The Behaviour of the Floquet Multipliers of Periodic Solutions Near a Homoclinic Orbit

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Report TW 287, December 1998

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For two-dimensional problems, for which the equilibrium point has two real eigenvalues, we derive an asymptotic estimate for the nontrivial Floquet multipliers in terms of the period. For more complicated situations (higher dimensional problems, complex conjugate pairs of eigenvalues), we perform numerical experiments for some model equations, showing that eigenvalues with large negative real part cause small Floquet multipliers.

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AMS(MOS) Classification : Primary : 65N35, Secondary : 35B32.

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Key Words Floquet multiplier, Periodic Solution, Homoclinic Bifurcation
AMS(MOS) subject classifications 65N12,35B10,35B32

1. Introduction

Several numerical methods (Beyn (1989, 1990), Kuznetsov (1990) Doedel et al.(1997)) for the computation of homoclinic bifurcations of ordinary differential equations have been developed. Typically, they are formulated as approximate boundary value problems with periodic or projected boundary value conditions. For a large-scale system of ODEs, arising from the space discretization of PDEs, these methods become prohibitively expensive. Recently, Newton-Picard schemes (Roose et al.(1995), Lust et al.(1997)) have been developed to solve large-scale periodic boundary value problem with low dimensional dynamics, i.e, when most of the Floquet Multiplers of the periodic solution have small modulus. This is the case for many ODE systems arising from the space discretization of a PDE. In this paper, we study the behaviour of the modulus of the Floquet multipliers when the periodic solutions approach a homoclinic orbit, and we show that most of the Floquet multipliers become small, such that a Newton-Picard scheme can be used to compute periodic solutions as an approximation to a homoclinic orbit. Specifically, we suppose that the autonomous ODE system

$$\dot{x}(t) = F(x(t), \lambda), \quad x \in \mathbb{R}^N, \; \lambda \in \mathbb{R}, \; N >> 1 \tag{1}$$

has a nondegenerate homoclinic orbit $(x^*(t), \lambda^*)$ connecting $p(\lambda^*)$, where $p(\lambda)$ is a hyperbolic fixed point of ODE system (1). As a starting point, we recall Theorem 3.1 from (Beyn (1989))

Theorem There exits $T_0 > 0$ large enough and $\delta > 0$, such that for any $T > T_0$, the approximate periodic boundary value problem

$$\begin{cases} \dot{x}(t) = F(x(t), \lambda), & t \in J = [-T, T] \\ x(-T) = x(T) \\ \dot{x}^*(0)^T (x(0) - x^*(0)) = 0 \end{cases} \tag{2}$$

has a unique solution $(x(t;T), \lambda(T))$ in

$$\{(x, \lambda) \;:\; ||x - x^*||_1 + |\lambda - \lambda^*| < \delta\}$$

Further, for any $\epsilon > 0$, the solution $(x(t;T), \lambda(T))$ satisfies the estimates

$$\begin{array}{l} ||x(t;T) - x^*(t)||_1 = O(\exp(-(\min(\mu_+, \mu_-) - \epsilon)T)) \\ |\lambda - \lambda^*| = O(\exp(-(2\min(\mu_+, \mu_-) - \epsilon)T)) \end{array} \tag{3}$$

with the critical exponents

$$\begin{array}{l} \mu_+ = Min\{Re(\nu) : \; \nu \; is \; an \; eigenvalue \; of \; A(\lambda^*) \; with \; Re(\nu) > 0\} \\ \mu_- = Min\{-Re(\nu) : \; \nu \; is \; an \; eigenvalue \; of \; A(\lambda^*) \; with \; Re(\nu) < 0\} \end{array} \tag{4}$$

and $A(\lambda) = F_x(p(\lambda), \lambda)$. □

In this paper, we relate the Floquet multipliers of the periodic solution $x(t;T)$ to the eigenvalues of the fixed point $p(\lambda^*)$ in terms of the period T. We show that low dimensional dynamics of the fixed point (most of the eigenvalues of $F_x(p(\lambda^*), \lambda^*)$ have a very large negative real part) produces low dimensional dynamics of the periodic solutions (most of the Floquet multipliers are very small in modulus).

2. Low dimensional dynamics of the fixed point and periodic solutions approaching an homoclinic orbit

We discuss the relation between the low dimensional dynamics of the fixed point $p(\lambda^*)$ and that of the periodic solutions which approach the homoclinic orbit. The discussion is based on the fact that near the fixed point, the vector field of (1) is essentially linear, and when periodic solutions approach the homoclinic orbit, the time they spend in the small linear region goes to infinity. The local linearization vector field of the fixed point plays an important role in our discussion. In fact, for three dimensional systems, Glendinning and Sparrow ([1984]) show that the eigenvalues of the local linearization vector field determines the stability of the periodic solutions when they approach the homoclinic orbit. For higher dimensional ODE systems, the periodic solutions in a neighbourhood of the homoclinic orbit have different bifurcation and stability behaviours, depending on the relative size of the critical exponents $\mu_{\pm}$ (4), see Kuznetsov Yu.A. [1997]. The estimate (3) also illustrate that the eigenvalues of the fixed point determine the approximate order of the solution of the approximate periodic boundary value problems. We discuss two different cases: a) all eigenvalues of the fixed point are real; b) there are some complex conjugate pairs of eigenvalues. Since the topology of the phase space of the local linearization vector field is qualitatively different in these two cases, we expect that the modulus of the Floquet multipliers will also vary in qualitatively different ways. In fact, Kuznetsov (1997) has shown that the Floquet multipliers may contract to zero or expand infinitely when the periodic solutions approach the homoclinic orbit. In this paper, we derive some more quanlitative information, e.g. how many Floquet multipliers contract to zero and how many expand to infinity. First, we obtain a prediction from the analytical analysis of the simplest situations with the help of Sil'nikov's standard analysis methods for the local behaviour near a homoclinic orbit (Sil'nikov (1965), (1970)); Then the prediction is further verified by numerical experiments.

2.1 All eigenvalues are real

When all eigenvalues are real, the simplest situation is a two - dimensional system whose hyperbolic fixed point $p(\lambda^*)$ has two simple real eigenvalues ν_1, ν_2 with $\nu_2 > 0$ and $\nu_1\nu_2 < 0$. For two-dimensional periodic solutions, no period-doubling bifurcations and bifurcations to torus can occur. Theorem 6.5 in Guckenheimer & Holmes (1983) implies that when the periodic solutions approach the homoclinic orbit, saddle-node bifurcations can not occur either. Thus the unique nontrivial Floquet multipliers always stays within or outside the unit circle. To relate their modulus to the eigenvalues of the fixed point, we compute an asymptotic estimate for it by Sil'nikov's standard analysis method. An important observation in Sil'nikov's analysis is that the flow near $p(\lambda)$ is essentially linear. Then the flow is divided into two regions as shown in Figure (1), the linear region bounded by a box B=$\{|x| \leq l,\ |y| \leq h\}$ and the nonlinear region outside this box. In the box B, there exist local coordinates (x, y) so that the local stable manifold $W^s_{loc} = \{y = 0\}$ and the local unstable manifold $W^u_{loc} = \{x = 0\}$. In the box B, the vector field can

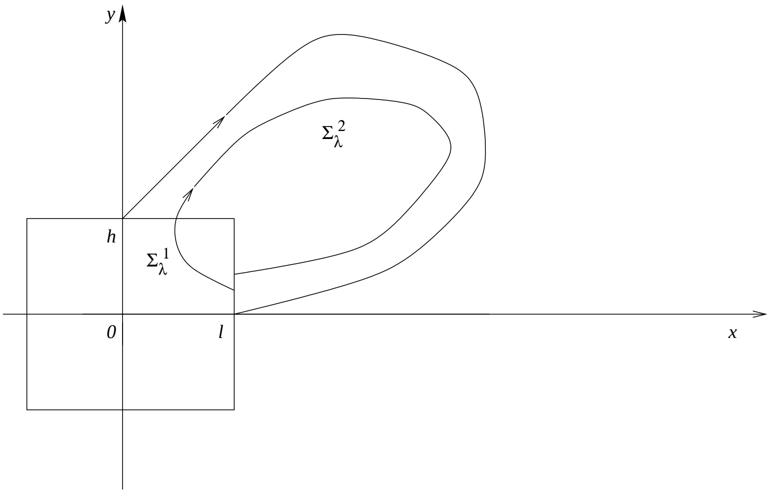

Figure 1: Within the box, the flow is essentially linear.

be written as

$$\begin{cases} \dot{x} = \nu_1 x + f(x,y,\lambda), \\ \dot{y} = \nu_2 y + g(x,y,\lambda) \end{cases} \tag{5}$$

where $f(x,y,\lambda)$ and $g(x,y,\lambda)$ are higher order terms. When $\lambda = \lambda^*$, there is a homoclinic orbit. Thus there exists a global nonlinear flow which takes trajectories close to the unstable manifold away from the linear region and then back into it. The global nonlinear flow induces a return map

$$\Sigma_\lambda = \Sigma^2_\lambda \circ \Sigma^1_\lambda : \ (l,y) \to (l, \phi(y,\lambda)) \tag{6}$$

With $|y| < h << 1$ and $|\lambda - \lambda^*| << 1$. Σ^1_λ maps the plane $x = l$ to the plane $y = h$ and is determined by the local linearised flow of (5). Thus it is easy to check that

$$\Sigma^1_\lambda : \ (l,y) \to (l(y/h)^\delta, h) \tag{7}$$

with $|y| < h << 1$ and $\delta = |\frac{\nu_1}{\nu_2}|$. The time needed to perform the linear map Σ^1_λ from $x = l$ to $y = h$ is

$$T_1(y) = \frac{1}{\nu_2} ln \frac{h}{y} \tag{8}$$

Σ^2_λ specifies how trajectories leave $y = h$ into the nonlinear region and then return to the linear region at $x = l$, and can be written as

$$\Sigma^2_\lambda : \ (x,h) \to (l, \psi(x,\lambda)) \tag{9}$$

where $|x| < l << 1$ and $|\lambda - \lambda^*| << 1$.

At $\lambda = \lambda^*$, we have $\psi(0, \lambda^*) = 0$. If the linear region is taken small enough, $\psi(x, \lambda)$ will be given by its Taylor expansion until the first order term, that is,

$$\psi(x, \lambda) = ax + b(\lambda - \lambda^*) \tag{10}$$

here $a = \frac{\partial\psi(0,\lambda^*)}{\partial x}$ and $b = \frac{\partial\psi(0,\lambda^*)}{\partial\lambda}$. Combining (7) with (10), we get the return map

$$\Sigma_\lambda : \ y \to cy^\delta + b(\lambda - \lambda^*) \tag{11}$$

with $c = \frac{al}{h^\delta}$, $|y| < h << 1$ and $|\lambda - \lambda^*| << 1$. The fixed point $y(\lambda)$ of the return map Σ_λ satisfies

$$cy^\delta + b(\lambda - \lambda^*) = y \tag{12}$$

In equation (12), ignoring the higher order terms of y, the fixed point can be taken to be

$$y(\lambda) = \begin{cases} b(\lambda - \lambda^*), & \text{if } \delta > 1 \\ \\ (\frac{b}{c}(\lambda^* - \lambda))^{\frac{1}{\delta}}, & \text{if } 0 < \delta < 1 \end{cases} \tag{13}$$

and the eigenvalue $\nu(\lambda)$ of the return map Σ_λ at the fixed point is

$$\nu(\lambda) = \begin{cases} c\delta(b(\lambda - \lambda^*))^{\delta-1}, & \text{if } \delta > 1 \\ \\ c\delta((\frac{b}{c}(\lambda^* - \lambda))^{\frac{\delta-1}{\delta}}, & \text{if } 0 < \delta < 1 \end{cases} \tag{14}$$

The relation (14) gives the stability information of the periodic solutions (see Guckenheimer & Holmes [1983]). To derive an asymptotic estimate for the nontrivial Floquet multiplier in terms of the period, we notice that if the linear region is small enough, the time spent in the linear region (8) is the main contribution to the period $T(\lambda)$ of the periodic solution corresponding to the fixed point $y(\lambda)$, and we can take $T(\lambda) \simeq T_1(y(\lambda))$. In view of (8) and (13), we have the following asymptotic estimate for the period

$$T(\lambda) \sim \begin{cases} -\frac{1}{\nu_2}\ln|\lambda - \lambda^*|, & \text{if } \delta > 1 \\ \\ \frac{1}{\nu_1}\ln|\lambda - \lambda^*|, & \text{if } 0 < \delta < 1 \end{cases} \tag{15}$$

which implies that

$$\lambda - \lambda^* \sim \begin{cases} e^{-\nu_2 T(\lambda)}, & \text{if } \delta > 1 \\ \\ e^{\nu_1 T(\lambda)}, & \text{if } 0 < \delta < 1 \end{cases} \tag{16}$$

Substituting (16) in (14), we obtain an asymptotic estimate for the nontrivial Floquet multiplier

$$\nu(\lambda) \sim e^{-\nu_2(\delta-1)T(\lambda)}, \qquad \text{if } \delta \neq 1 \tag{17}$$

Thus the nontrivial Floquet multiplier varies exponentially with exponent $-\nu_2(\delta - 1)$ as $T \to \infty$. If $\delta > 1$, the nontrivial Floquet multiplier contracts to zero; in particular, the contraction is very strong when $|\nu_1|$ is very large. If $0 < \delta < 1$, the nontrivial Floquet multiplier expands to ∞. Thus depending on the relative size of the eigenvalues of the fixed point, the nontrivial Floquet multiplier would either expand infinitely or contract toward zero. Based on the analysis above, we expect that for higher dimensional systems, the moduli of the Floquet multipliers vary in the same way. More precisely, a typical case is that only one of the eigenvalues ν_i, $i = 1, 2, \cdots, N$, say ν_1, is positive. Then depending on whether $\delta_i = |\frac{\nu_i}{\nu_1}|$, $i = 2,\ 3, \cdots,\ N$ is larger or smaller than 1, a Floquet multiplier will contract or expand with the exponent $-\nu_1(\delta_i - 1)$. To verify this, we have done numerical experiments.

To make sure that the numerical computation is accurate enough and reliable, we give some details of the numerical methods for computing Floquet multipliers and of our computational strategies. Following (Fairgrieve (1994)), a numerical method for computing Floquet multipliers is considered to be robust if the error in the computed Floquet multipliers is directly proportional to the truncation error of the computed periodic solution and is not badly influenced by the roundoff error in the eigenvalue calculation. The numerical method implemented in the AUTO97 software package is robust (Fairgrieve et al.(1991), (1994)). In the AUTO97, a spline collocation scheme with adaptive mesh selection is applied to compute the periodic solution. The truncation error (Ascher et al.(1979) and Houstis (1978)) is determined by the number of mesh subintervals and the number of collocation points per subinterval. In our numerical computations, we have used set 4 collocation points per subinterval and we have chosen the number of mesh subintervals large but not too large so that the roundoff error does not dominate the truncation error. In addition, we set all the relative convergence tolerances in the Newton/Chord methods small enough such that the error in the computed periodic solution is dominated by the truncation error. Moreover, throughout the numerical computation, we always use adaptive mesh selection strategy and repeat the computation with different number of mesh subintervals. These are effective safeguards to avoid spurious solutions. The trivial Floquet multiplier is monitored and controlled within an acceptable accuracy, and in most runs, is obtained to maximal accuracy. Except that some of the Floquet multipliers become so small that numerical noise becomes visible, the Floquet multipliers computed within AUTO are accurate and reliable.

For the case of three real eigenvalues ν_i, $i = 1, 2, 3$, if considering the ODE system in reverse time, we only need to verify the situation with $\nu_1 > 0$, $\nu_3 < \nu_2 <$

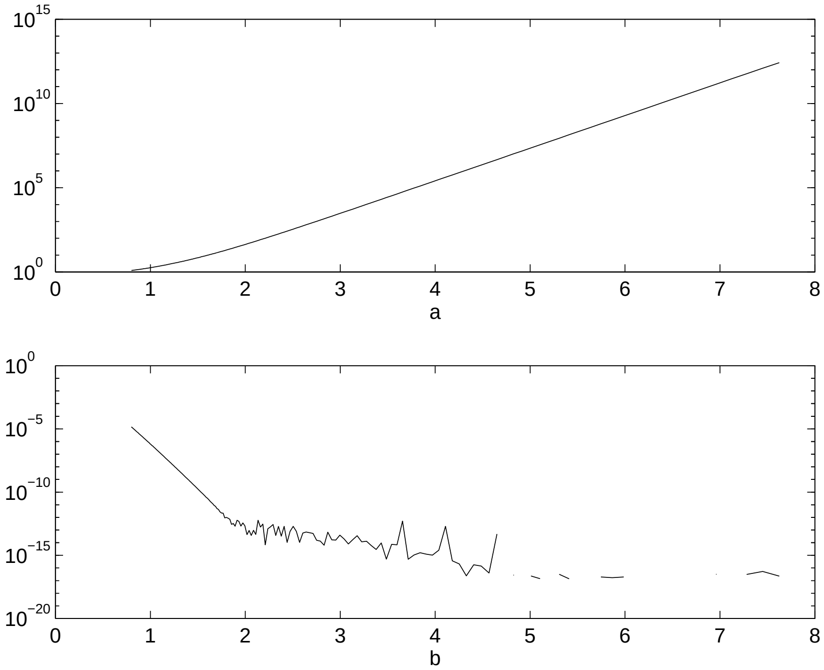

Figure 2: Periodic Solutions of the Lorenz equations: period versus logarithm of the moduli of the Floquet multipliers, indicating that the nontrivial Floquet multipliers vary exponentially with the period. (a) Floquet multiplier which expands to ∞; (b) Floquet multiplier which contracts to 0.

0. As our computational model, we take the well-known Lorenz equations

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz \end{cases} \tag{18}$$

As usual, we take $\sigma = 10,\ b = 8/3$ and $0 < r < \infty$. When $r = \bar{r} \simeq 13.92$, there is a homoclinic orbit connecting the origin and the three real eigenvalues of the origin at $\bar{r}$ are $(\nu_1,\ \nu_2,\ \nu_3) = (7.1299,\ -2.6667,\ -18.1299)$.

We trace a branch of periodic solutions, which bifurcates from a Hopf point at $r \simeq 24.74$ and terminates at the homoclinic orbit. Since this branch does not contain any further local bifurcations, the Floquet multipliers do not cross the unit circle. Their modulus should change in the same way as two-dimensional systems. More precisely, since the eigenvalues of the origin for $\bar{r}$ satisfy the relation $\delta_1 = |\frac{\nu_3}{\nu_1}| > 1$, the two dimensional analysis predicts that the modulus of one of the two nontrivial Floquet multipliers should contract to zero with exponent $-\nu_1(\delta_1 - 1) \simeq -11.0$. The fast contraction is due to the large negative real eigenvalue $\nu_3 \simeq -18.13$. The modulus of the other nontrivial Floquet multiplier should expand to ∞ with exponent $-\nu_1(\delta_2 - 1) \simeq 4.4632$ since $\delta_2 = |\frac{\nu_2}{\nu_1}| < 1$.

The computed nontrivial Floquet multipliers are shown in Figure 2, demonstrating that the nontrivial Floquet multipliers indeed expand or contract exponentially with the period T. The computed smaller nontrivial Floquet multiplier quickly becomes so small that the computational results are subject to noise for $T > 2$. Using least squares fitting, we that the exponents for the smaller and larger nontrivial Floquet multiplier is -14.847 and 4.012, respectively. Compared with the predicted exponents -11.0 and 4.4632, the computed exponent for the larger one is more accurate than that for the smaller one.

2.1 A pair of complex conjugate eigenvalues

For the case where there are complex conjugate pairs of eigenvalues, the analysis is more complicated. A three-dimensional system with a pair of complex conjugate eigenvalues is the simplest situation we need to consider. We assume that the three eigenvalues $\nu_i,\ i = 1, 2, 3$, satisfy $Re(\nu_3) = Re(\nu_2) < 0$ and $\nu_1 > 0$. Using Sil'nikov's standard analysis method, Glendinning and Sparrow (1984) discuss the stability of periodic solutions of 3-dimensional ODE systems which approach the homoclinic orbit. Their results state that if $\delta = |\frac{Re(\nu_2)}{\nu_1}| > 1$, there is a branch of stable periodic solutions which approach the homoclinic orbit without any local bifurcations and the two nontrivial Floquet multipliers contract to zero; if $\delta = |\frac{Re(\nu_2)}{\nu_1}| < 1$, there is a branch of periodic solutions which approach the homoclinic orbit through an infinite sequence of pairs of two successive saddle-node and two successive period-doubling bifurcations. The two nontrivial Floquet multipliers will oscillate around 1 and -1, respectively. It is not easy to compute their asymptotic estimates in terms of the period T as we did in the two dimensional setting. Instead, following the stability analysis in Glendinning and Sparrow (1984), we have performed numerical experiments to show the relation between the two nontrivial Floquet multipliers and the period. To find out how large or how small the modulus of the Floquet multipliers would be, we take Arneodo's equations for $\delta > 1$ and Chua's circuit equations for $\delta < 1$ as model problems.

Arneodo's equations can be written as

$$\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = -z - by + cx - x^2 \end{cases} \tag{19}$$

Arneodo et al.(1982) have located homoclinic orbits at some values of the parameters b and c. We fix $b = 0.5$ and take $c > 0$ as a free parameter. For $c = c^* \simeq 0.9641494$, there is a homoclinic orbit connecting the fixed point at the origin. The eigenvalues (ν_1, ν_2, ν_3) of the origin at c^* are $(0.629, -0.814 \pm i0.928)$. The periodic solutions arising from the Hopf point do not undergo any local bifurcations when they approach the homoclinic orbit. Thus the moduli of the Floquet multipliers do not cross the unit circle, but in this case, the moduli of the Floquet multipliers contract to zero in an oscillatory way in function of the period. For $T > 28$, the smallest Floquet multipliers (Figure 3b) is computed inaccurately due

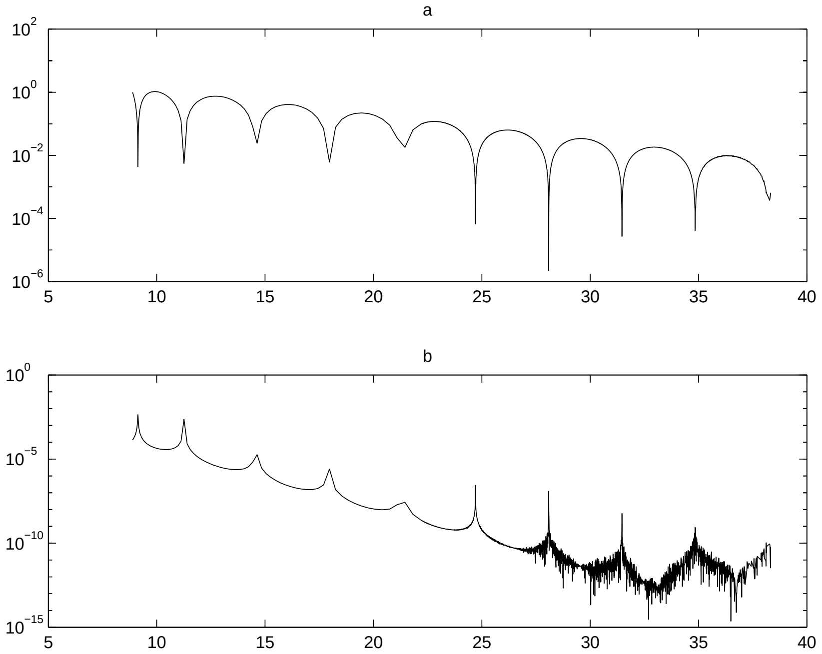

Figure 3: The modulus of the Floquet multipliers of the periodic solutions of Arneodo's equation in function of the period. Both of the Floquet multipliers contract to zero. The peaks in curve (b) indicate an increasing trend, but they become more and more sharp when the period $\to \infty$.

to numerical noise and is omitted from the Figure, see Figure 3. The oscillations of the moduli of the Floquet multipliers have peaks. Although in Figure 3b, the peaks indicate a increasing trend as the period goes to infinity, both the modulus and the width of the peaks become smaller so that numerically it is not easy to find the sharp points by solving the approximate periodic boundary value problem (2). Ignoring the sharp points, we can say that this situation is similar to the case with three real eigenvalues with both $\delta_1 > 1$ and $\delta_2 > 1$.

For the case of complex conjugate eigenvalue with $\delta < 1$, we deal with Chua's circuit equations

$$\left\{ \begin{array}{l} \dot{x} = \alpha(y - \phi(x)) \\ \dot{y} = x - y + z \\ \dot{z} = -\beta z \end{array} \right. \tag{20}$$

Following Huang, et al.(1996), we take $\phi(x) = (x^3/16 - x/6)$, $\alpha = 8.0$ and β to be the free parameter. The system has three fixed points, i.e, $P_0 = (0, 0, 0)$, $P_\pm = ((8/3)^{1/2}, 0, \pm(8/3)^{1/2})$ for any $\beta \in R$. When $\beta = 4.684658$, $P_\pm$ are Hopf bifurcation points, from which a branch of periodic solutions bifurcates. The stability analysis in (Glendinning et al.(1984)) shows that as the periodic solutions approach the homoclinic orbit, a saddle-node bifurcation and a period-doubling bifurcation occur successively and between them, there is a short section of unstable periodic solutions. This pattern of stability changes is repeated until the periodic solutions end at the homoclinic orbit when $\beta = \bar{\beta} \simeq 4.5965$. The periodic

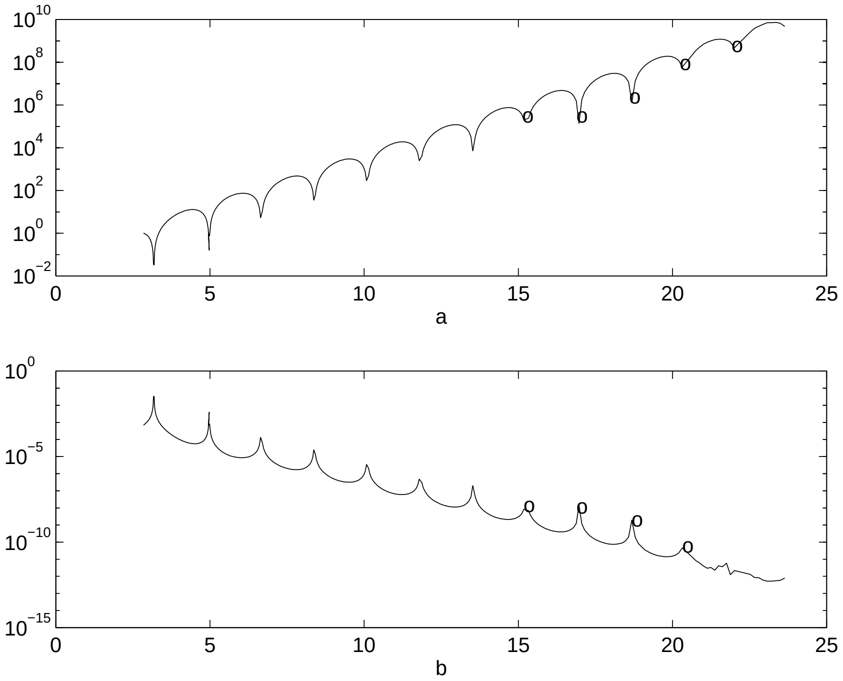

Figure 4: If the sharp points are ignored, the modulus of the Floquet multiplier of Chua's equations in (a) expand to infinity; the one in (b) contracts to zero. They vary in a somewhat similar way to the real eigenvalues case shown in Figure (2).

solutions approach the homoclinic orbit through a sequence of local bifurcations, contrary to the situations in Lorenz's equations and Arneodo's equations. The fixed point on the homoclinic orbit is the origin, whose eigenvalues at $\beta = \bar{\beta}$ are $(\nu_1, \nu_2, \nu_3) = (2.2776, -1.0891 \pm 1.6357i)$.

Since there is an infinite sequence of saddle-node and period-doubling bifurcations, the moduli of the Floquet multipliers do not increase or decrease in a monotonous way as in Lorenz's equations, but oscillate around ± 1. The computed moduli of the Floquet multipliers, presented in Figure 4, indeed reflects the bifurcation information. The peaks of the curves of the computed modulus result from the saddle-node and period-doubling bifurcations. The intervals between these two successive bifurcations are so small that the two nontrivial Floquet multipliers cross ± 1 almost for the same value of the period; thus the moduli of the Floquet multipliers will change very quickly. In addition, these intervals will become smaller and smaller when the period tends to infinity. Successive saddle-node and period doubling bifurcations and the unstable periodic solutions between them are not found by numerical computation with AUTO when T is larger than 15, even with $T = 10^{-5}$. Ignoring the peaks, caused by the local bifurcations, the general behaviours of the Floquet multipliers is as follows: one fo the Floquet multipliers expands to infinity, similar to the case of real eigenvalues with $\delta < 1$; the other Floquet multipliers contracts to zero, similar to the case of real eigenvalues with $\delta > 1$.

The discussion above has shown that for a pair of complex conjugate eigenval-

Eigenvalues of fixed point		*Real*
Real part	*Imaginary part*	*Floquet multipliers*
7.8152504E+00	-1.0452187E+02	9.9997384E-01
7.8152504E+00	1.0452187E+02	-3.6455148E-13
-3.5554418E+01	0.0000000E+00	-8.5525090E-20
-4.3258700E+02	0.0000000E+00	6.8230858E-18
-1.6085723E+03	0.0000000E+00	1.4981425E-19
-3.9081603E+03	0.0000000E+00	7.1584459E-18
-7.8728384E+03	0.0000000E+00	5.1133315E-19
-1.4111208E+04	0.0000000E+00	4.4722867E+11

Table 1: Kuramoto-Sivashinsky equation: The eigenvalues of the fixed point and the Floquet multipliers of the periodic solutions with $\alpha = 3.5174067E+01$.

ues, numerically the moduli of the Floquet multipliers vary in a similar way as in the case of three real eigenvalues, i.e., depending on the relative size of the positive and negative real part of the eigenvalues, the Floquet multipliers will expand infinitely or contract to zero. For higher dimensional systems, we expect that the same conclusion is true. A typical case is when there is only one pair of complex conjugate eigenvalues with positive real part, say, $\nu_i,\ i = 1, 2$, with Re(ν_1)=Re(ν_2)> 0, and $|\nu_3| \leq |\nu_4| \leq \cdots \leq |\nu_N|$ are negative real eigenvalues. Consider this system in reverse time and let $\delta_i = |\frac{\nu_i}{Re\nu_1}|,\ i = 3,\ \cdots,\ N$, then we predict that if $\delta_3 < 1$, two of the nontrivial Floquet multipliers expand to infinity; if $\delta_1 > 1$, one of nontrivial Floquet multipliers expands infinitely and another nontrivial Floquet multiplier will contract to zero. Moreover, depending on $\delta_i\ (i \geq 3) < 1$ *or* > 1, one of the other nontrivial Floquet multipliers will expand or contract. When one eigenvalue has a large real part, one of the nontrivial Floquet multipliers becomes quickly very small. To verify this further, we consider an infinite dimensional model, Kuramoto-Sivashinsky equation (K-S Equation)

$$\frac{\partial u}{\partial t} + 4\frac{\partial^4 u}{\partial x^4} + \alpha\left(\frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x}\right) = 0 \tag{21}$$

subject to spatial periodic boundary conditions $u(x, t) = u(x + 2\pi, t)$ with $0 \leq x \leq 2\pi$. Using the traditional Galerkin method to discretize the space variable, Jolly et al.(1990)) obtain the following ODE system

$$\dot{x}_k = (-4k^4 + \alpha k^2)x_k - \alpha\beta_k^m,\ \ 1 \leq k \leq m \tag{22}$$

where

$$\beta_k^m = 1/2\sum_{j=1}^{m} jx_j[x_{j+k} + sgn(k-j)x_{|k-j|}] \tag{23}$$

The index m represents the number of modes used in the discretization. We set $x_j = 0$ if $j < 0$ and $j > m$, respectively. At $\alpha = 34.299$, there are two Hopf

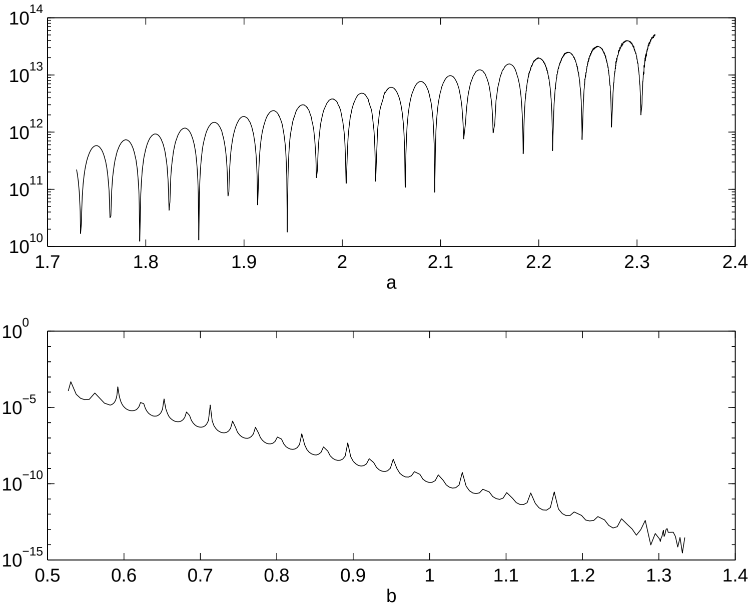

Figure 5: The modulus of the two nontrivial Floquet multipliers of K-S equation, which cross through the unit circle. The two modulus curves are similar to those of Chua's equations as shown in Figure(4).

points H_i, $i = 1, 2$. Numerical computations (Jolly et al.(1990)) indicate that the periodic solutions bifurcating from both Hopf points terminate at a homoclinic orbit at $\alpha = \alpha^* \simeq 35.174066$. We focus on one of the homoclinic orbits. As Jolly et al. (1990) have shown, the periodic solutions approach to the homoclinic orbit in the same way as Chua's equations. The fixed point on the homoclinic orbit has a complex conjugate pair of eigenvalues with positive real part, and the other eigenvalues are large negative real numbers, as shown in Table 1. Due to $\delta_3 > 1$, two of the Floquet multipliers, which cross through the unit circle, should vary in a similar way as for Chua's equations. Figure 5 presents the computed moduli of the Floquet multipliers for the 8-mode discretization. It indeed looks similar to Figure 4. Moreover due to $\delta_i > 1$, $i = 4, \cdots, N$, the rest of the nontrivial Floquet multipliers should contract to zero. In fact, since the negative real eigenvalues are very large in modulus, the rest of the computed Floquet multipliers become quickly so small that they are not obtained with a reasonable accuracy in our computation where the trivial Floquet multiplier has only 4 significant digits. To show the order of these small Floquet multipliers, in Table(1), we list them for a periodic solution close to the homoclinic orbit. Our computational results agree with the prediction that eigenvalues with a large negative real part make the Floquet multipliers contract very quickly. The computed results for discretizations of the K-S equation with 4, 16 and 32 modes also further confirm the prediction.

3. Conclusions

We have related the eigenvalues of the fixed point having a homoclinic orbit with the Floquet multipliers of the periodic solutions which approach to the homoclinic orbit. We have shown that the Floquet multipliers expand to infinity or contract to zero, depending on the relative size of the positive and negative real parts of the eigenvalues. Specifically, we have studied the following two typical

- if all eigenvalues $\nu_i,\ i = 1, 2, \cdots, N$ are real and only one of them, say ν_1, is positive, then depending on whether $\delta_i = |\frac{\nu_i}{\nu_1}|$, $i = 2,\ 3, \cdots,\ N$ is larger or smaller than 1, a Floquet multiplier will contract or expand asymptotically with exponent $-\nu_1(\delta_i - 1)$;
- Now suppose that there is only one pair of complex conjugate eigenvalues with positive real part, say, $\nu_i,\ i = 1, 2$, with Re(ν_1)=Re(ν_2)> 0, and that ν_i $(i \geq 3)$ are negative real eigenvalues. Then if $\delta_3 < 1$, two of the nontrivial Floquet multipliers expand to infinity; if $\delta_3 > 1$, one of nontrivial Floquet multipliers expands infinitely and another nontrivial Floquet multiplier will contract to zero. Moreover, depending on δ_i $(i > 3) < 1$ *or* > 1, one of the other nontrivial Floquet multipliers will expand or contract. Here $\delta_i = |\frac{\nu_i}{Re(\nu_1)}|$, $i = 3,\ \cdots,\ N$.

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