State-space realization and orthogonal rational functions

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Abstract

In this article we give a state space approach to orthogonal rational functions and how they can be used in system identification. The main result is a recursive algorithm to find the minimal balanced realization of a product of successive rational inner functions. It generalizes an algorithm by Heuberger et al. who considered the case of powers of a fixed rational matrix inner function. We extend this to the general block form which allows to consider a product of inner matrix functions. This realization can be used to find an orthogonal basis of rational functions.

Keywords : orthogonal rational functions, state space realization, balanced realization, matrix inner function

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State-space realization and orthogonal rational functions

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In this article we give a state space approach to orthogonal rational functions and how they can be used in system identification. The main result is a recursive algorithm to find the minimal balanced realization of a product of successive rational inner functions. It generalizes an algorithm by Heuberger et al. who considered the case of powers of a fixed rational matrix inner function. We extend this to the general block form which allows to consider a product of inner matrix functions. This realization can be used to find an orthogonal basis of rational functions.

1. Introduction

In recent years, there has been an increasing interest in the use of orthogonal rational functions in linear system theory [1,2,5,9,6,8,12].

If we want to represent for example a stable transfer function, we can use the classical basis $\{z^k\}$, which leads to a Laurent expansion of the function which uses Markov parameters. Typical approximants are then obtained by truncation or (minimal) partial realization methods. However, in general, these methods do not give good norm-approximation and are numerically unstable. To overcome this problem one may switch to an orthogonal polynomial basis. This is better for least squares approximation and for numerical stability. But still those bases may give some convergence problems for the supremum norm, unless special summation techniques (e.g. Fejér- or φ-summation [7]) are used and if convergence occurs, it may be very slow. The latter problem is especially crucial if there is a pole close to the imaginary axis or the unit circle (depending on whether one considers a continuous time or a discrete time system). This was the reason why one started to use approximants belonging to subspaces of rational functions (rather than polynomials) whose poles are prescribed in advance and which preferrably should be close to the actual pole of the system. Good approximations may already be obtained with only few terms in the orthogonal expansion which may lead to a considerable data reduction. We also mention here that Fejér-summation can be generalized to orthogonal rational functions with respect to an arbitrary measure μ on the unit circle (instead of the Lebesgue measure). See [10].

Let us consider a discrete time system, with a stable proper transfer function f. Thus f is in the Hilbert space $H_2(\mathbb{E})$ of square integrable functions, analytic in $\mathbb{E} \cup \mathbb{T}$, where $\mathbb{E} = \{z \in \mathbb{C} : |z| > 1\}$ and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

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The polynomial basis $\{z^{-k}\}_{k=0}^{\infty}$ is orthogonal and complete in $H_2(\mathbb{E})$, i.e. any function in $H_2(\mathbb{E})$ can be approximated arbitrary close in L_2-norm by a polynomial in z^{-1}. The truncated expansion

$$\sum_{k=0}^{n} c_k z^{-k}, \quad c_k = \left\langle f, z^{-k} \right\rangle := \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{ik\theta}\mathrm{d}\theta,$$

is the best (in least squares sense) approximation of the transfer function f in the subspace $\mathcal{L}_n = \mathrm{span}\{1, z^{-1}, \ldots, z^{-n}\}$.

The idea is to generalize this to the setting where

$$\mathcal{L}_n = \left\{ \frac{p_n(z)}{\prod_{k=1}^{n}(z-\alpha_k)} : p_n \in \Pi_n \right\},$$

where Π_n is the set of polynomials in z of degree at most n and all α_k lie in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Remark that if all $\alpha_k = 0$, then $\mathcal{L}_n$ reduces to the set of polynomials in z^{-1} of degree at most n as before.

Now we want to construct a set of orthogonal rational functions ϕ_k such that $\phi_n \in \mathcal{L}_n \setminus \mathcal{L}_{n-1}$ and $\phi_n \perp \mathcal{L}_{n-1}$. The most popular among the orthogonal rational functions are the Kautz and Laguerre systems. For a survey of the history and applications, see [5,9,8,1]. In the Laguerre case, the α_k are all equal to some $\alpha \in \mathbb{D}$. They are appropriate for modeling dominant first order dynamics. For second-order dynamics, the Kautz-system is used where there are only two different α_k, which are complex conjugates and repeated an infinite number of times.

In [5,8] this idea is generalized a step further in the sense that there are d different α_k that are cyclically repeated infinitely. A further generalization is achieved in [6], where an arbitrary sequence $\underline{\alpha} = \{\alpha_1, \alpha_2, \ldots\} \subset \mathbb{D}$ denotes the prescribed poles. The orthogonal rational functions are known as the Malmquist basis [11, p. 224]. In a sequence of papers, summarized in the monograph [3], more general orthogonal rational functions are considered. They are more general in the sense that they are orthogonal with respect to an arbitrary measure μ on $\mathbb{T}$. These orthogonal rational functions were first considered by M.M. Djrbashian, see [4] for a survey and references of his work.

It is our intention of this paper to show that the state space approach that was used in [5] to treat a finite number of poles that is cyclically repeated (henceforth called the *cyclic* case) can be generalized to an arbitrary sequence of poles.

We follow rather closely the approach used by Heuberger, Van den Hof and Bosgra in their paper [5]. This paper shows that their ideas fit in a more general setting.

This article is built up as follows. In the second section some notations and results are summarized. Our main results, Theorems 3.1 and 3.2, are stated in section three. The fourth section discusses the minimal balanced realization of a square inner function of degree 1. This could be used in a recursive algorithm to construct the minimal balanced realization of a product of inner functions. In section five, such a general block form algorithm is given, which is also the generalization of what is done in [5]. Section six contains the proof of our main results, Theorems 3.1 and 3.2. In the last section we give our conclusions and some special cases.

2. Preliminaries

In this article we will only consider discrete time signals and systems. Here are some notations that are used troughout this article.

$(.)^*$	Hermitian conjugate = complex conjugate of the transpose: $\overline{(.)}^T$;
$\mathbb{R}^{p\times q}$	set of $p \times q$ real matrices;
$\mathbb{C}^{p\times q}$	set of $p \times q$ complex matrices;
$\mathbb{Z}$	set of integers;
$\mathbb{N}$, $\mathbb{N}_0$	set of positive, resp. stricly positive integers;
$\mathbb{T}$, $\mathbb{D}$, $\mathbb{E}$	the unit circle, its interior (the unit disc) and its exterior, resp.;
$\ell_2^{p\times q}[0,\infty)$	space of square summable $p \times q$ matrices on $\mathbb{N}$; $\ell_2^p[0,\infty) = \ell_2^{1\times p}[0,\infty)$;
$H_2^{p\times q}(\mathbb{E})$	space of all $p \times q$ matrix functions that are square-integrable on the unit circle and analytic in $\mathbb{E}$; $H_2^p(\mathbb{E}) = H_2^{1\times p}(\mathbb{E})$;
$RH_2^{p\times q}(\mathbb{E})$	The rational functions in $H_2^{p\times q}(\mathbb{E})$;
$\\|\cdot\\|_2$	spectral norm of a matrix (maximum singular value);
$\mathcal{L}_n^{p\times q}$	set of all rational $p\times q$ matrix functions of McMillan degree at most n, with prescribed poles $\alpha_1,\ldots,\alpha_n$;
e_i	the ith Euclidean basis vector in $\mathbb{R}^n$;
I_n	the $n \times n$ identity matrix;
δ_{ij}	Kronecker delta, i.e. 1 if $i = j$ and 0 otherwise;
$\otimes$	Kronecker matrix product;
$\mathcal{H}(f)$	(Block) Hankel matrix with symbol f, i.e., whose (i,j)th (block) element is $\mathcal{H}_{ij}(f) = f_{i+j-1}$, with $f(z) = \sum_{k=0}^{\infty} f_k z^{-k}$, the Laurent expansion of the stable transfer function f.

The rational MIMO transfer function $f(z) \in RH_2^{q\times q}(\mathbb{E})$ has a n-dimensional state space realization (A,B,C,D) with $A \in \mathbb{C}^{n\times n}$, $B, C^* \in \mathbb{C}^{n\times q}$ and $D \in \mathbb{C}^{q\times q}$ if $f(z) = D + C(zI - A)^{-1}B$. A realization is minimal if the size of A is minimal. This minimal degree is called the McMillan degree. The realization is called stable if all eigenvalues of A lie strictly within the unit circle. The controllability Gramian P and observability Gramian Q are defined by the Lyapunov equations: $APA^* + BB^* = P$ and $A^*QA + C^*C = Q$. A stable realization is called balanced if $P = Q = \Sigma$, with $\Sigma = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_n)$, $\sigma_1 \geq \ldots \geq \sigma_n$, where σ_i are the positive Hankel singular values. A function is called inner if it satisfies $[f(1/z^*)]^* f(z) = I_q$.

In the article by Heuberger et al. [5] the construction of a generalized orthonormal basis is developed. It is done by using the minimal balanced state space realization (A,B,C,D) of an inner transfer function G. The following result was shown.

Theorem 2.1. *Let G be a $q \times q$ inner transfer function with McMillan degree $n > 0$, having a Laurent expansion $G(z) = \sum_{k=0}^{\infty} G_k z^{-k}$ with $\|G_0\| < 1$, and let (A,B,C,D) be a balanced realization of G. Denote*

$$V_k(z) = z(zI - A)^{-1} B G^k(z).$$

Then the set of functions $\{e_i^T V_k(z)\}_{i=1,\ldots,n;k=0,\ldots,\infty}$ constitutes an orthonormal basis of the function space $H_2^q(\mathbb{E})$.

Consider the SISO case ($q = 1$) and assume $n = 1$. Then the inner function G is just a Blaschke factor $\zeta(z) = (1 - \alpha^* z)/(z - \alpha)$ ($\alpha \in \mathbb{D}$). Thus, what this theorem actually does in this case is orthogonalizing the sequence $\{\zeta^k\}_{k=1}^{\infty}$. The resulting orthogonal rational functions are the well known

Laguerre functions

$$\phi_k(z) = \sqrt{1-|\alpha|^2} z \frac{(1-\alpha^* z)^{k-1}}{(z-\alpha)^k}, \quad k = 1, 2, \ldots.$$

This is easily generalized to orthonormalizing a sequence of basis functions, which are not the powers of one Blaschke factor, but of a Blaschke product, i.e., a product of n different Blaschke factors with poles $\{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{D}$. Thus these poles are repeated periodically. This can be useful for system identification when one has information about n poles of the system. In that case G is a scalar inner function containing these n poles. In mathematical terms we have $G(z) = \zeta_1(z) \cdots \zeta_n(z)$, with $\zeta_k(z) = (1-\alpha_k^* z)/(z-\alpha_k)$. By taking powers of G, these n poles are cyclically repeated. This is the SISO case with $n > 1$ of Theorem 2.1. See [9].

What we want to do is generalize this idea in the sense that we do not repeat the poles, but add new poles at each step, which may or may not be a repetition of a previous one. In this case we will need to put some restrictions on the placement of the poles, see (2.4).

By $\underline{\alpha}$ we denote the sequence of numbers $\{\alpha_1, \alpha_2, \alpha_3, \ldots\} \subset \mathbb{D}$.

The Blaschke factors ζ_k are defined as

$$\zeta_k(z) = \frac{1-\alpha_k^* z}{z-\alpha_k}, \quad k = 1, 2, \ldots. \tag{2.1}$$

Note that this definition is somewhat different from the one used in [3] where the poles were assumed outside the unit disc. The reason is that we want our poles to lie inside the unit disc to get stability in a system theoretic sense.

The Blaschke products G_k are defined as

$$G_0(z) = 1; \quad G_k(z) = \zeta_1(z) \cdots \zeta_k(z), \quad k = 1, 2 \ldots. \tag{2.2}$$

Recall that we consider here the scalar (SISO) case where $q = 1$. Then a scalar inner function $H_k = \zeta_k$ of degree one, is added to the system in step k.

The G_k have Laurent-series expansions around ∞, given by

$$G_k(z) = \sum_{l=0}^{\infty} G_{k,l} z^{-k}. \tag{2.3}$$

In the scalar case it is most convenient to set $\alpha_1 = 0$ to keep in touch with the theory in [3]. It does not affect our results much, but makes the proofs a lot easier.

The spaces $\mathcal{L}_k$ of rational functions of degree k are defined as

$$\mathcal{L}_k = \mathrm{span}\{G_0, G_1, \ldots, G_k\}.$$

The union of all the $\mathcal{L}_k$ is denoted by $\mathcal{L}$.

$$\mathcal{L} = \bigcup_{k=0}^{\infty} \mathcal{L}_k.$$

The condition for the poles not approaching the boundary too fast, is reflected in the Blaschke condition

$$\sum_{k=1}^{\infty} (1-|\alpha_k|) = \infty. \tag{2.4}$$

This condition is equivalent with the divergence of the Blaschke products G_k: they go to zero uniformly in $|z| \geq r > 1$. It is also a necessary and sufficient condition for the density of $\mathcal{L}$ in $H_2(\mathbb{E})$ [3,6].

The main theorem concerning the scalar case is formulated below as Theorem 3.1.

In the matrix case, we can consider square inner functions H_k of size $q \times q$ having a McMillan degree n. There are of course many possibilities. For example $\zeta_k(z)I_q$ is an inner function with McMillan degree equal to q with a pole α_k of multiplicity q. This can be generalized a bit more by considering $\mathrm{diag}(\zeta_1, \zeta_1, \ldots, \zeta_q)$ which is also a degree q inner function. Another interesting case is a general $q \times q$ inner function of McMillan degree one with one prescribed pole in $\alpha \in \mathbb{D}$. The most general form of such an inner function is given now.

Lemma 2.2. *A general $q \times q$ inner function of McMillan degree 1, with a pole in $\alpha \in \mathbb{D}$ is given by*

$$Z(z) = N\left[I_q - \frac{z-1}{(z-\alpha)(1-\alpha^*)}uu^*\right], \tag{2.5}$$

with $NN^* = N^*N = I_q$ and $u \in \mathbb{C}^{q\times 1}$ satisfying $u^*u = 1 - |\alpha|^2$. The factor N is a normalization putting: $Z(1) = N$.

Proof. See appendix. □

Thus, if we let $H_k = Z_k$ be such an inner function with pole α_k, then we can again construct inner functions as in the scalar case:

$$G_0(z) = I_q; \quad G_k(z) = H_k(z)\cdots H_1(z), \quad k = 1, 2 \ldots. \tag{2.6}$$

Note that the matrix Blaschke factor H_k is *pre*multiplied. Again we assume Laurent series

$$H_k(z) = \sum_{j=0}^{\infty} H_{k,j}z^{-j} \quad \text{and} \quad G_k(z) = \sum_{j=0}^{\infty} G_{k,j}z^{-j}.$$

Of course the same can be said about a general factor H_k which is inner, of size $q\times q$ and of degree m_k. In [5], all the H_k were the same. The main result about this general case is formulated in Theorem 3.2.

Before concluding this section, we give some additional results needed for our main results in the next section. We have the following remarkable fact for a Hankel matrix of an inner function [5, Proposition 4.2].

Proposition 2.3. *Let $G(z)$ be a square inner function with McMillan degree $n > 0$. Then a singular value decomposition of $\mathcal{H}(G)$ satisfies*

$$\mathcal{H}(G) = UV^*,$$

with U, $V \in \mathbb{C}^{\infty\times n}$ unitary, and the pair (U, V) is unique modulo postmultiplication with a unitary matrix $T \in \mathbb{C}^{n\times n}$.

Thus all the Hankel singular values of an inner function are equal to 1.

Another property is the following.

Lemma 2.4. *Let G be a $q \times q$ inner function with Laurent expansion $G(z) = \sum_{k=0}^{\infty} G_k z^{-k}$. Then $\sum_{k=0}^{\infty} G_{k+i}^* G_k = \delta_{i0} I_q$, $i \in \mathbb{N}$, and also*

$$\|G_k\|_2 \leq 1, \quad k \in \mathbb{N}. \tag{2.7}$$

Proof. The first part is given in [5, Proposition 4.1]. It immediately follows from the fact that for $z \in \mathbb{T}$ we have for an inner function G that $G^*G = I_q$. Replacing G by its Laurent series gives the result. In particular we have $\sum_{k=0}^{\infty} G_k^* G_k = I_q$, so that for all $k \in \mathbb{N}$, $I_q - G_k^* G_k \geq 0$ (inequality in the sense of positive definite matrices). Hence $\|G_k\|_2 \leq 1$. □

We have a nice property for the Laurent coefficients of the limit of a sequence of inner functions.

Proposition 2.5. *Let $H_k \in RH_2^{q\times q}(\mathbb{E})$ be a sequence of inner functions of finite McMillan degree m_k. Then $G_n = H_n \cdots H_1$ is an inner function of McMillan degree $\sum_{k=1}^{n} m_k$. Suppose also that $\lim_{n\to\infty} G_n(z) = 0$ uniformly in $\bar{\mathbb{E}}_r = \{z \in \mathbb{C} : 1 < r \leq |z| \leq \infty\}$. Assume $G_n(z) = \sum_{k=0}^{\infty} G_{n,k} z^{-k}$. Then $G_{n,k} \to 0$ for all $k \in \mathbb{N}$ as $n \to \infty$.*

Proof. We know that the Laurent coefficients of G_n are given by

$$G_{n,k} = \frac{1}{2\pi i} \oint_{\mathbb{T}_r} G_n(z) z^{k-1} \mathrm{d}z$$

where $\mathbb{T}_r := \{z \in \mathbb{C} : |z| = r > 1\}$. Choose $k \in \mathbb{N}$. Because $\lim_{n\to\infty} G_n(z) = 0$, we find for every $\epsilon > 0$ a sufficiently large n, such that (for any norm) $\|G_n(z)\| < \frac{\epsilon}{r^k}$, uniformly in $\bar{\mathbb{E}}_r$. Thus for large enough n, we can make $\|G_{n,k}\|$ arbitrarily small. Thus $G_{n,k} \to 0$ as $n \to \infty$. This proves the statement. □

3. Main results

The first theorem we can prove is the following.

Theorem 3.1. *Let $\{G_k\}_{k=0}^{\infty}$ be an infinite sequence of scalar Blaschke products, defined as in (2.2). These are inner functions with prescribed poles in $\underline{\alpha}$. Suppose that these G_k have Laurent-series expansions (2.3). Let (A_k, B_k, C_k, D_k) be a minimal balanced realization of $G_k(z)$ with $k \geq 1$. Suppose that the poles $\underline{\alpha}$ satisfy (2.4) and denote*

$$W_k(z) = z(zI - A_k)^{-1} B_k.$$

If we let k tend to ∞, then the entries of W_k converge to a complete orthonormal system for the function space $H_2(\mathbb{E})$.

The previous theorem shows how one pole, i.e. one Blaschke factor, can be introduced at a time. In [5] there are n different poles that are cyclically repeated. These n poles are added simultaneously so that in each update, the n different poles are repeated. A generalization that we give here is that we add in the kth update m_k poles. The number of poles m_k can differ in each step and the poles introduced may or may not be a repetition of previously introduced poles. At the same time, we shall also allow the inner functions to be square of size $q \times q$ as it was formulated in [5].

As suggested by Proposition 2.5, we replace the condition (2.4) by the more general condition

$$\lim_{n\to\infty} G_n(z) = 0, \quad \text{uniformly for all } z \in \bar{\mathbb{E}}_r = \{z \in \mathbb{C} : |z| \geq r > 1\}. \tag{3.1}$$

It is needed to guarantee that the resulting orthogonal functions will span the whole Hardy space. In the scalar case $q = 1$, then $G_{k,0} = \prod_{j=1}^{k} (-\alpha_j^*)$. It is well known that the divergence of $\prod |\alpha_j|$ is equivalent with the divergence of the series $\sum(1 - |\alpha_j|)$. For example if all the α_k are in a compact subset of $\mathbb{D}$, then $|G_{n,0}| \leq \prod_j |H_{j,0}| \to 0$.

In [5] the H_k are all equal and it is required that $\|H_{k,0}\| < 1$. In that case also $\|G_{n,0}\| \leq \|H_{k,0}\|^n$ will vanish as $n \to \infty$.

The general block form of the theorem is then as follows.

Theorem 3.2. *Let $\{H_k(z)\}_{k=1}^{\infty}$ be a sequence of $q \times q$ rational inner functions, with minimal balanced realizations $(\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k, \mathcal{D}_k)$, having McMillan degree $m_k > 0$. From this sequence we generate*

$$G_n(z) = H_n(z) \cdots H_1(z).$$

The function G_n will be inner and have McMillan degree $d_n = \sum_{k=1}^{n} m_k$. Suppose that the sequence $\{H_k(z)\}_{k=1}^{\infty}$ is such that (3.1) holds. Denote

$$V_n(z) = z(zI - \mathcal{A}_n)^{-1}\mathcal{B}_n G_{n-1}(z) \in RH_2^{m_n \times q}(\mathbb{E}).$$

Then the set of functions $\{e_i^* V_k(z)\}_{i=1,\ldots,m_k;k=1,2,\ldots}$ constitutes an orthonormal basis of the function space $H_2^q(\mathbb{E})$.

We can also formulate the analog of [5, Corollary 3.2].

Corollary 3.3. *Let G_n and V_n be as in the previous theorem. Then there exist $D_0 \in \mathbb{C}^{p\times q}$ and for $k \geq 0$, there are $L_k \in \mathbb{C}^{p\times m_k}$ such that for $f \in H_2^{p\times q}(\mathbb{E})$ we have $f(z) = D_0 + z^{-1}\sum_{k=0}^{\infty} L_k V_k(z)$.*

Proof. See Appendix. □

4. Minimal balanced realization of inner functions of degree 1

In this section we look at the minimal balanced realization of an inner function of degree 1.

Lemma 4.1. *The elementary degree one inner function given in (2.5) has a minimal balanced realization given by*

$$(A, B, C, D) = \left(\alpha, u^*, \frac{1-\alpha}{1-\alpha^*} N u, N(I - \frac{uu^*}{1-\alpha^*})\right). \tag{4.1}$$

Proof. This is immediately checked by direct computation. □

In the case $q = 1$, $\alpha = \alpha_k$, and setting $N = \frac{1-\alpha^*}{1-\alpha}$ this is a Blaschke factor ζ_k. A minimal balanced realization then specializes to

$$(A, B, C, D) = (\alpha_k, \sqrt{1 - |\alpha_k|^2}, \sqrt{1 - |\alpha_k|^2}, -\alpha_k^*) \tag{4.2}$$

which is of course easily derived in a direct way by rewriting the Blaschke factor as

$$\zeta_k(z) = \frac{1 - \alpha_k^* z}{z - \alpha_k} = -\alpha_k^* + \frac{1 - |\alpha_k|^2}{z - \alpha_k}.$$

Note that in the general expression, D has the singular values $(1, 1, \ldots, 1, |\alpha|)$ so that $\|D\|_2 = 1$.

We see that if the H_k are rank 1 inner functions of size $q \times q$, with $q > 1$ then the spectral norms of the factors are $\|H_{k,0}\|_2 = 1$ while it may well be that $G_{n,0}$ goes to zero. This is why we need the more general criterion (3.1).

5. A recursive algorithm to find the minimal balanced state space realization of a product of inner functions

In this section we prove the general block form of a recursive algorithm to generate the minimal balanced realization of the inner function G_n. The theorem is as follows.

Theorem 5.1. *Let $\{H_k(z)\}_{k=1}^{\infty}$ be a sequence of $q \times q$ inner transfer functions, with minimal balanced realization $(\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k, \mathcal{D}_k)$ and with McMillan degree m_k. Denote*

$$G_n(z) = H_n(z) \cdots H_1(z).$$

Then $\{G_n(z)\}_{n=1}^{\infty}$ is also a sequence of $q \times q$ inner transfer functions. We denote their minimal balanced realization as (A_n, B_n, C_n, D_n).

1. *The McMillan degree of G_n is $d_n := \sum_{i=1}^{n} m_i$.*
2. *The following recursion is valid.*

$$\begin{aligned} A_n &= \begin{bmatrix} A_{n-1} & 0 \\ \mathcal{B}_n C_{n-1} & \mathcal{A}_n \end{bmatrix}; \\ B_n &= \begin{bmatrix} B_{n-1} \\ \mathcal{B}_n D_{n-1} \end{bmatrix}; \\ C_n &= \begin{bmatrix} \mathcal{D}_n C_{n-1} & \mathcal{C}_n \end{bmatrix}; \\ D_n &= \mathcal{D}_n D_{n-1}, \end{aligned}$$

with $(A_1, B_1, C_1, D_1) = (\mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1, \mathcal{D}_1)$.

Before we start the proof, we have to give the following lemmas.

Lemma 5.2. *With the notation of Theorem 5.1, we have*

$$(zI - A_n)^{-1} = \begin{bmatrix} (zI - A_{n-1})^{-1} & 0 \\ (zI - \mathcal{A}_n)^{-1} \mathcal{B}_n C_{n-1} (zI - A_{n-1})^{-1} & (zI - \mathcal{A}_n)^{-1} \end{bmatrix}.$$

Proof. The proof is a simple calculation. □

We also need the following result [5, Proposition 5.2.].

Lemma 5.3. *(A_k, B_k, C_k, D_k) is a balanced realization iff*

$$\Sigma^* \Sigma = I_{n+q} = \Sigma \Sigma^*,$$

where Σ is the system matrix

$$\Sigma = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

This happens iff

$$\begin{aligned} A_k C_k^* + B_k D_k^* &= 0; \quad & C_k C_k^* + D_k D_k^* &= I; \\ B_k^* A_k + D_k^* C_k &= 0; \quad & B_k^* B_k + D_k^* D_k &= I. \end{aligned} \tag{5.1}$$

Proof of Theorem 5.1.

1. This follows directly from the the size of the matrix A_n constructed above and the fact that the realization is minimal.

2. We prove this by induction on n.
For $n = 1$ it is valid, because of the starting values $(A_1, B_1, C_1, D_1) = (\mathcal{A}_1, \mathcal{B}_1, \mathcal{C}_1, \mathcal{D}_1)$.
If it is valid for $i < n$, then we have to show its validity for n.
First we prove that it is a realization of G_n.

$$\begin{aligned}
&C_n(zI - A_n)^{-1}B_n + D_n \\
&= \begin{bmatrix} \mathcal{D}_n C_{n-1} & \mathcal{C}_n \end{bmatrix} \begin{bmatrix} (zI - A_{n-1})^{-1} & 0 \\ (zI - \mathcal{A}_n)^{-1}\mathcal{B}_n C_{n-1}(zI - A_{n-1})^{-1} & (zI - \mathcal{A}_n)^{-1} \end{bmatrix} B_n + D_n \\
&= \begin{bmatrix} (\mathcal{D}_n + \mathcal{C}_n(zI - \mathcal{A}_n)^{-1}\mathcal{B}_n)C_{n-1}(zI - A_{n-1})^{-1} & \mathcal{C}_n(zI - \mathcal{A}_n)^{-1} \end{bmatrix} \begin{bmatrix} B_{n-1} \\ \mathcal{B}_n D_{n-1} \end{bmatrix} + \mathcal{D}_n D_{n-1} \\
&= H_n(z)C_{n-1}(zI - A_{n-1})^{-1}B_{n-1} + H_n(z)D_{n-1} \\
&= H_n(z)G_{n-1}(z) = G_n(z).
\end{aligned}$$

We only need to prove its balancedness. This means

$$\begin{aligned}
A_n A_n^* + B_n B_n^* &= I; \\
A_n^* A_n + C_n^* C_n &= I.
\end{aligned}$$

Therefore we will need (5.1) which is valid for *every* minimal balanced state space realization of an inner transfer function. We will only prove the second relation. The first can be proven similarly.

$$\begin{aligned}
A_n^* A_n + C_n^* C_n &= \begin{bmatrix} A_{n-1}^* & C_{n-1}^* \mathcal{B}_n^* \\ 0 & \mathcal{A}_n^* \end{bmatrix} \begin{bmatrix} A_{n-1} & 0 \\ \mathcal{B}_n C_{n-1} & \mathcal{A}_n \end{bmatrix} + \begin{bmatrix} C_{n-1}^* \mathcal{D}_n^* \\ \mathcal{C}_n^* \end{bmatrix} \begin{bmatrix} \mathcal{D}_n C_{n-1} & \mathcal{C}_n \end{bmatrix} \\
&= \begin{bmatrix} A_{n-1}^* A_{n-1} + C_{n-1}^* \mathcal{B}_n^* \mathcal{B}_n C_{n-1} & C_{n-1}^* \mathcal{B}_n^* \mathcal{A}_n \\ \mathcal{A}_n^* \mathcal{B}_n C_{n-1} & \mathcal{A}_n^* \mathcal{A}_n \end{bmatrix} \\
&\quad + \begin{bmatrix} C_{n-1}^* \mathcal{D}_n^* \mathcal{D}_n C_{n-1} & C_{n-1}^* \mathcal{D}_n^* \mathcal{C}_n \\ \mathcal{C}_n^* \mathcal{D}_n C_{n-1} & \mathcal{C}_n^* \mathcal{C}_n \end{bmatrix} \\
&= \begin{bmatrix} A_{n-1}^* A_{n-1} + C_{n-1}^* (\mathcal{B}_n^* \mathcal{B}_n + \mathcal{D}_n^* \mathcal{D}_n) C_{n-1} & C_{n-1}^* (\mathcal{B}_n^* \mathcal{A}_n + \mathcal{D}_n^* \mathcal{C}_n) \\ (\mathcal{A}_n^* \mathcal{B}_n + \mathcal{C}_n^* \mathcal{D}_n) C_{n-1} & \mathcal{A}_n^* \mathcal{A}_n + \mathcal{C}_n^* \mathcal{C}_n \end{bmatrix} = I.
\end{aligned}$$

Here we used the balancedness of $(A_{n-1}, B_{n-1}, C_{n-1}, D_{n-1})$ and $(\mathcal{A}_n, \mathcal{B}_n, \mathcal{C}_n, \mathcal{D}_n)$. □

By substituting the appropriate matrices, this can be specialized to the case where the McMillan degree of H_k is one i.e., when we choose $H_k = Z_k$ as described above. Another special case is $q = m_k = 1$, so that H_k is just a Blaschke factor ζ_k.

6. Proof of the main results

We now are able to prove the main theorems. We first prove the simplest scalar case separately. The G_k are then the Blaschke products. We assume as before that $\alpha_1 = 0$ so that $(A_1, B_1, C_1, D_1) =$

$(\mathcal{A}_1, \mathcal{B}_1, C_1, \mathcal{D}_1) = (0, 1, 1, 0)$. Also we know that $(\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k, \mathcal{D}_k) = (\alpha_k, \sqrt{1-|\alpha_k|^2}, \sqrt{1-|\alpha_k|^2}, -\alpha_k^*)$ and the orthogonal functions are given by [6]

$$\phi_k(z) = z\frac{\sqrt{1-|\alpha_k|^2}}{z-\alpha_k}G_{k-1}(z), \quad k \geq 1, \tag{6.1}$$

and $\text{span}\{\phi_1, \phi_2, \ldots\}$ is dense in $H_2(\mathbb{E})$ iff (2.4) holds.

Proof of Theorem 3.1.
We prove the theorem by induction on k.

For $k = 1$, we have from section 2 (recall $\alpha_1 = 0$)

$$W_1(z) = z\frac{\sqrt{1-|\alpha_1|^2}}{z-\alpha_1} = 1 = \phi_1(z).$$

This is in correspondence with (6.1).

If it is correct for all $i < k$, then we have for $i = k$ that

$$\begin{aligned} W_k(z) &= z(zI - A_k)^{-1}B_k \\ &= z\begin{bmatrix} (zI - A_{k-1})^{-1} & 0 \\ (z-\alpha_k)^{-1}\sqrt{1-|\alpha_k|}C_{k-1}(zI - A_{k-1})^{-1} & (z-\alpha_k)^{-1} \end{bmatrix}\begin{bmatrix} B_{k-1} \\ \sqrt{1-|\alpha_k|^2}D_{k-1} \end{bmatrix} \\ &= \begin{bmatrix} z(zI - A_{k-1})^{-1}B_{k-1} \\ z\frac{\sqrt{1-|\alpha_k|^2}}{z-\alpha_k}(C_{k-1}(zI - A_{k-1})^{-1}B_{k-1} + D_{k-1}) \end{bmatrix} \\ &= \begin{bmatrix} W_{k-1}(z) \\ z\frac{\sqrt{1-|\alpha_k|^2}}{z-\alpha_k}G_{k-1}(z) \end{bmatrix} = \begin{bmatrix} W_{k-1}(z) \\ \phi_k(z) \end{bmatrix}. \end{aligned}$$

Here we used Lemma 5.2. □

If we denote with $(\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k, \mathcal{D}_k)$ the minimal balanced realization of the Blaschke factor ζ_k (this is consistent with the block form notation in the case $H_k = \zeta_k$), then it is easy to see that

$$\phi_k(z) = z(zI - \mathcal{A}_k)^{-1}\mathcal{B}_k G_{k-1}(z),$$

with G_n the Blaschke product as defined above.

This can be generalized to the general block form. This is what Theorem 3.2 states. Before we prove this theorem, we make the connection between the V- and the W-notation.

Lemma 6.1. *Consider the inner functions H_k and $G_n = H_n \cdots H_1$ with minimal balanced realizations $(\mathcal{A}_k, \mathcal{B}_k, \mathcal{C}_k, \mathcal{D}_k)$ and (A_n, B_n, C_n, D_n) respectively. Recall the definition $V_k = z(zI - \mathcal{A}_n)^{-1}\mathcal{B}_n G_{n-1}(z)$, and define*

$$W_n(z) = \begin{bmatrix} V_1(z) \\ \vdots \\ V_n(z) \end{bmatrix}.$$

Then $W_n(z) = z(zI - A_n)^{-1}B_n$.

Proof. For $n = 1$ this is trivial.

If it is valid for $i < n$, then we find for n

$$
\begin{aligned}
z(zI - A_n)^{-1}B_n &= z \begin{bmatrix} (zI - A_{n-1})^{-1} & 0 \\ (zI - \mathcal{A}_n)^{-1}\mathcal{B}_n C_{n-1}(zI - A_{n-1})^{-1} & (zI - \mathcal{A}_n)^{-1} \end{bmatrix} \begin{bmatrix} B_{n-1} \\ \mathcal{B}_n D_{n-1} \end{bmatrix} \\
&= z \begin{bmatrix} (zI - A_{n-1})^{-1}B_{n-1} \\ (zI - \mathcal{A}_n)^{-1}\mathcal{B}_n (C_{n-1}(zI - A_{n-1})^{-1}B_{n-1} + D_{n-1}) \end{bmatrix} \\
&= \begin{bmatrix} W_{n-1}(z) \\ z(zI - \mathcal{A}_n)^{-1}\mathcal{B}_n G_{n-1}(z) \end{bmatrix} = W_n(z).
\end{aligned}
$$

□

Now we take a look at the Hankel matrix $\mathcal{H}(G_k)$. We set by definition

$$
\mathcal{H}(G_k) := \begin{bmatrix} G_{k,1} & G_{k,2} & G_{k,3} & \cdots \\ G_{k,2} & G_{k,3} & G_{k,4} & \cdots \\ G_{k,3} & G_{k,4} & G_{k,5} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}. \tag{6.2}
$$

We have the following lemma (compare with [5, Lemma A5]).

Lemma 6.2. *Let $\{H_i(z)\}_{i=1}^{\infty}$ and $\{G_i(z)\}_{i=1}^{\infty}$ be as in Theorem 3.2. Suppose that (3.1) is satisfied. Use the notation $\mathcal{H}_k$ to denote the Hankel matrix $\mathcal{H}(G_k)$ as defined in (6.2). Then for all $i \in \mathbb{N}_0$, we have*

$$
\lim_{k\to\infty} \max_{j\in\mathbb{N}_0} \| (\mathcal{H}_k^*\mathcal{H}_k)_{ij} - \delta_{ij} I_q \| = 0.
$$

Proof. Consider the (i, j)th block element of $\mathcal{H}_k^*\mathcal{H}_k$. For $j \geq i$ we find

$$
\begin{aligned}
(\mathcal{H}_k^*\mathcal{H}_k)_{ij} &= \sum_{l=0}^{\infty} G_{k,i+l}^* G_{k,j+l} \\
&= \delta_{ij} I_q - \sum_{l=0}^{i-1} G_{k,l}^* G_{k,l+j-i}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\| (\mathcal{H}_k^*\mathcal{H}_k)_{ij} - \delta_{ij} I_q \| &= \left\| \sum_{l=0}^{i-1} G_{k,l}^* G_{k,l+j-i} \right\| \\
&\leq \sum_{l=0}^{i-1} \|G_{k,l}\| =: R_k(i-1).
\end{aligned}
$$

Because $\lim_{n\to\infty} G_{n,j} = 0$, we find that $\lim_{k\to\infty} R_k(i-1) = 0$.

For $j < i$ we get by the same arguments

$$
\| (\mathcal{H}_k^*\mathcal{H}_k)_{ij} - \delta_{ij} I_q \| \leq R_k(j-1).
$$

Because $R_k(j-1) \leq R_k(i-1)$, we find that for all $i, j \in \mathbb{N}_0$

$$
\| (\mathcal{H}_k^*\mathcal{H}_k)_{ij} - \delta_{ij} I_q \| \leq R_k(i-1) \to 0, \qquad k \to \infty.
$$

This completes the proof. □

Now we are able to prove the main theorem.

Proof of Theorem 3.2.
We know from Proposition 2.3 that for every k the Hankel matrix $\mathcal{H}(G_k)$ has a singular value decomposition

$$\mathcal{H}(G_k) = \Gamma_k^o \Gamma_k^c,$$

with $\Gamma_k^o, (\Gamma_k^c)^* \in \mathbb{C}^{\infty \times d_k}$ and unitary, i.e., $(\Gamma_k^o)^*\Gamma_k^o = I_{d_k} = \Gamma_k^c(\Gamma_k^c)^*$. These matrices are the observability and controllability matrix of the system. This can be seen as follows. Recall that the matrix functions $V_k(z)$ and $W_k(z)$ from Lemma 6.1 are related by

$$W_k(z) = \begin{bmatrix} V_1(z) \\ V_2(z) \\ \vdots \\ V_k(z) \end{bmatrix}.$$

Suppose $W_k(z) = \sum_{i=0}^{\infty} W_{k,i} z^{-i}$ and define the matrix $W_k = \begin{bmatrix} W_{k,0}\, W_{k,1} \ldots \end{bmatrix}$. Then we prove that $\Gamma_k^c = W_k$. Recall that $W_k(z)$ has d_k scalar rows. We also show that the functions $\phi_k(z) = \sum_{i=0}^{\infty} \phi_{k,i} z^{-i}$, defined by

$$W_k = \begin{bmatrix} \phi_{1,0} & \phi_{1,1} & \cdots \\ \phi_{2,0} & \phi_{2,1} & \cdots \\ \vdots & & \\ \phi_{d_k,0} & \phi_{k,1} & \cdots \end{bmatrix}, \quad \phi_{k,l} \in \mathbb{C}^{1 \times q},$$

for $k = 1, 2, \ldots$ are complete in $H_2^q(\mathbb{E})$.

First we take a look at $G_k(z)$. It has a minimal balanced realization (A_k, B_k, C_k, D_k). Thus

$$G_k(z) = D_k + C_k(zI - A_k)^{-1}B_k = \sum_{i=0}^{\infty} G_{k,i} z^{-i},$$

with

$$\begin{aligned} G_{k,0} &= D_k, \\ G_{k,i} &= C_k A_k^{i-1} B_k, \quad i \geq 1. \end{aligned} \tag{6.3}$$

We know that $W_k(z) = z(zI - A_k)^{-1}B_k$ from Lemma 6.1. Thus

$$W_{k,i} = A_k^i B_k, \quad i \geq 0. \tag{6.4}$$

Because the realization (A_k, B_k, C_k, D_k) is balanced, it is easy to see that W_k is unitary because

$$W_k W_k^* = \sum_{i=0}^{\infty} A_k^i B_k B_k^* (A_k^*)^i = I.$$

For Γ_k^o, we can use (6.2) to obtain that

$$\Gamma_{k,i}^o = C_k A_k^i, \quad i \geq 0.$$

By again using the balancedness, we find that also Γ_k^o is unitary since

$$(\Gamma_k^o)^*\Gamma_k^o = \sum_{i=0}^{\infty} (A_k^*)^i C_k^* C_k A_k^i = I.$$

Thus the rows of $W_k \in \mathbb{C}^{d_k \times \infty}$ form a set of d_k orthonormal sequences $\{\phi_i\}_{i=1}^{d_k}$ in $\ell_2^q[0, \infty)$. Because of the fact that the $\mathcal{Z}$-transform is an isometric isomorphism from $\ell_2^q[0, \infty) \to H_2^q(\mathbb{E})$, it is clear that the functions $\{\phi_i(z)\}_{i=1}^{d_k}$ are a set of d_k orthonormal functions. We still need to prove that the set $\{\phi_i(z)\}_{i=1}^{\infty}$ is complete in $H_2^q(\mathbb{E})$. We have

$$W_k(z) = \begin{bmatrix} \phi_1(z) \\ \phi_2(z) \\ \vdots \\ \phi_{d_k}(z) \end{bmatrix}.$$

By definition $\{\phi_i(z)\}_{i=1}^{\infty}$ is complete in $H_2^q(\mathbb{E})$ iff $\forall x = (x_k)_{k=0}^{\infty} \in \ell_2^q[0,\infty) : \langle \phi_k, x \rangle = 0 \ \forall k \Rightarrow x = 0$.

With $x^* = [x_0 \ x_1 \ \ldots]^*$, we have that if $\langle \phi_k, x \rangle = 0$ for all $k \geq 1$, then $W_k x^* = 0$ for all $k \in \mathbb{N}_0$, hence also $W_k^* W_k x^* = 0$ for all $k \in \mathbb{N}_0$. When e_i is the ith unit vector, and $E_i = e_i \otimes I_q$, then the ith block row of this equation is $[W_k^* W_k]_{i*} x^* := E_i^* [W_k^* W_k]_{i*} x^* = 0$ (we assume $iq \leq d_k$). Thus

$$\|x_i\| = \|([W_k^* W_k]_{i*} - E_i^*) x^*\| \leq \|[W_k^* W_k]_{i*} - E_i^*\| \, \|x\|.$$

Since $W_k^* W_k = W_k^* (\Gamma_k^o)^* \Gamma_k^o W_k = (\mathcal{H}(G_k))^* \mathcal{H}(G_k)$, it follows from Lemma 6.2 that $\lim_{k\to\infty} \|[W_k^* W_k]_{i*} - E_i^*\| = 0$, which implies that $x_i = 0$. Because this holds for arbitrary $i \in \mathbb{N}$, we may conclude that $x = 0$. This proves the theorem. □

7. Conclusions

We have shown that there exists a recursive method to find the minimal balanced state space realization of an inner function given as a product of inner factors. This realization gives rise to an orthonormal basis of the function space $H_2^q(\mathbb{E})$, which can be of great use in system identification. The theory we describe here is a generalization of the one presented in [5]. To conclude, we mention 3 special cases of this theory in the scalar case, i.e., when $q = 1$.

1. First we take $m_i = 1$ and $\alpha_i = 0$ for all i. This means that all $H_i(z)$ are equal to $1/z$. Then we find the pulse basis.

2. When we take $m_i = 1$ and $\alpha_i = a$ for all i, with $a \in \mathbb{D}$, then we have that $H_i(z) = \frac{1-a^*z}{z-a}$. We now find the Laguerre basis.

3. When we take $m_i = 2$, $\alpha_{2k+1} = a$ and $\alpha_{2k} = a^*$ for all i, with $a \in \mathbb{D}$, then we find the Kautz basis.

8. Appendix

Proof of Lemma 2.2.
The most general function with pole $z = \alpha$ is $Z(z) = Z_0 + Z_1/(z-\alpha)$. Thus for a degree one function we need a rank one matrix for Z_1. We write it as $Z_1 = uv^*$. We normalize it by making $Z(1) = N$ with $N^*N = I_q$. This is obtained by setting

$$Z_0 = N - \frac{uv^*}{1-\alpha}.$$

So that

$$Z(z) = N + \frac{(1-z)uv^*}{(z-\alpha)(1-\alpha)}.$$

We express that $[Z(1/z^*)]^*Z(z) = I_q$ and use $N^*N = I_q$ to get after multiplying out the denominator

$$\frac{N^*uv^*(1-z)}{(z-\alpha)(1-\alpha)} + \frac{vu^*N(1-1/z)}{(1/z-\alpha^*)(1-\alpha^*)} + \frac{vu^*uv^*(1-1/z)(1-z)}{(1/z-\alpha^*)(z-\alpha)(1-\alpha^*)(1-\alpha)} = 0$$

or, after rearranging terms

$$\frac{t}{z} + zt^* - (t+t^*) = 0, \quad \text{with} \quad t = (1-\alpha^*)N^*uv^* + \alpha(1-\alpha)vu^*N - vu^*uv^*.$$

Since this has got to be true for any z, we have $t = 0$. Thus, since

$$t - t^* = (1-\alpha^*)^2 N^*uv^* - (1-\alpha)^2 vu^*N = 0,$$

we get after multilpying with v from the right

$$(1-\alpha^*)^2(v^*vN^*u = (1-\alpha)^2(u^*Nv)v,$$

so that

$$N^*u = \left(\frac{1-\alpha}{1-\alpha^*}\right)^2 \frac{u^*Nv}{v^*v}v$$

or $N^*u = \rho v$ for short. Because $u^*Nv = \rho^*v^*v$, we see that

$$\rho = \left(\frac{1-\alpha}{1-\alpha^*}\right)^2 \rho^*.$$

Use this in $t + t^* = 0$ to get

$$[(1-(\alpha^*)^2)\rho + (1-\alpha^2)\rho^* - 2(u^*u)]vv^* = 0.$$

Hence, using $u^*u = u^*NN^*u = |\rho|^2v^*v$,

$$(1-(\alpha^*)^2) + (1-\alpha^2)\frac{\rho^*}{\rho} = 2\rho^*(v^*v).$$

Thus

$$\rho = \frac{(1-\alpha)(1-|\alpha|^2)}{(1-\alpha^*)(v^*v)}.$$

We therefore have from $u = \rho Nv$

$$Z(z) = N + \frac{(1-z)uv^*}{(z-\alpha)(1-\alpha)} = N + \frac{(1-z)(1-|\alpha|^2)Nvv^*}{(z-\alpha)(1-\alpha^*)(v^*v)}.$$

By normalizing $v^*v = 1-|\alpha|^2$, we finally have

$$Z(z) = N\left[I_q + \frac{(1-z)vv^*}{(z-\alpha)(1-\alpha^*)}\right], \quad v^*v = 1-|\alpha|^2.$$

□

Proof of Corollary 3.3.

Define $D_0 = \lim_{z\to\infty} f(z)$ and define $h(z) = f(z) - D_0$. Because we can consider each row of $zh(z) \in H_2^{p\times q}(\mathbb{E})$ as an element in $H_2^q(\mathbb{E})$, and because the ϕ_i form an orthogonal basis for $H_2^q(\mathbb{E})$, there is a sequence of complex numbers $c = (c_k)_{k=1}^{\infty} \in \ell_2$ such that for row i of $zh(z)$ we have

$$zh_{i*}(z) = \sum_{k=1}^{\infty} c_{ik}\phi_k(z) = \sum_{j=1}^{\infty}(L_j)_{i*}V_j(z),$$

where $(L_j)_{i*}$ denotes row i of a matrix $L_j \in \mathbb{C}^{p \times m_j}$

$$(L_j)_{i*} = [c_{i,d_{j-1}+1}, \ldots, c_{i,d_j}], \quad \text{and} \quad V_j(z) = \begin{bmatrix} \phi_{d_{j-1}+1}(z) \\ \vdots \\ \phi_{d_j}(z) \end{bmatrix}.$$

Putting the rows of h together proves the corollary. □

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