq 0,\ \sum_{i=1}^n j_i \leq q \rangle.$$

Consider

$$\overline{M}_d^{n,\ell}[\mathbf{n}] := \langle f \text{ such that } f \in \overline{G}_d^{n,\ell} \text{ and } \mathcal{R}[f] = 0 \text{ for all rules } \mathcal{R} \in [\mathbf{n}] \rangle$$

and

$$\overline{N}^n_q[\mathbf{n}] := \langle f \text{ such that } f \in \overline{S}^n_q \text{ and } \mathcal{R}[f] = 0 \text{ for all rules } \mathcal{R} \in [\mathbf{n}] \rangle.$$

Lemma 3.1 *It holds that*

$$\begin{aligned} \overline{M}_d^{n,\ell}[\mathbf{n}] &= \overline{N}^n_{\lfloor \frac{d}{2} \rfloor}[\mathbf{n}] \oplus r\overline{N}^n_{\lfloor \frac{d-\ell}{2} \rfloor}[\mathbf{n}] \quad && \textit{if } d \geq \ell \\ \overline{M}_d^{n,\ell}[\mathbf{n}] &= \overline{N}^n_{\lfloor \frac{d}{2} \rfloor}[\mathbf{n}] && \textit{if } d < \ell. \end{aligned}$$

Proof. For $d < \ell$, the equality is clear.

Let $d \geq \ell$. The inclusion in the left direction follows from $\mathcal{R}[f + rg] = \mathcal{R}[f] + \mathcal{R}[r]\mathcal{R}[g]$ for all polynomials f and g. We prove now the inclusion in the right direction. Let $f \in \overline{M}_d^{n,\ell}[\mathbf{n}]$. Then, $f \in \overline{G}_d^{n,\ell}$ and there exist $f_1 \in \overline{S}^n_{\lfloor \frac{d}{2} \rfloor}$ and $f_2 \in \overline{S}^n_{\lfloor \frac{d-\ell}{2} \rfloor}$ such that $f = f_1 + rf_2$. One has $\mathcal{R}[f_1] + \mathcal{R}[rf_2] = 0$ for all rules $\mathcal{R} \in [\mathbf{n}]$. Let $[\mathbf{n}] = [n_1, \ldots, n_k]$. Then, $\mathcal{R}[f_1] = -\sqrt{n_1\alpha_1^2 + \cdots + n_k\alpha_k^2}\mathcal{R}[f_2]$ for arbitrary but distinct $\alpha_1, \ldots, \alpha_k$ where $\mathcal{R}[f_1]$ and $\mathcal{R}[f_2]$ are polynomials in the variables $\alpha_1, \ldots, \alpha_k$. Thus $\mathcal{R}[f_1] = \mathcal{R}[f_2] = 0$ for all $\mathcal{R} \in [\mathbf{n}]$. □

Let $\mu_1, \ldots, \mu_t$ be a set of t integers satisfying $0 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_t \leq D$. Recall that $D + 1$ is the number of different classes possible. The *parameter limited spaces* are defined by

$$\overline{B}_{\mu_1,\ldots,\mu_t}(n, \ell, d) := \overline{G}_d^{n,\ell} \cap \overline{M}_d^{n,\ell}[\mathbf{n}_{\mu_1}] \cap \cdots \cap \overline{M}_d^{n,\ell}[\mathbf{n}_{\mu_t}].$$

Obviously, there are 2^{D+1} spaces to consider. From Lemma 3.1 and equation (3.5) follows

$$\overline{B}_{\mu_1,\ldots,\mu_t}(n, \ell, d) = \overline{A}_{\mu_1,\ldots,\mu_t}(n, \lfloor \frac{d}{2} \rfloor) \oplus r\overline{A}_{\mu_1,\ldots,\mu_t}(n, \lfloor \frac{d-\ell}{2} \rfloor) \tag{3.6}$$

where

$$\overline{A}_{\mu_1,\ldots,\mu_t}(n,q) := \overline{S}_q^n \cap \overline{N}_q^n[\mathbf{n}_{\mu_1}] \cap \cdots \cap \overline{N}_q^n[\mathbf{n}_{\mu_t}].$$

For $q < 0$ we set $\overline{A}_{\mu_1,\ldots,\mu_t}(n,q) := \{0\}$. Associated with each parameter limited space there is a subsystem of equations

$$Q[f] = I[f],\ f \in \overline{B}_{\mu_1,\ldots,\mu_t}(n,\ell,d). \tag{3.7}$$

The number of independent equations in (3.7), denoted by $NE_{\mu_1,\ldots,\mu_t}(n,\ell,d)$, is given by

$$NE_{\mu_1,\ldots,\mu_t}(n,\ell,d) = \dim\ \overline{B}_{\mu_1,\ldots,\mu_t}(n,\ell,d). \tag{3.8}$$

From (3.6) and (3.8) follows

$$NE_{\mu_1,\ldots,\mu_t}(n,\ell,d) = \dim\ \overline{A}_{\mu_1,\ldots,\mu_t}(n,\lfloor\frac{d}{2}\rfloor) + \dim\ \overline{A}_{\mu_1,\ldots,\mu_t}(n,\lfloor\frac{d-\ell}{2}\rfloor). \tag{3.9}$$

The number of parameters, i.e., the number of unknowns (generators and weights), on the left hand side of (3.7) is denoted by $NP_{\mu_1,\ldots,\mu_t}(n)$. Now, we define the set of consistency conditions as the set of 2^{D+1} inequalities

$$NP_{\mu_1,\ldots,\mu_t}(n) \geq NE_{\mu_1,\ldots,\mu_t}(n,\ell,d) = \dim\ \overline{A}_{\mu_1,\ldots,\mu_t}(n,\lfloor\frac{d}{2}\rfloor)+\dim\ \overline{A}_{\mu_1,\ldots,\mu_t}(n,\lfloor\frac{d-\ell}{2}\rfloor). \tag{3.10}$$

Equation (3.10) gives the gross set of consistency conditions. This set can be reduced eliminating redundant inequalities. Indeed, the number of parameters depends on the parameter limited spaces $\overline{B}_{\mu_1,\ldots,\mu_t}(n,\ell,d)$, or equivalently on the functions in $\overline{A}_{\mu_1,\ldots,\mu_t}(n,\lfloor\frac{d}{2}\rfloor)$ and $\overline{A}_{\mu_1,\ldots,\mu_t}(n,\lfloor\frac{d-\ell}{2}\rfloor)$. Since $\overline{A}_{\mu_1,\ldots,\mu_t}(n,\lfloor\frac{d-\ell}{2}\rfloor) \subset \overline{A}_{\mu_1,\ldots,\mu_t}(n,\lfloor\frac{d}{2}\rfloor)$ one deduces that the number of parameters depends on the functions in $\overline{A}_{\mu_1,\ldots,\mu_t}(n,\lfloor\frac{d}{2}\rfloor)$. Thus the gross set of consistency conditions in (3.10) is the same as in the polynomial case, but with a different number of independent equations. In the polynomial case, and in the reduction procedure to pass from the gross set of consistency conditions to the net set, only properties of the subspace $\overline{A}_{\mu_1,\ldots,\mu_t}(n,q)$ are used, and no properties of the number of independent equations, see [5]. Thus we conclude that the net set of consistency conditions for our space of functions $R_k^{n,\ell}$, $k \geq \ell$, is equal to the net set of consistency conditions for polynomials, but with the number of independent equations given by (3.9).

4 Construction of formulas for $\ell = 2$

In this section we fix $\ell = 2$. We will write the consistency conditions for $n = 1$ and $n = 2$ explicitly and we construct some minimal cubature formulas of low degree to illustrate the theoretical results of the previous section. For other values of n or ℓ, one can proceed in a similar way to obtain the consistency conditions. The construction of cubature formulas of higher degree will be more difficult. Obviously the practical difficulties will be larger than in the polynomial case.

4.1 The one dimensional case

For one dimension, our aim is to construct quadrature formulas exact in $R^{1,2}_{2s+1} = P^1_{2s+1} + \mid x \mid P^1_{2s-1}$ for centrally symmetric integrals

$$I[f] = \int_\Omega f(x)\omega(x)dx.$$

We only consider the case that Ω is a symmetric interval, finite or infinite, say $(-a, a)$. Taking into account the consistency conditions for polynomials in one dimension (see, e.g., [2]) and the remark after equation (3.10), one can deduce the following.

Theorem 4.1 *The consistency conditions for one dimensional fully symmetric quadrature formulas exact in* $R^{1,2}_{2s+1}$ *for a fully symmetric integral are*

$$\begin{array}{rcl} K_0 + 2K_1 & \geq & 2s+1 \\ 2K_1 & \geq & 2s-1 \\ K_0 & \leq & 1. \end{array}$$

A fully symmetric quadrature formula that satisfies these conditions has $N = K_0 + 2K_1$ *nodes. See Table 3 for the definition of* K_0 *and* K_1.

The optimal solution, the one which uses the lowest number of nodes, (optimal solutions are not necessarily unique) of the integer programming problem in Theorem 4.1 is $[K_0\ K_1] = [1\ s]$. A quadrature formula with this structure uses $N = 2s+1$ nodes. If s is even, then $\dim h^{1,2}_s = s+1$. If s is odd, then $\dim h^{1,2}_s = s$. Consequently, according to Theorem 2.4, this quadrature formula uses precisely the minimal number of nodes. Observe that if the quadrature rule has degree $d = 2s+1$, then it also has degree $d = 2s$ and so Theorem 2.1 can be applied, giving a worse bound, $N \geq 2s$.

Structure $[1\ s]$ corresponds to a quadrature formula of the form

$$Q[f] = w_0 f(0) + w_1[f(a_1) + f(-a_1)] + \cdots + w_s[f(a_s) + f(-a_s)]. \tag{4.1}$$

The next step to construct the quadrature formula is to solve the nonlinear system

$$Q[f] = I[f],\ f \text{ basis elements of } \overline{G}^{1,2}_{2s+1}, \tag{4.2}$$

where w_0 and $w_i,\ a_i,\ i = 1, \ldots, s$, are the unknowns. We prove now that such a system has a unique solution for all degree $d = 2s+1, s \in I\!N$. Without loss of generality, we assume $a_i > 0,\ i = 1, \ldots, s$, in (4.1). We also prove that $a_i \in (0,\ a),\ i = 1, 2, \ldots, s$ and $w_i > 0,\ i = 0, \ldots, s$.

Let

$$J[f] := \int_0^a f(x)x\omega(x)dx.$$

Since the weight function $\omega(x)$ is even

$$\begin{array}{rll} I[x^{2k}] = & 2J[x^{2k-1}], & k = 1, 2, \ldots, \\ I[x^{2k} \mid x \mid] = & 2J[x^{2k}], & k = 0, 1, 2, \ldots. \end{array} \tag{4.3}$$

The system (4.2) for the basis $\{1, x^2, \ldots, x^{2s}, \mid x \mid, \ldots \mid x \mid x^{2s-2}\}$ of $\overline{G}^{1,2}_{2s+1}$, can be written in the form

$$\begin{array}{lcll} I[1] & = & w_0 + 2\sum_{i=1}^{s} w_i & \\ J[x^{2k}] & = & a_1^{2k+1} w_1 + \cdots + a_s^{2k+1} w_s, & k = 0, \ldots, s-1, \\ J[x^{2k-1}] & = & a_1^{2k} w_1 + \cdots + a_s^{2k} w_s, & k = 1, \ldots, s. \end{array} \tag{4.4}$$

Let $t_i := a_i$ and $c_i := a_i w_i$, $i = 1, \ldots, s$. The last $2s$ equations in (4.4) can be written as

$$J[x^k] = t_1^k c_1 + \cdots + t_s^k c_s, \ k = 0, \ldots, 2s-1. \tag{4.5}$$

The system (4.5) is the one that is obtained when one looks for the Gauss quadrature formula of polynomial type of degree $d = 2s - 1$ for the integral J. Thus, system (4.5) has solution $0 < t_i < a$ and $c_i > 0$, $i = 1, 2, \ldots, s$. Then, there is solution $0 < a_i = t_i < a$, $w_i = c_i/t_i > 0$, $i = 1, 2, \ldots, s$ and $w_0 = I[1] - 2\sum_{i=1}^{s} w_i$ for system (4.2). Furthermore, $w_0 > 0$. Indeed, $p(x) := \Pi_{i=1}^{s}(\mid x \mid - a_i)$ is such that $p^2 \in R^{1,2}_{2s+1}$. So, $I[p^2] = Q[p^2] = w_0(p(0))^2$, from which follows that $w_0 > 0$. The solution of system (4.2) is unique since the solution of system (4.5) is unique.

4.2 The n-dimensional case

For an arbitrary dimension n, the bound for the number of nodes provided by Theorem 2.3 for degree $d = 3$ is sharp, i.e., one can construct cubature rules with degree $d = 3$ with the minimal number of nodes $N = 2 \dim\ h_1^{n,2} = 2n$ for particular weight functions. Indeed, consider the integral I given by (1.1) where $\Omega = I\!R^n$ and $\omega(\mathbf{x})$ is one of the weight functions $\omega_1(\mathbf{x})$, $\omega_2(\mathbf{x})$ or $\omega_3(\mathbf{x})$. We propose the cubature formula

$$\begin{aligned} Q[f] = w_1[f(a_1, 0, \ldots, 0) + f(-a_1, 0, \ldots, 0) + \cdots + f(0, \ldots, 0, a_1, 0) + \\ f(0, \ldots, 0, -a_1, 0)] + w_2[f(0, \ldots, 0, a_2) + f(0, \ldots, 0, -a_2)]. \end{aligned} \tag{4.6}$$

Q has degree 3 if

$$Q[f] = I[f], \text{ for all} f \in \{1, r, x_1^2, x_n^2\}. \tag{4.7}$$

Let $m_1 := I[1]$, $m_2 := I[r]$ and $m_3 := I[x_1^2] = \cdots = I[x_n^2]$, see Table 2. The solution to the system (4.7), if a solution exists, can be expressed by

$$\begin{aligned} a_1 &= \frac{-m_3[m_2(n-1) + \sqrt{(n-1)[nm_1m_3 - m_2^2]}]}{m_1m_3 - m_2^2} \\ a_2 &= \frac{m_3[\sqrt{(n-1)[nm_1m_3 - m_2^2]} - m_2]}{(n-1)m_1m_3 - m_2^2} \\ w_1 &= \frac{(n-2)m_2^2 + m_1m_3n - 2m_2\sqrt{(n-1)[nm_1m_3 - m_2^2]}}{2(n-1)m_3n^2} \\ w_2 &= \frac{(n-1)m_1m_3n - (n-2)m_2^2 + 2m_2\sqrt{(n-1)[nm_1m_3 - m_2^2]}}{2m_3n^2} \end{aligned} \tag{4.8}$$

Taking the opposite sign for the square roots in (4.8), a second solution is obtained. One can verify that for the weight functions we are interested in and using Table 2, $nm_1m_3 - m_2^2 \geq 0$. (To deduce this for $\omega_2(\mathbf{x})$, take into account that for $0 < a < 1$ is $\Gamma(x+a) \leq x^a\Gamma(x)$, see e.g. [1, p. 17]. Taking $a = 1/2$ and $x = n/2$, the desired inequality follows.) For the weight functions $\omega_1(\mathbf{x})$ and $\omega_2(\mathbf{x})$ a solution therefore exists, since in these cases the denominators of a_1 and a_2 are different from zero. For the weight function $\omega_3(\mathbf{x})$ there is however no solution to (4.7), i.e, there is no cubature formula of degree 3 of the form (4.6). This is due to $(n-1)m_1m_3 - m_2^2 = 0$ for all n in this case. This illustrates that a better lower bound than the one of Theorem 2.3 must take into account information about the weight function. Such bounds are however not yet known, also for the purely algebraic case ($\ell > d$). This shows a fundamental difference between the univariate and the multivariate case.

For an arbitrary fully symmetric integral, one can construct cubature rules with degree $d = 3$ with $2n+1$ nodes, that is, one node more than our lower bound. Let

$$\begin{aligned} Q[f] = w_0 f(0,\ldots,0) + w_1[&f(a_1,0,\ldots,0) + f(-a_1,0,\ldots,0) + \cdots \\ &+ f(0,\ldots,0,a_1) + f(0,\ldots,0,-a_1)]. \end{aligned}$$

According to Theorem 3.1, Q has degree 3 if $Q[f] = I[f]$ for all $f \in \{1, r, x_1^2\}$. This system possesses a solution:

$$a_1 = \frac{nI[x_1^2]}{I[r]}, \; w_0 = I[1] - \frac{I^2[r]}{nI[x_1^2]}, \; w_1 = \frac{I^2[r]}{2n^2I[x_1^2]}.$$

4.3 The two-dimensional case

We consider now the case of two dimensions. By virtue of the consistency conditions for polynomials in two dimensions, see e.g. [6, 3], and the remark after equation (3.10) one deduces

Theorem 4.2 *The consistency conditions for two dimensional fully symmetric cubature formulas exact in* $R_{2s+1}^{2,2}$ *for a fully symmetric integral are*

$$\begin{array}{rcl} 3K_3 & \geq & \nu(s) + \nu(s+1) - 2s - 1 \\ 2K_2 + 3K_3 & \geq & \nu(s) + \nu(s+1) \\ 2K_1 + 3K_3 & \geq & \nu(s) + \nu(s+1) \\ K_0 + 2K_1 + 2K_2 + 3K_3 & \geq & \nu(s) + \nu(s+1) + 2s + 1 \\ K_0 & \leq & 1 \end{array}$$

where

$$\nu(p) = \begin{cases} \frac{(p-1)^2}{4}, & \text{if } p \text{ is odd} \\ \frac{p}{2}(\frac{p}{2} - 1), & \text{if } p \text{ is even.} \end{cases}$$

A fully symmetric cubature formula that satisfies these conditions has $N = K_0 + 4K_1 + 4K_2 + 8K_3$ *nodes. See Table 4 for the definition of* K_0, K_1, K_2 *and* K_3.

Let us first consider cubature formulas of degree $d = 3$. The bounds for the number of nodes given by Theorem 2.1 and Theorem 2.3 are respectively 3 and 4. There is however no guarantee that for a given region and weight function the lower bounds of Theorem 2.1 and Theorem 2.3 will be attained, as illustrated in Section 4.2. In general Theorem 4.2 will lead to cubature formulas with more nodes. That is the price one pays for demanding more symmetry. For degree $d = 3$, an optimal solution of the integer programming problem in Theorem 4.2 is $[K_0\ K_1\ K_2\ K_3] = [1\ 1\ 0\ 0]$. This structure implies $N = 5$ nodes and a cubature formula of this form is already presented at the end of Section 4.2.

For degree $d = 5$, an optimal solution of the integer programming problem in Theorem 4.2 is $[K_0\ K_1\ K_2\ K_3] = [0\ 2\ 1\ 0]$. This structure corresponds to a quadrature formula with 12 nodes of the form

$$Q[f] = \sum_{i=1,2} w_i Q_i[f] + w_3 Q_3[f]$$

where

$$\begin{array}{l} Q_i[f] = f(a_i, 0) + f(-a_i, 0) + f(0, a_i) + f(0, -a_i),\ i = 1, 2 \\ Q_3[f] = f(a_3, a_3) + f(a_3, -a_3) + f(-a_3, a_3) + f(-a_3, -a_3). \end{array}$$

Theorem 2.1 and Theorem 2.3 assert $N \geq 7$ and $N \geq 9$ respectively. From Theorem 3.1, Q integrates correctly any function in $R_5^{2,2}$ if $Q[f] = I[f]$ for all $f \in \{1, x^2, x^4, x^2y^2, r, x^2r\}$. This gives the following system of 6 equations in 6 unknowns:

$$\begin{aligned} 4w_1 + 4w_2 + 4w_3 &= I[1] \\ 2w_1a_1^2 + 2w_2a_2^2 + 4w_3a_3^2 &= I[x^2] \\ 2w_1a_1^4 + 2w_2a_2^4 + 4w_3a_3^4 &= I[x^4] \\ 4w_3a_3^4 &= I[x^2y^2] \\ 4w_1a_1 + 4w_2a_2 + 4\sqrt{2}w_3a_3 &= I[r] \\ 2w_1a_1^3 + 2w_2a_2^3 + 4\sqrt{2}w_3a_3^3 &= I[x^2r]. \end{aligned}$$

For the weight function $\omega_3(x, y)$ this system has 2 real solutions, both with all weights positive. The solutions are presented in Table 5.

Table 5: Cubature formulas of degree 5 for ω_3

	solution 1	solution 2
a_1	0.481480270891123	0.480957264477765
a_2	8.07280870801217	2.70376901671846
a_3	1.91441714368777	5.69094224452976
w_1	1.21554508008228	1.21480593408938
w_2	0.00442277404639526	0.351497723742847
w_3	0.350828472666223	0.00449266896267402

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