

**Translation of the Russian paper
“Orthogonal systems of rational
functions on the unit circle” by
M.M. Džrbašian**

Karsten Müller Adhemar Bultheel

Report TW 253, February 1997



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Abstract

We give the English translation of the paper “Orthogonal systems of rational functions on the unit circle” by M.M. Džrbašian which appeared in two parts in *Izvestiya Akademii Nauk Armyanskoj SSR, ser. Matematika* 1 (1966) 3–24, 2 (1966) 106–125. This paper is one of the first to deal with orthogonal rational functions on the unit circle whose poles are fixed. These rational functions generalize the orthogonal polynomials of Szegő.

We have tried to keep as close as possible to the original text, yet writing readable English.

Keywords : orthogonal rational functions, rational approximation, Szegő polynomials

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Orthonormal sets of rational functions on the unit circle

M.M. Džrbashian [Džrbašian]

Izvestiya Akademii Nauk Armyanskoi SSR, ser. Matematika
1 (1966) 3–24, 2 (1966) 106–125

Abstract

This paper reveals the algebraic properties of sets of rational functions which are orthogonal on the unit circle with respect to the weight $(2\pi)^{-1}d\alpha(x)$, and whose poles are among a given sequence of points situated outside the unit circle.

In case when all the poles of the set under consideration coincide with the point at infinity, the theorems proved here concur with the well known assertions of the theory of orthogonal (with respect to the weight function) polynomials developed by Szegő [2], [3].

1 Structure of orthogonal systems. Functional properties of kernels of the distribution $(2\pi)^{-1}d\alpha(x)$

1.1. Let $\{\alpha_k\}_0^\infty$ ($|\alpha_k| < 1$) be an arbitrary sequence of complex numbers which may appear as numbers of finite or even infinite multiplicity (and also not necessarily in succession).

Two sequences $\{\nu_k\}_0^\infty$ and $\{p_k\}_0^\infty$ of positive integers are associated with the sequence $\{\alpha_k\}_0^\infty$, where (for $k \geq 0$) ν_k denotes the multiplicity of appearance of the number α_k in the group of numbers $\{\alpha_0, \alpha_1, \dots, \alpha_k\}$ and p_k is defined from the condition

$$p_k = \begin{cases} 1, & \text{if } \alpha_k \neq 0 \\ \nu_k, & \text{if } \alpha_k = 0 \end{cases} \quad (k = 0, 1, 2, \dots). \quad (1.1)$$

Finally, with the sequence $\{\alpha_k\}_0^\infty$ of complex numbers we associate the sequence

$$\left\{ \frac{z^{p_k-1}}{(1 - \overline{\alpha_k}z)^{\nu_k}} \right\}_0^\infty \quad (1.2)$$

of rational functions.

Note that (for given $k \geq 0$) the function $z^{p_k-1}(1 - \overline{\alpha_k}z)^{-\nu_k}$ has only one pole of order ν_k in the point $z = \frac{1}{\alpha_k}$ ($|\frac{1}{\alpha_k}| > 1$), if $\alpha_k \neq 0$. If $\alpha_k = 0$, this pole is situated in the point $z = \infty$ and has the order $p_k - 1$.

Observe also that in the last case, where $\alpha_k = 0$ ($k = 0, 1, 2, \dots$), it is obvious that $\nu_k = p_k = k + 1$ ($k = 0, 1, 2, \dots$) so that with the null sequence $\{0\}_0^\infty$ we associate the sequence of powers $\{z^k\}_0^\infty$.

With the sequence of numbers $\{\alpha_k\}_0^\infty$, we associate a system of rational functions $\{\phi_k(z)\}_0^\infty$ defined as

$$\begin{aligned}\phi_0(z) &= \frac{(1 - |\alpha_0|^2)^{1/2}}{1 - \overline{\alpha_0}z}, \\ \phi_n(z) &= \frac{(1 - |\alpha_n|^2)^{1/2}}{1 - \overline{\alpha_n}z} \prod_{k=0}^{n-1} \frac{\alpha_k - z}{1 - \overline{\alpha_k}z} \frac{|\alpha_k|}{\alpha_k} \quad (n = 1, 2, 3, \dots)\end{aligned}\quad (1.3)$$

where $\frac{|\alpha_k|}{\alpha_k} = \frac{\overline{\alpha_k}}{|\alpha_k|} = -1$ for $\alpha_k = 0$. Such a system is called *Malmquist system*.

It is well known that this system is orthonormal on the unit circle in the sense that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_n(z) \overline{\phi_m(z)} dx = \delta_{n,m} = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases} \quad (n, m = 0, 1, 2, \dots; z = e^{ix}). \quad (1.4)$$

From this, it follows by elementary arguments that the system of functions $\{\phi_n(z)\}_0^\infty$ is obtained by orthogonalization of the ordered sequence of rational functions (1.2) on the unit circle $z = e^{ix}$, $-\pi \leq x \leq \pi$ with respect to the weight function $\frac{1}{2\pi} dx$. Hence it follows easily, that the representation

$$\begin{aligned}\frac{z^{p_k-1}}{(1 - \overline{\alpha_n}z)^{\nu_k}} &= \sum_{k=0}^n \alpha_k^{(n)} \phi_k(z), \quad \alpha_n^{(n)} \neq 0, \\ \phi_n(z) &= \sum_{k=0}^n b_k^{(n)} \frac{z^{p_k-1}}{(1 - \overline{\alpha_k}z)^{\nu_k}}, \quad b_n^{(n)} \neq 0\end{aligned}\quad (1.5)$$

holds true for arbitrary $n \geq 0$.

1.2. Let $\alpha(x)$ be an arbitrary bounded nondecreasing function in the interval $[-\pi, \pi]$ with an infinite set of points of increase.

We will orthogonalize the ordered sequence of rational functions (1.2) on the unit circle $z = e^{ix}$ ($-\pi \leq x \leq \pi$) with respect to the weight function $(2\pi)^{-1} d\alpha(x)$.

But it obviously follows from formula (1.5), that the orthogonalization process with respect to the weight function $(2\pi)^{-1} d\alpha(x)$ on the unit circle applied to the ordered sequences of functions

$$\left\{ \frac{z^{p_k-1}}{(1 - \overline{\alpha_k}z)^{\nu_k}} \right\}_0^\infty \quad \text{and} \quad \{\phi_k(z)\}_0^\infty$$

leads to one and the same orthogonal system.

Thus we get a sequence of rational functions $\{\Phi_n(z)\}_0^\infty$ satisfying conditions which determine the functions of our systems in a unique way:

a) $\Phi_n(z)$ is a "polynomial of degree n " of the first $n + 1$ Malmquist functions

$$\Phi_n(z) = \sum_{k=0}^n c_{k,n} \phi_k(z), \quad c_{1,n} > 0, \quad (1.6)$$

b)

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_n(z) \overline{\Phi_m(z)} d\alpha(x) = \delta_{n,m}, \quad z = e^{ix} \quad (n, m = 0, 1, 2, \dots). \quad (1.7)$$

We denote

$$(\phi_p, \phi_q) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_p(z) \overline{\phi_q(z)} d\alpha(x), \quad z = e^{ix} \quad (p, q = 0, 1, 2, \dots). \quad (1.8)$$

and introduce the Gramian as

$$D_0 = (\phi_0, \phi_0), \quad D_n = \begin{vmatrix} (\phi_0, \phi_0) & \dots & (\phi_0, \phi_n) \\ (\phi_1, \phi_0) & \dots & (\phi_1, \phi_n) \\ \vdots & & \vdots \\ (\phi_n, \phi_0) & \dots & (\phi_n, \phi_n) \end{vmatrix} \quad (n = 1, 2, \dots). \quad (1.9)$$

In view of the fact that every finite part $\{\phi_k(z)\}_0^\infty$ of the countable systems of functions $\{\phi_k(z)\}_0^\infty$ is linearly independent, it can be asserted, that all determinants D_n ($n = 0, 1, 2, \dots$) are positive.

Hence, using the orthogonalization formulas of E. Schmidt, the desired functions of the systems $\{\Phi_k(z)\}_0^\infty$ can be represented in the form

$$\Phi_0(z) = \frac{\phi_0(z)}{\sqrt{D_0}},$$

$$\Phi_n(z) = \frac{1}{\sqrt{D_{n-1}D_n}} \begin{vmatrix} (\phi_0, \phi_0) & \dots & (\phi_0, \phi_{n-1}) & (\phi_0, \phi_n) \\ (\phi_1, \phi_0) & \dots & (\phi_1, \phi_{n-1}) & (\phi_1, \phi_n) \\ \vdots & & \vdots & \vdots \\ (\phi_{n-1}, \phi_0) & \dots & (\phi_{n-1}, \phi_{n-1}) & (\phi_{n-1}, \phi_n) \\ \phi_0(z) & \dots & \phi_{n-1}(z) & \phi_n(z) \end{vmatrix}. \quad (1.10)$$

For our purpose we introduce a special notation for the extreme coefficients of the function $\Phi_n(z)$ in its decomposition (1.6) in terms of the Malmquist functions. Namely, we put

$$k_n = c_{n,n} \quad \text{and} \quad l_n = c_{0,n}. \quad (1.11)$$

Obviously we have by formula (1.10)

$$k_n = (D_{n-1}D_n)^{1/2}, \quad (1.12)$$

$$l_n = (-1)^n (D_{n-1}D_n)^{-1/2} \begin{vmatrix} (\phi_0, \phi_1) & \dots & (\phi_0, \phi_n) \\ (\phi_1, \phi_1) & \dots & (\phi_1, \phi_n) \\ \vdots & & \vdots \\ (\phi_{n-1}, \phi_0) & \dots & (\phi_{n-1}, \phi_n) \end{vmatrix}, \quad (1.13)$$

where we set $D_{-1} = 1$.

Finally we mention that in the particular case where $\alpha_k = 0$ ($k = 0, 1, 2, \dots$), then the Malmquist system becomes the system of powers $\{z^k\}_0^\infty$. When orthogonalizing with respect to the weight $(2\pi)^{-1}d\alpha(x)$, the system $\{\Phi_k(z)\}_0^\infty$ becomes the orthogonal system of Szegő polynomials $\{P_k(z)\}$ with respect to the same weight

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P_n(z) \overline{P_m(z)} d\alpha(z) = \delta_{n,m}, \quad z = e^{ix} \quad (n, m = 0, 1, 2, \dots). \quad (1.14)$$

1.3 Now we give some lemmas about optimizing properties of orthogonal systems $\{\Phi_k(z)\}_0^\infty$ and the associated systems

$$S_n(\xi; z) = \sum_{k=0}^n \overline{\Phi_k(\xi)} \Phi_k(z) \quad (n = 0, 1, 2, \dots). \quad (1.15)$$

which are called *kernels of the distribution* $(2\pi)^{-1}d\alpha(x)$.

Note that these properties are obviously analogous to the corresponding well-known optimality properties of the Szegő polynomials.

First let us agree on the following notation. For an integer $n \geq 0$, we denote by $M\{\alpha_k\}_0^n$ all possible linear combinations of the system of Malmquist functions $\{\phi_k(z)\}_0^\infty$. Thus it is the set of rational functions of the form

$$R_n(z) = \sum_{k=0}^n c_k \phi_k(z) , \quad (1.16)$$

where $\{c_k\}_0^n$ are arbitrary complex numbers. Such functions are called *generalized Malmquist polynomials of degree n*.

Furthermore, observe that a general aspect of the theory of orthogonal systems implies, that not only for an arbitrary $n \geq 0$

$$\Phi_n(z) = \sum_{k=0}^n c_{k,n} \phi_k(z) , \quad c_{n,n} > 0 , \quad (1.17')$$

but conversely, also

$$\phi_n(z) = \sum_{k=0}^n d_{k,n} \Phi_k(z) , \quad d_{n,n} \neq 0 . \quad (1.17'')$$

Thus an arbitrary function $R_n(z) \in M\{\alpha_k\}_0^n$ can be represented in the form

$$R_n(z) = \sum_{k=0}^n d_k \Phi_k(z) . \quad (1.18)$$

L e m m a 1. *In the class $M\{\alpha_k\}_0^n$ of functions of the form*

$$Q_n(z) = \phi_n(z) + \alpha_1 \phi_{n-1}(z) + \dots + \alpha_n \phi_0(z) , \quad (1.19)$$

the minimum of the functional

$$\mu(Q_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |Q_n(z)|^2 d\alpha(x) , \quad z = e^{ix} , \quad (1.20)$$

is obtained for the function

$$Q_n^{(0)}(z) = k_n^{-1} \Phi_n(z) , \quad (1.21)$$

where

$$\min \mu(Q_n) = \mu(Q_n^{(0)}) = \mu_n = k_n^{-2} = \frac{D_n}{D_{n-1}} . \quad (1.22)$$

Indeed, from (1.17'') we can conclude that the function $Q_n(z)$ can be represented in the form

$$Q_n(z) = \sum_{p=0}^n v_p \Phi_p(z) , \quad (1.19')$$

where the coefficients $\{v_p\}_0^{n-1}$ are arbitrary, but $v_n = k_n^{-1}$.

In view of the orthonormality of the system $\{\Phi_k(z)\}_0^\infty$ in the sense of (1.7), it follows from the representation (1.19'):

$$\mu(Q_n) = \sum_{p=0}^{n-1} |v_p|^2 + k_n^{-2} \geq k_n^{-2} ,$$

where the inequality becomes an equality only in the case $v_0 = v_1 = \dots = v_{n-1} = 0$. This obviously completes the proof.

L e m m a 2. *Let $\zeta \neq 1/\bar{\alpha}_k$ ($k = 0, 1, \dots, n$) be an arbitrary constant number. In the family of functions $\{P(z)\} \in M\{\alpha_k\}_0^n$ satisfying the additional condition*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |P_n(z)|^2 d\alpha(x), \quad z = e^{ix}, \quad (1.23)$$

the maximum of the functional

$$L(P_n) = |P_n(\zeta)|^2 \quad (1.24)$$

is realized by the function

$$P_n^{(0)} = \epsilon \frac{S_n(\zeta; z)}{\sqrt{S_n(\zeta; \zeta)}} \quad (|\epsilon| = 1), \quad (1.25)$$

where, according to (1.15),

$$S_n(\zeta; z) = \sum_{k=0}^n \overline{\Phi_k(\zeta)} \Phi_k(z),$$

and

$$L(P_n^{(0)}) = \max L(P_n) = S_n(\zeta; \zeta). \quad (1.26)$$

Indeed, with the representation of the function $P_n(z) \in M\{\alpha_k\}_0^n$ in the form

$$P_n(z) = \sum_{k=0}^n p_k \Phi_k(z),$$

and by condition (1.23), we get

$$\sum_{k=0}^n |p_k|^2 = 1.$$

But then according to Cauchy's inequality, we have

$$|P_n(\zeta)|^2 \leq \sum_{k=0}^n |p_k|^2 \cdot \sum_{k=0}^n |\Phi_k(\zeta)|^2 = S_n(\zeta; \zeta),$$

where equality is possible only when

$$p_k = c \overline{\Phi_k(\zeta)} \quad (k = 0, 1, 2, \dots),$$

i.e., if

$$P_n(z) = P_n^{(0)}(z) = c \sum_{k=0}^n \overline{\Phi_k(\zeta)} \Phi_k(z) = c S_n(\zeta; z),$$

where c is a constant.

The value of this constant is determined from condition (1.13)

$$\mu(P_n^{(0)}) = |c|^2 \sum_{k=0}^n |\Phi_k(\zeta)|^2 = 1,$$

from which follows, that

$$c = \frac{\epsilon}{\sqrt{S_n(\zeta; \zeta)}} \quad (|\epsilon| = 1).$$

This completes the proof of the lemma.

L e m m a 3. *Let $\zeta \neq 1/\alpha_k$ ($k = 0, 1, \dots, n$) and A_0 be arbitrary but fixed constants. In the family of functions $R_n(z) \in M\{\alpha_k\}_0^n$ satisfying the additional condition*

$$R_n(\zeta) = A_0, \quad (1.26')$$

the minimum of the functional

$$\mu(R_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |R_n(z)|^2 d\alpha(x), \quad z = e^{ix}, \quad (1.27)$$

is obtained for the function

$$R_n^{(0)}(z) = A_0 \frac{S_n(\zeta; z)}{S_n(\zeta; \zeta)}, \quad (1.28)$$

where

$$\mu(R_n^{(0)}) = \min_{\{R_n\}} \mu(R_n) = \frac{|A_0|^2}{S_n(\zeta; \zeta)}. \quad (1.29)$$

With the representation of the the function $R_n(z) \in M\{\alpha_k\}_0^n$ in the form (1.19'), we have

$$\mu(R_n) = \sum_{k=0}^n |v_k|^2.$$

Then condition (1.26) means that

$$\sum_{k=0}^n v_k \Phi_k(\zeta) = A_0.$$

Hence by the Cauchy inequality we get

$$|A_0|^2 \leq \sum_{k=0}^n |v_k|^2 \cdot \sum_{k=0}^n |\Phi_k(\zeta)|^2 = \mu(R_n) S_n(\zeta; \zeta),$$

i.e.,

$$\mu(R_n) \geq \frac{|A_0|^2}{S_n(\zeta; \zeta)},$$

where equality is only possible when

$$v_k = c \overline{\Phi_k(\zeta)} \quad (k = 0, 1, 2, \dots, n).$$

Therefore the optimizing function has the form

$$R_n(z) = R_n^{(0)}(z) = c \sum_{k=0}^n \overline{\Phi_k(\zeta)} \Phi_k(z) = c S_n(\zeta; z).$$

The value of the constant c is defined by condition (1.26'):

$$R_n^{(0)}(\zeta) = c S_n(\zeta; \zeta) = A_0,$$

i.e.,

$$c = \frac{A_0}{S_n(\zeta; \zeta)},$$

and the lemma is proved.

The kernel $S_n(\zeta; z)$ of the distribution $(2\pi)^{-1}d\alpha(x)$ which is involved in solutions of the two last optimization problems, could be characterized by the following important properties.

L e m m a 4. *Assume $\rho(\zeta; z) \in M\{\alpha_k\}_0^n$ for any value of the parameter*

$$\zeta \neq \frac{1}{\alpha_k} \quad (k = 0, 1, 2, \dots, n) .$$

Then the identity

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \rho(\zeta; z) \overline{g(z)} d\alpha(x) = \overline{g(\zeta)} , \quad z = e^{ix} , \quad (1.30)$$

holds for any function $g(z) \in M\{\alpha_k\}_0^n$, if and only if

$$\rho(\zeta; z) \equiv S_n(\zeta; z) . \quad (1.31)$$

Noting that any function $g(z) \in M\{\alpha_k\}_0^n$ can be represented in the form

$$g(z) = \sum_{k=0}^n \alpha_k \Phi_k(z) ,$$

and using condition (1.31), we arrive at the identity (1.30) since

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \rho(\zeta; z) \overline{g(z)} d\alpha(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \sum_{k=0}^n \overline{\Phi_k(\zeta)} \Phi_k(z) \right\} \overline{\left\{ \sum_{k=0}^n \alpha_k \Phi_k(z) \right\}} d\alpha(x) = \\ &= \sum_{k=0}^n \overline{\alpha_k} \overline{\Phi_k(\zeta)} = \overline{g(\zeta)} . \end{aligned}$$

Thus condition (1.31) is sufficient for the realization of the identity (1.30).

Note that the function $\rho(\zeta; z)$ can obviously be represented in the form

$$\rho(\zeta; z) = \sum_{k=0}^n C_k(\zeta) \Phi_k(z) .$$

Furthermore, if the condition (1.30) is fulfilled, then it holds in particular for the choice

$$g(z) = \Phi_p(z) \in M\{\alpha_k\}_0^n \quad (p = 0, 1, \dots, n) .$$

Therefore it follows from (1.30) that

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \rho(\zeta; z) \overline{\Phi_p(z)} d\alpha(x) &= \sum_{k=0}^n C_k(\zeta) \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_k(z) \overline{\Phi_p(z)} d\alpha(x) = \\ &= C_p(\zeta) \equiv \overline{\Phi_p(\zeta)} \quad (p = 0, 1, \dots, n) , \end{aligned}$$

i.e.,

$$\rho(\zeta; z) \equiv \sum_{k=0}^n \overline{\Phi_p(\zeta)} \Phi_k(z) = S_n(\zeta; z) .$$

2 Functional equation for kernels of the distribution $(2\pi)^{-1}d\alpha(x)$

2.1. At first let us prove some main identities* which holds for Malmquist systems $\{\phi_k(z)\}_0^\infty$ associated with a given sequence $\{\alpha_k\}_0^\infty$ of complex numbers.

Setting in the rest of the paper

$$B_{n+1} = \prod_{k=0}^n \frac{\alpha_k - z}{1 - \overline{\alpha_k}z} \frac{|\alpha_k|}{\alpha_k} \quad \dagger, \quad (2.1)$$

we have established

L e m m a 5. *For arbitrary values of the variables z and ζ , we have the identity*

$$\frac{1}{1 - \zeta z} = \sum_{k=0}^n \overline{\phi_k(\zeta)} \phi_k(z) + \frac{\overline{B_{n+1}(\zeta)} B_{n+1}(z)}{1 - \zeta z} \quad (n = 0, 1, 2, \dots). \quad (2.2)$$

Proof. Obvious it is sufficient to prove the correctness of the identity (2.2) for $|z| < 1$ and $|\zeta| < 1$.

Note that for $|\zeta| < 1$, we have

$$\frac{1}{2\pi} \int_{|t|=1} \frac{\overline{\phi_\nu(t)}}{1 - \zeta t} |dt| = \left\{ \frac{1}{2\pi} \int_{|t|=1} \frac{\phi_\nu(t)}{t - \zeta} dt \right\} = \overline{\phi_\nu(\zeta)} \quad (\nu = 0, 1, \dots, n).$$

Therefore, setting

$$\Phi_n(\zeta; z) = \frac{1}{1 - \zeta z} - \sum_{k=0}^n \overline{\phi_k(\zeta)} \phi_k(z), \quad (2.3)$$

we get the equality

$$\frac{1}{2\pi} \int_{|t|=1} \Phi_n(t; \zeta) \overline{\phi_\nu(t)} |dt| = 0 \quad (\nu = 0, 1, \dots, n). \quad (2.4)$$

But by (1.3), we have for $|t| = 1$

$$\frac{\overline{\phi_\nu(t)}}{t} = \frac{(1 - |\alpha_\nu|^2)^{1/2}}{t - \alpha_\nu} \prod_{k=0}^{\nu-1} \frac{1 - \overline{\alpha_k}t}{\alpha_k - 1} \frac{|\alpha_k|}{\overline{\alpha_k}} \quad (\nu = 0, 1, \dots), \quad (2.5)$$

where, for the value $\nu = 0$, the product $\prod_{k=0}^{\nu-1}$ should be replaced by one.

*Such identities and their generalizations were often used in interpolation problems of function theory (see, e.g., Dsh. Uolsh [5], P. Lagrange [6], E. Lammell [7]).

In our investigation, this identity is mainly used to deal with convergence of Fourier series in Malmquist systems [8].

†Just as in formula (1.3), it should be assumed that if $\alpha_k = 0$ then

$$\frac{|\alpha_k|}{\alpha_k} = \frac{\overline{\alpha_k}}{|\alpha_k|} = -1.$$

Since $|dt| = \frac{dt}{it}$ for $|t| = 1$, and using (2.5), the condition (2.4) can be written in the form

$$\frac{1}{2\pi i} \int_{|t|=1} \Phi_n(t; \zeta) \frac{\prod_{k=0}^{\nu-1} (1 - \overline{\alpha_k} t)}{\prod_{k=0}^{\nu-1} (\alpha_k - t)} dt = 0 \quad (\nu = 0, 1, \dots, n). \quad (2.4')$$

Note further, that the rational function under the integral in (2.4') is defined for an arbitrary $\nu \geq 0$. We conclude easily that the condition (2.4) is equivalent to the following:

$$\Phi_n^{(r)}(\alpha_\nu; \zeta) = \frac{r!}{2\pi i} \int_{|t|=1} \frac{\Phi_n(t; \zeta)}{(t - \alpha_\nu)^{1+r}} dt = 0 \quad \left(\begin{array}{l} r = 0, 1, \dots, p_\nu - 1 \\ \nu = 0, 1, \dots, n \end{array} \right), \quad (2.6)$$

where as above $p_\nu \geq 1$, denotes the multiplicity of the appearance of the number α_ν in the group of numbers $\{\alpha_0, \alpha_1, \dots, \alpha_n\}$.

But from equality (2.6) and taking definition (2.1) of the function $B_{n+1}(z)$ into account, we conclude that the quotient

$$\Phi_n(z; \zeta) / B_{n+1}(z)$$

is holomorphic in the closed disc $|z| \leq 1$. Therefore the integral formula

$$\Phi_n(z; \zeta) = \frac{B_{n+1}(z)}{2\pi} \int_{|t|=1} \frac{\Phi_n(t; \zeta) B_{n+1}^{-1}(t)}{1 - z\bar{t}} |dt| \quad (|z| < 1) \quad (2.7)$$

is valid.

Noting that for $|t| = 1$

$$B_{n+1}^{-1}(t) = \overline{B_{n+1}(t)},$$

we further have from (2.3),

$$\begin{aligned} \frac{1}{2\pi} \int_{|t|=1} \frac{\Phi_n(t; \zeta) B_{n+1}^{-1}(t)}{1 - z\bar{t}} |dt| &= \overline{\left\{ \frac{1}{2\pi i} \int_{|t|=1} \frac{B_{n+1}(t)}{(1 - \bar{z}t)(t - \zeta)} dt \right\}} - \\ &- \sum_{k=0}^n \phi_k(\zeta) \left\{ \frac{1}{2\pi i} \int_{|t|=1} \frac{B_{n+1}(t) \overline{\phi_k(t)}}{t(1 - \bar{z}t)} dt \right\}. \end{aligned} \quad (2.8)$$

Since the function

$$\frac{B_{n+1}(t)}{1 - \bar{z}t} \quad (|z| < 1)$$

is holomorphic in the closed disc $|t| \leq 1$, we have

$$\overline{\left\{ \frac{1}{2\pi i} \int_{|t|=1} \frac{B_{n+1}(t)}{(1 - \bar{z}t)(t - \zeta)} dt \right\}} = \frac{B_{n+1}(\zeta)}{1 - \bar{\zeta}z}. \quad (2.9)$$

On the other hand, by (2.1) and (2.5) we have for $|t| = 1$

$$B_{n+1}(t) \frac{\overline{\phi_\nu(t)}}{t} = - \left\{ (1 - |\alpha_\nu|^2)^{1/2} \prod_{k=\nu}^n \frac{|\alpha_k|}{\alpha_k} \right\} \frac{\prod_{k=\nu+1}^n (\alpha_k - t)}{\prod_{k=\nu}^n (1 - \overline{\alpha_k} t)} \quad (\nu = 0, 1, \dots, n),$$

where the left-hand side is holomorphic in the closed disc $|t| \leq 1$.

Hence it follows that

$$\frac{1}{2\pi i} \int_{|t|=1} \frac{B_{n+1}(t) \overline{\phi_k(t)}}{t(1-\bar{z}t)} dt = 0 \quad (k = 0, 1, \dots, n) \quad (2.10)$$

for $|z| < 1$.

Finally, from (2.8), (2.9) and (2.10) follows that

$$\frac{1}{2\pi} \int_{|t|=1} \frac{\Phi_n(t; \zeta) B_{n+1}^{-1}(t)}{1-z\bar{t}} |dt| = \frac{\overline{B_{n+1}(\zeta)}}{1-\bar{\zeta}z}. \quad (2.11)$$

From (2.7) and (2.11) we have

$$\Phi_n(z; \zeta) = \frac{\overline{B_{n+1}(\zeta)} B_{n+1}(z)}{1-\bar{\zeta}z}.$$

From this and formula (2.3), the identity (2.2) follows. The Lemma is proved.

Let now

$$S_n(\zeta; z) = \sum_{k=0}^n \overline{\phi_k(\zeta)} \phi_k(z) \quad (2.12)$$

denote the kernel of the distribution $(2\pi)^{-1} dx$.

Then it follows from Lemma 5 that in particular

C o r o l l a r y. *For arbitrary values of variables z and ζ , the kernel $S_n(\zeta; z)$ satisfies the functional equation*

$$S_n(\zeta; z) = \frac{\overline{B_{n+1}(\zeta)} B_{n+1}(z)}{\bar{\zeta}z} S_n\left(\frac{1}{\bar{z}}; \frac{1}{\bar{\zeta}}\right). \quad (2.13)$$

2.2. Now we prove that the functional equation (2.13) also holds for kernels of an arbitrary distribution $(2\pi)^{-1} d\alpha(x)$.

T h e o r e m 1. *For arbitrary $n \geq 0$, z and ζ , the kernel $S_n(\zeta; z)$ of the distribution $(2\pi)^{-1} d\alpha(x)$ satisfies the functional equation*

$$S_n(\zeta; z) = \frac{\overline{B_{n+1}(\zeta)} B_{n+1}(z)}{\bar{\zeta}z} S_n\left(\frac{1}{\bar{z}}; \frac{1}{\bar{\zeta}}\right). \quad (2.14)$$

Proof. According to Lemma 4, we have in particular the identity

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} S_n(\zeta; z) \overline{\phi_\nu(z)} d\alpha(x) = \overline{\phi_\nu(\zeta)} \quad (\nu = 0, 1, \dots, n) \quad (2.15)$$

which is valid for arbitrary $\zeta \neq 1/\bar{\alpha}_k$ ($k = 0, 1, \dots, n$).

Now observe that we have $S_n(\zeta; z) \in M\{\alpha_k\}_0^n$ for arbitrary $\zeta \neq 1/\bar{\alpha}_k$ ($k = 0, 1, \dots, n$) so that we have the representation

$$S_n(\zeta; z) = \sum_{k=0}^n A_k(\zeta) \phi_k(z). \quad (2.16)$$

From this and from formula (2.15), it follows that the coefficients $\{A_k(\zeta)\}_0^n$ satisfy the conditions

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \sum_{k=0}^n A_k(\zeta) \phi_k(z) \right\} \overline{\phi_\nu(z)} d\alpha(x) = \\ & = \sum_{k=0}^n A_k(\zeta) (\phi_k, \phi_\nu) = \overline{\phi_\nu(\zeta)} \quad (\nu = 0, 1, \dots, n). \end{aligned} \quad (2.17)$$

If we write now the formulas (2.16) and (2.17) in the form

$$\begin{cases} \sum_{k=0}^n A_k(\zeta) (\phi_k, \phi_\nu) - \phi_\nu(\zeta) = 0, \\ \sum_{k=0}^n A_k(\zeta) \phi_k(z) - S_n(\zeta; z) = 0 \end{cases} \quad (\nu = 0, 1, \dots, n), \quad (2.18)$$

we can assert that the system of linear orthogonal equations (2.18) has the non-trivial solution

$$\{A_0, A_1, \dots, A_n, -1\}.$$

Consequently, the determinant of this system must be identically zero, i.e.,

$$\begin{vmatrix} (\phi_0, \phi_0) & (\phi_1, \phi_0) & \dots & (\phi_n, \phi_0) & \overline{\phi_0(\zeta)} \\ (\phi_0, \phi_1) & (\phi_1, \phi_1) & \dots & (\phi_n, \phi_1) & \overline{\phi_1(\zeta)} \\ \vdots & \vdots & & \vdots & \vdots \\ (\phi_0, \phi_n) & (\phi_1, \phi_n) & \dots & (\phi_n, \phi_n) & \overline{\phi_n(\zeta)} \\ \phi_0(z) & \phi_1(z) & \dots & \phi_n(z) & S_n(\zeta; z) \end{vmatrix} \equiv 0.$$

Furthermore, keeping in mind the definition (1.9) of the determinant D_n , we get the following representation for the function $S_n(\zeta; z)$:

$$S_n(\zeta; z) = -\frac{1}{D_n} \begin{vmatrix} (\phi_0, \phi_0) & \dots & (\phi_n, \phi_0) & \overline{\phi_0(\zeta)} \\ (\phi_0, \phi_1) & \dots & (\phi_n, \phi_1) & \overline{\phi_1(\zeta)} \\ \vdots & & \vdots & \vdots \\ (\phi_0, \phi_n) & \dots & (\phi_n, \phi_n) & \overline{\phi_n(\zeta)} \\ \phi_0(z) & \dots & \phi_n(z) & 0 \end{vmatrix}. \quad (2.19)$$

Now we introduce the following system of rational functions for a fixed value $n \geq 0$:

$$\phi_k^{(n)}(z) = -\frac{B_{n+1}(z)}{z} \overline{\phi_{n-k}} \left(\frac{1}{z} \right) \quad (k = 0, 1, \dots, n). \quad (2.20)$$

Then we have

$$\begin{aligned} \phi_0^{(n)}(z) &= \frac{|\alpha_n| (1 - |\alpha_n|^2)^{1/2}}{\alpha_n (1 - \overline{\alpha_n} z)}, \\ \phi_k^{(n)}(z) &= \prod_{p=n-k}^n \left\{ \frac{|\alpha_p|}{\alpha_p} \right\} \frac{(1 - |\alpha_{n-k}|^2)^{1/2}}{1 - \overline{\alpha_{n-k}} z} \sum_{p=n-k+1}^n \frac{\alpha_p - z}{1 - \overline{\alpha_p} z} \frac{|\alpha_p|}{\alpha_p} \\ & \quad (k = 1, 2, \dots, n). \end{aligned} \quad (2.21)$$

From definition (2.20) it follows that

$$(\phi_p^{(n)}, \phi_q^{(n)}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_p^{(n)}(z) \overline{\phi_q^{(n)}(z)} d\alpha(x) =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_{n-q}(z) \overline{\phi_{n-p}(z)} d\alpha(x) = (\phi_{n-q}, \phi_{n-p}) \quad (p, q = 0, 1, \dots, n), \quad (2.22)$$

since $|B_{n+1}(z)| = 1$ for $|z| = 1$.

In particular, for $\alpha(x) \equiv x$, it follows from formula (2.22), that the system of rational functions $\{\phi_k^{(n)}(z)\}_0^n$ is orthonormal on the unit circle with respect to the weight $(2\pi)^{-1}dx$. Therefore it presents a finite Malmquist system associated with the sequence of numbers $\{\alpha_n, \alpha_{n-1}, \dots, \alpha_0\}$.

Now we orthogonalize this new finite system of functions $\{\phi_k^{(n)}(z)\}_0^n$ on the unit circle with respect to the weight $(2\pi)^{-1}d\alpha(x)$. Now we get a finite system of rational functions $\{\Phi_k^{(n)}(z)\}_0^n$, which satisfies the conditions:

a) $\Phi_k^{(n)}(z)$ is a "polynomial of degree k " in the first $k+1$ functions $\{\phi_k^{(n)}(z)\}_0^n$, i.e.,

$$\Phi_k^{(n)}(z) = \sum_{p=0}^k c_p^{(n)} \phi_p^{(n)}(z), \quad c_k^{(n)} > 0;$$

b)

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_p^{(n)}(z) \overline{\Phi_q^{(n)}(z)} d\alpha(x) = \delta_{p,q}, \quad z = e^{ix} \quad (p, q = 0, 1, 2, \dots, n).$$

Denoting further

$$S_n^{(n)}(\zeta; z) = \sum_{k=0}^n \overline{\Phi_k^{(n)}(\zeta)} \Phi_k^{(n)}(z), \quad (2.23)$$

we have in analogy with formula (2.19)

$$S_n^{(n)}(\zeta; z) = -\frac{1}{D_n^{(n)}} \begin{vmatrix} (\phi_0^{(n)}, \phi_0^{(n)}) & \dots & (\phi_n^{(n)}, \phi_0^{(n)}) & \overline{\phi_0^{(n)}(\zeta)} \\ (\phi_0^{(n)}, \phi_1^{(n)}) & \dots & (\phi_n^{(n)}, \phi_1^{(n)}) & \overline{\phi_1^{(n)}(\zeta)} \\ \vdots & & \vdots & \vdots \\ (\phi_0^{(n)}, \phi_n^{(n)}) & \dots & (\phi_n^{(n)}, \phi_n^{(n)}) & \overline{\phi_n^{(n)}(\zeta)} \\ \phi_0^{(n)}(z) & \dots & \phi_n^{(n)}(z) & 0 \end{vmatrix}. \quad (2.19')$$

where

$$D_n^{(n)} = \begin{vmatrix} (\phi_0^{(n)}, \phi_0^{(n)}) & \dots & (\phi_0^{(n)}, \phi_n^{(n)}) \\ (\phi_1^{(n)}, \phi_0^{(n)}) & \dots & (\phi_1^{(n)}, \phi_n^{(n)}) \\ \vdots & & \vdots \\ (\phi_n^{(n)}, \phi_0^{(n)}) & \dots & (\phi_n^{(n)}, \phi_n^{(n)}) \end{vmatrix} \quad (1.9')$$

is the Gramian for the collection of functions $\{\phi_k^{(n)}(z)\}_0^n$.

Let us prove that the identity

$$S_n^{(n)}(\zeta; z) \equiv S_n(\zeta; z) \quad (2.24)$$

holds. To this end, we now assume for the sake of simplicity of our arguments, that the group of numbers $\{\alpha_0, \dots, \alpha_n\}$ satisfies the condition

$$\alpha_i \neq 0, \quad \alpha_i \neq \alpha_j \quad (i \neq j; \quad i, j = 0, 1, \dots, n), \quad (2.25)$$

i.e., that all these numbers are different from each other and different from zero.

Then the corresponding finite Malmquist system $\{\phi_k(z)\}_0^n$ is the result of orthogonalization of the system of rational functions

$$\left\{ \frac{1}{1 - \bar{\alpha}_k z} \right\}_0^n$$

on the circle $z = e^{ix}$ with respect to the weight $(2\pi)^{-1} dx$.

That means that in the considered case, the set $M\{\alpha_k\}_0^n$ coincides with the set of linear combinations of “polynomials of degree n ” of the form

$$R_n(z) = \sum_{k=0}^n \frac{\alpha_k}{1 - \bar{\alpha}_k z}. \quad (2.26)$$

Furthermore it is obvious that the finite Malmquist system $\{\phi_k^{(n)}(z)\}_0^n$, which is orthogonal on the circle $z = e^{ix}$ with respect to the weight $(2\pi)^{-1} dx$, as we have constructed it above, is obtained by orthonormalizing the ordered system of rational functions

$$\left\{ \frac{1}{1 - \bar{\alpha}_{n-k} z} \right\}_0^n.$$

Hence the set $M\{\alpha_{n-k}\}_0^n$ of all generalized “polynomials of degree n ” of the form

$$\sum_{k=0}^n c_k \phi_k^{(n)}(z)$$

also coincides with the set of rational functions (2.26), i.e., $M\{\alpha_{n-k}\}_0^n \equiv M\{\alpha_k\}_0^n$.

Now we turn to the optimization problem, whose solution was mentioned in Lemma 3, assuming that parameter $A_0 = 1$.

Since $M\{\alpha_k\}_0^n \equiv M\{\alpha_{n-k}\}_0^n$, optimization over these families of functions gives optimizing functions and corresponding minima which must be identical.

That means, that

$$R_n^{(0)}(z) = \frac{S_n(\zeta; z)}{S_n(\zeta; \zeta)} \equiv \frac{S_n^{(n)}(\zeta; z)}{S_n^{(n)}(\zeta; \zeta)},$$

but also

$$\mu(R_n^{(0)}) = \frac{1}{S_n(\zeta; \zeta)} \equiv \frac{1}{S_n^{(n)}(\zeta; \zeta)}$$

for all z and $\zeta \neq 1/\bar{\alpha}_k$ ($k = 0, 1, \dots, n$).

Hence the desired identity (2.24) follows, but for the moment only for the restriction (2.25) on the collection of complex numbers $\{\alpha_k\}_0^n$.

In order to get rid of this restriction on the collection of numbers $\{\alpha_k\}_0^n$, we consider an other collection $\{\tilde{\alpha}_k\}_0^n$ ($0 < |\tilde{\alpha}_k| < 1$) which does satisfy the condition (2.25).

Let $\{\tilde{\phi}_k(z)\}_0^n$ and $\{\tilde{\phi}_k^{(n)}(z)\}_0^n$ be Malmquist systems associated with the ordered groups of numbers $\{\tilde{\alpha}_k\}_0^n$ and $\{\tilde{\alpha}_{n-k}\}_0^n$, respectively.

Furthermore, let $\{\tilde{\Phi}_k(z)\}_0^n$ and $\{\tilde{\Phi}_k^{(n)}(z)\}_0^n$ be systems of functions which are obtained from the systems $\{\tilde{\phi}_k(z)\}_0^n$ and $\{\tilde{\phi}_k^{(n)}(z)\}_0^n$ respectively by orthogonalization on the unit circle $z = e^{ix}$ with respect to the weight $(2\pi)^{-1} d\alpha(x)$.

Finally, let

$$\tilde{S}_n(\zeta; z) = \sum_{k=0}^n \overline{\tilde{\Phi}_k(\zeta)} \tilde{\Phi}_k(z) \quad \text{and} \quad \tilde{S}_n^{(n)}(\zeta; z) = \sum_{k=0}^n \overline{\tilde{\Phi}_k^{(n)}(\zeta)} \tilde{\Phi}_k^{(n)}(z)$$

be the kernels of the systems $\{\tilde{\Phi}_k(z)\}_0^n$ and $\{\tilde{\Phi}_k^{(n)}(z)\}_0^n$ respectively. Because condition (2.25) is assumed to hold for the group of numbers $\{\tilde{\alpha}_k\}_0^n$, it holds as before that the identity

$$\tilde{S}_n^{(n)}(\zeta; z) \equiv \tilde{S}_n(\zeta; z) \quad (2.24')$$

is true for ζ and $z \neq 1/\overline{\tilde{\alpha}_k}$ ($k = 0, 1, 2, \dots, n$).

But it is obvious that if

$$\lim \tilde{\alpha}_k = \alpha_k \quad (k = 0, 1, \dots, n), \quad (2.27)$$

then the limit relations

$$\lim \tilde{\phi}_k(z) = \phi_k(z), \quad \lim \tilde{\phi}_k^{(n)}(z) = \phi_k^{(n)}(z) \quad (k = 0, 1, \dots, n)$$

hold. They even hold uniformly for all z in the plane outside sufficiently small neighborhoods of the distinct points in $\{1/\overline{\tilde{\alpha}_k}\}_0^n$. In particular it holds on the unit circle $z = e^{ix}$.

Therefore we also have

$$\lim(\tilde{\phi}_p, \tilde{\phi}_q) = (\phi_p, \phi_q) \quad \text{and} \quad \lim(\tilde{\phi}_p^{(n)}, \tilde{\phi}_q^{(n)}) = (\phi_p^{(n)}, \phi_q^{(n)}) \quad (p, q = 0, 1, \dots, n).$$

Thus for the corresponding orthogonal systems and kernels with respect to the weight $(2\pi)^{-1}d\alpha(x)$ it holds that under condition (2.27), the limit relation

$$\lim \tilde{\Phi}_k(z) = \Phi_k(z) \quad (k = 0, 1, \dots, n),$$

and also

$$\begin{aligned} \lim \tilde{\Phi}_k^{(n)}(z) &= \Phi_k^{(n)}(z), \\ \lim \tilde{S}_n(\zeta; z) &= S_n(\zeta; z), \\ \lim \tilde{S}_n^{(n)}(\zeta; z) &= S_n^{(n)}(\zeta; z) \end{aligned} \quad (2.28)$$

hold true.

It is sufficient now to take the limit in identity (2.24') under the condition (2.27). By formula (2.28), the correctness of identity (2.24) is already verified without any additional restrictions on the collection $\{\alpha_k\}_0^n$.

Now we see that the Gramian $D_n^{(n)}$ for system of functions $\{\phi_k^{(n)}(z)\}_0^n$ is also positive. Therefore, using formulas (2.22)

$$(\phi_p^{(n)}, \phi_q^{(n)}) = (\phi_{n-q}, \phi_{n-p}) = \overline{(\phi_{n-p}, \phi_{n-q})} \quad (p, q = 0, 1, 2, \dots, n),$$

we get from (1.9')

$$D_n^{(n)} = \overline{D_n^{(n)}} = \begin{vmatrix} \overline{(\phi_n, \phi_n)} & \dots & \overline{(\phi_n, \phi_0)} \\ \overline{(\phi_{n-1}, \phi_n)} & \dots & \overline{(\phi_{n-1}, \phi_0)} \\ \vdots & & \vdots \\ \overline{(\phi_0, \phi_n)} & \dots & \overline{(\phi_0, \phi_0)} \end{vmatrix} = \begin{vmatrix} (\phi_n, \phi_n) & \dots & (\phi_0, \phi_n) \\ (\phi_n, \phi_{n-1}) & \dots & (\phi_0, \phi_{n-1}) \\ \vdots & & \vdots \\ (\phi_n, \phi_0) & \dots & (\phi_0, \phi_0) \end{vmatrix}.$$

It is obvious that reflection in the anti diagonal for the last determinant will not change its value. By virtue of (1.9), we get the equality

$$D_n^{(n)} = D_n . \quad (2.29)$$

Finally, from (2.19'), (2.24) and (2.29), and by using formulas (2.22), we arrive at the following representation for the kernel

$$S_n(\zeta; z) = -\frac{1}{D_n} \begin{vmatrix} (\phi_n, \phi_n) & \dots & (\phi_n, \phi_0) & \overline{\phi_0^{(n)}(\zeta)} \\ (\phi_{n-1}, \phi_n) & \dots & (\phi_{n-1}, \phi_0) & \phi_1^{(n)}(\zeta) \\ \vdots & & \vdots & \vdots \\ (\phi_0, \phi_n) & \dots & (\phi_0, \phi_0) & \overline{\phi_n^{(n)}(\zeta)} \\ \phi_0^{(n)}(z) & \dots & \phi_n^{(n)}(z) & 0 \end{vmatrix} . \quad (2.30)$$

But according to definition (2.21) of the system of functions $\{\phi_k^{(n)}(z)\}_0^n$, we have

$$\begin{aligned} \phi_k^{(n)}(z) &= -\frac{B_{n+1}(z)}{z} \overline{\phi_{n-k}} \left(\frac{1}{z} \right) , \\ \overline{\phi_k^{(n)}(\zeta)} &= -\frac{\overline{B_{n+1}(\zeta)}}{\bar{\zeta}} \phi_{n-k} \left(\frac{1}{\bar{\zeta}} \right) \quad (k = 0, 1, \dots, n) . \end{aligned}$$

From this and from (2.30) it follows that

$$S_n(\zeta; z) = -\frac{\overline{B_{n+1}(\zeta)} B_{n+1}(z)}{\bar{\zeta} z D_n} \begin{vmatrix} (\phi_n, \phi_n) & \dots & (\phi_n, \phi_0) & \phi_n(\frac{1}{\zeta}) \\ (\phi_{n-1}, \phi_n) & \dots & (\phi_{n-1}, \phi_0) & \phi_{n-1}(\frac{1}{\zeta}) \\ \vdots & & \vdots & \vdots \\ (\phi_0, \phi_n) & \dots & (\phi_0, \phi_0) & \phi_0(\frac{1}{\zeta}) \\ \overline{\phi_n(\frac{1}{z})} & \dots & \overline{\phi_0(\frac{1}{z})} & 0 \end{vmatrix} . \quad (2.30')$$

Observe now that the identity

$$B_{n+1} \left(\frac{1}{z} \right) \overline{B_{n+1}(z)} \equiv 1$$

is valid for arbitrary z . From (2.30') we get further

$$\frac{\overline{B_{n+1}(\zeta)} B_{n+1}(z)}{\bar{\zeta} z} S_n \left(\frac{1}{\bar{z}}; \frac{1}{\zeta} \right) = -\frac{1}{D_n} \begin{vmatrix} (\phi_n, \phi_n) & \dots & (\phi_n, \phi_0) & \phi_n(z) \\ (\phi_{n-1}, \phi_n) & \dots & (\phi_{n-1}, \phi_0) & \phi_{n-1}(z) \\ \vdots & & \vdots & \vdots \\ (\phi_0, \phi_n) & \dots & (\phi_0, \phi_0) & \phi_0(z) \\ \overline{\phi_n(\zeta)} & \dots & \overline{\phi_0(\zeta)} & 0 \end{vmatrix} . \quad (2.31)$$

On the other hand, we have from (2.19)

$$S_n(\zeta; z) = -\frac{(-1)^{\frac{n(n+1)}{2}}}{D_n} \begin{vmatrix} (\phi_n, \phi_0) & \dots & (\phi_0, \phi_0) & \overline{\phi_0(\zeta)} \\ (\phi_n, \phi_1) & \dots & (\phi_0, \phi_1) & \overline{\phi_1(\zeta)} \\ \vdots & & \vdots & \vdots \\ (\phi_n, \phi_n) & \dots & (\phi_0, \phi_n) & \overline{\phi_n(\zeta)} \\ \phi_n(z) & \dots & \phi_0(z) & 0 \end{vmatrix} =$$

$$= -\frac{1}{D_n} \begin{vmatrix} (\phi_n, \phi_n) & \dots & (\phi_0, \phi_n) & \overline{\phi_n(\zeta)} \\ (\phi_n, \phi_{n-1}) & \dots & (\phi_0, \phi_{n-1}) & \overline{\phi_{n-1}(\zeta)} \\ \vdots & & \vdots & \vdots \\ (\phi_n, \phi_0) & \dots & (\phi_0, \phi_0) & \overline{\phi_0(\zeta)} \\ \phi_n(z) & \dots & \phi_0(z) & 0 \end{vmatrix} .$$

Finally, reflecting the last determinant in its main diagonal and keeping formula (2.31) in mind, we get the identity (2.14). Thus the theorem is proved.

Note that in the particular case when $\alpha_0 = \alpha_1 = \dots = \alpha_n = \dots = 0$, the system of functions (1.2) or the system of corresponding Malmquist functions $\{\phi_k(z)\}_0^\infty$ reduce itself to the systems of powers $\{z^k\}_0^\infty$.

Therefore it is obvious, that, in the case

$$\alpha_k = 0 \quad (k = 0, 1, 2, \dots) ,$$

the system of functions $\{\Phi_k(z)\}_0^\infty$ goes over into the system of Szegő polynomials $\{P_k(z)\}_0^\infty$ which are orthogonal on the unit circle $z = e^{ix}$ with respect to the weight $(2\pi)^{-1}d\alpha(x)$.

Finally, since for $\alpha_k = 0$ ($k = 0, 1, 2, \dots$), we have

$$B_{n+1}(z) = z^{n+1} \quad (n = 0, 1, 2, \dots) ,$$

so that Theorem 1 implies in particular a well-known formula of Szegő which is proved very easily.

C o r o l l a r y . *For systems of polynomials $\{P_k(z)\}_0^\infty$ which are orthogonal onto the unit circle with respect to the weight $(2\pi)^{-1}d\alpha(x)$, the corresponding kernel*

$$\sigma_n(\zeta; z) = \sum_{k=0}^n \overline{P_k(\zeta)} P_k(z)^\dagger \quad (2.32)$$

satisfies the functional equation

$$\sigma_n(\zeta; z) = (\bar{\zeta}z)^n S_n \left(\frac{1}{z}; \frac{1}{\bar{\zeta}} \right) . \quad (2.33)$$

2.3 In addition to the main identity (2.14) in the previous subsection, we deduce several results which are important to derive further formulas concerning the kernel $S_n(\zeta; z)$ of an arbitrary distribution $(2\pi)^{-1}d\alpha(x)$.

L e m m a 6. *For arbitrary $n \geq 0$, the following formulas hold:*

1°.

$$S_n(\alpha_n; z) = -\frac{|\alpha_n|}{\alpha_n} \frac{k_n}{(1 - |\alpha_n|^2)^{1/2}} \frac{B_{n+1}(z)}{z} \overline{\Phi_n} \left(\frac{1}{z} \right) , \quad (2.34)$$

$$S_n(\alpha_n; \alpha_n) = \frac{k_n^2}{1 - |\alpha_n|^2} , \quad (2.34')$$

where $k_n > 0$ is the coefficient for $\phi_n(z)$ in the representation

$$\Phi_n(z) = k_n \phi_n(z) ; \quad (2.35)$$

[†]Note from trans.: In the original paper formula (2.32) was identical to (2.33). We have replaced it by the correct one.

[‡]In the original paper formula (2.32) was identical to (2.33). We have replaced it by the correct one.

2°.

$$S_n(\alpha_0; z) = -\frac{|\alpha_0|}{\alpha_0} \frac{k_n^{(n)}}{(1-|\alpha_0|^2)^{1/2}} \frac{B_{n+1}(z)}{z} \overline{\Phi_n^{(n)}} \left(\frac{1}{z} \right), \quad (2.36)$$

$$S_n(\alpha_0; \alpha_0) = \frac{(k_n^{(n)})^2}{1-|\alpha_0|^2}, \quad (2.36')$$

where $k_p^{(n)}$ is the coefficient for $\phi_p^{(n)}(z)$ in the representation

$$\Phi_p^{(n)}(z) = k_p^{(n)} \phi_p^{(n)}(z) + \dots + l_p^{(n)} \phi_0^{(n)}(z) \quad (p = 0, 1, \dots, n); \quad (2.37)$$

3°. Between coefficients $\{l_p^{(n)}\}_0^n$ and k_n , and also between $\{l_p\}_0^n$ and $k_n^{(n)}$, the relations

$$\sum_{p=0}^n |l_k^{(n)}|^2 = k_n^2, \quad (2.38)$$

$$\sum_{p=0}^n |l_k|^2 = (k_n^{(n)})^2 \quad (2.39)$$

hold.

Proof. 1°. From formula (2.20), it follows for arbitrary $p \geq 0$, that

$$\overline{\phi_k} \left(\frac{1}{z} \right) = -\frac{z}{B_{p+1}(z)} \phi_{p-k}^{(p)}(z) \quad (k = 0, 1, \dots, p),$$

where

$$B_{p+1}(z) = \prod_{k=0}^p \frac{\alpha_k - z}{1 - \overline{\alpha_k} z} \frac{|\alpha_k|}{\alpha_k}.$$

Hence, from (2.35), we have the representation

$$\begin{aligned} \overline{\Phi_p} \left(\frac{1}{\zeta} \right) &= k_p \overline{\phi_p} \left(\frac{1}{\zeta} \right) + \dots + \overline{l_p} \overline{\phi_0} \left(\frac{1}{\zeta} \right) = \\ &= -\frac{\zeta}{B_{p+1}(\zeta)} \{k_p \phi_0^{(p)}(\zeta) + \dots + \overline{l_p} \phi_0^{(p)}(\zeta)\}, \end{aligned}$$

whence we get for $0 \leq p \leq n$

$$\frac{B_{p+1}(\zeta)}{\zeta} \overline{\Phi_p} \left(\frac{1}{\zeta} \right) = -\prod_{k=p+1}^n \frac{\alpha_k - \zeta}{1 - \overline{\alpha_k} \zeta} \frac{|\alpha_k|}{\alpha_k} \{k_p \phi_0^{(p)}(\zeta) + \dots + \overline{l_p} \phi_0^{(p)}(\zeta)\}, \quad (2.40)$$

where the symbol $\prod_{k=p+1}^n$ must be replaced by the identity for $p = n$.

From formula (2.40), we obviously get

$$\lim_{\zeta \rightarrow \alpha_n} \frac{B_{p+1}(\zeta)}{\zeta} \overline{\Phi_p} \left(\frac{1}{\zeta} \right) = 0 \quad (p = 0, 1, \dots, n-1). \quad (2.41)$$

Further, since from the explicit expression (2.21) of the system $\{\phi_k^{(n)}(z)\}_0^n$ follows, that

$$\phi_0^{(n)}(\alpha_n) = \frac{|\alpha_n|}{\alpha_n} \frac{1}{(1-|\alpha_n|^2)^{1/2}}, \quad \phi_k^{(n)}(\alpha_n) = 0 \quad (k = 1, 2, \dots, n), \quad (2.42)$$

we get from (2.40) for $p = n$

$$\lim_{\zeta \rightarrow \alpha_n} \frac{B_{p+1}(\zeta) \overline{\Phi_n} \left(\frac{1}{\zeta} \right)}{\zeta} = -\frac{|\alpha_n| k_n}{\alpha_n (1 - |\alpha_n|^2)^{1/2}}. \quad (2.43)$$

Finally, according to identity (2.14) of Theorem 1, we have

$$\begin{aligned} S_n(\zeta; z) &= \frac{\overline{B_{n+1}(\zeta)} B_{n+1}(z)}{\overline{\zeta} z} \sum_{p=0}^n \overline{\Phi_p \left(\frac{1}{\zeta} \right)} \Phi_p \left(\frac{1}{z} \right) = \\ &= \frac{B_{n+1}(z)}{z} \sum_{p=0}^n \left\{ \overline{\frac{B_{n+1}(\zeta)}{\zeta} \Phi_p \left(\frac{1}{\zeta} \right)} \right\} \overline{\Phi_p \left(\frac{1}{z} \right)}. \end{aligned}$$

From this we get by using the relations (2.41) and (2.43), the formula (2.34) of the lemma when taking the limit under $\zeta \rightarrow \alpha_n$. Furthermore, by the relation (2.43), we get (2.34') from (2.34) after taking the limit for $z \rightarrow \alpha_n$.

2°. Observe that the system of functions $\{\Phi_k(z)\}_0^n$ was associated with the ordered group of numbers $\{\alpha_0, \alpha_1, \dots, \alpha_n\}$, while the system $\{\Phi_k^{(n)}(z)\}_0^n$ was associated with the ordered group $\{\alpha_n, \alpha_{n-1}, \dots, \alpha_0\}$.

Therefore, with an appropriate replacement of parameters appearing in formulas (2.34) and (2.34') for the kernel $S_n^{(n)}(\zeta; z)$ of the system $\{\Phi_k^{(n)}(z)\}_0^n$, it is obvious that we get the formulas

$$\begin{aligned} S_n^{(n)}(\alpha_0; z) &= -\frac{|\alpha_0|}{\alpha_0} \frac{k_n^{(n)}}{(1 - |\alpha_0|^2)^{1/2}} \frac{B_{n+1}(z) \overline{\Phi_n^{(n)}} \left(\frac{1}{z} \right)}{z}, \\ S_n^{(n)}(\alpha_0; \alpha_0) &= \frac{(k_n^{(n)})^2}{1 - |\alpha_0|^2}. \end{aligned}$$

Consequently, it is sufficient to use identity (2.24) in order to get formulas (2.36) and (2.36') of the lemma and, hence also formula (2.42).

3°. Using formula (2.42), we get from (2.37)

$$\Phi_p^{(n)}(\alpha_n) = l_p^{(n)} \phi_0^{(n)}(\alpha_n) = \frac{l_p^{(n)} |\alpha_n|}{\alpha_n (1 - |\alpha_n|^2)^{1/2}}.$$

Therefore, in view of identity (2.24) which is established in the proof of Theorem 1, we have

$$S_n(\alpha_n; \alpha_n) = S_n^{(n)}(\alpha_n; \alpha_n) = \sum_{p=0}^n |\Phi_p^{(n)}(\alpha_n)|^2 = \frac{1}{1 - |\alpha_n|^2} \sum_{p=0}^n |l_p^{(n)}|^2.$$

From this and from (2.34), the formula (2.38) follows.

Thus the lemma is proved completely.

From this lemma, it follows in the particular case when $\alpha_k = 0$ ($k = 0, 1, 2, \dots$), that

C o r o l l a r y. *For a system of Szegő polynomials $\{P_k(z)\}_0^\infty$ which is orthogonal on the unit circle $z = e^{ix}$ with respect to the weight $(2\pi)^{-1} d\alpha(x)$, the kernel $\sigma_n(\zeta; z)$ satisfies the conditions*

$$\begin{aligned} \sigma_n(0; z) &= \sum_{k=0}^n \overline{P_k(0)} P_k(z) = \sum_{k=0}^n l_k P_k(z) = k_n z^n \overline{P_n} \left(\frac{1}{z} \right), \\ \sigma_n(0; 0) &= \sum_{k=0}^n |P_k(0)|^2 = \sum_{k=0}^n |l_k|^2 = k_n^2 = \frac{D_{n-1}}{D_n}. \end{aligned} \quad (2.44)$$

Indeed, in the considered case, we have

$$\phi_k(z) = z^k, \quad B_{n+1}(z) = z^{n+1}, \quad P_k(0) = l_k.$$

Therefore, in particular, the assertion (2.44) follows from formulas (2.34) and (2.34').

3 Formula of Christoffel type. Recurrence relations

3.1. According to Lemma 5, we have that for the kernel

$$S_n(\zeta; z) = \sum_{k=0}^n \overline{\phi_k(\zeta)} \phi_k(z) \quad (n \geq 0) \quad (3.1)$$

of a Malmquist system, the representation

$$S_n(\zeta; z) = \frac{1 - \overline{B_{n+1}(\zeta)} B_{n+1}(z)}{1 - \bar{\zeta} z} \quad (3.2)$$

holds.

We transform the right-hand side of the formula (3.2). To this end, we first observe that

$$\begin{aligned} \phi_{n+1}(z) &= \frac{(1 - |\alpha_{n+1}|^2)^{1/2}}{1 - \overline{\alpha_{n+1}} z} \prod_{k=0}^n \frac{\alpha_k - z}{1 - \overline{\alpha_k} z} \frac{|\alpha_k|}{\alpha_k} = \\ &= \frac{(1 - |\alpha_{n+1}|^2)^{1/2}}{1 - \overline{\alpha_{n+1}} z} B_{n+1}(z) = \frac{|\alpha_{k+1}| (1 - |\alpha_{n+1}|^2)^{1/2}}{\alpha_{k+1} (\alpha_{n+1} - z)} B_{n+2}(z). \end{aligned}$$

Hence, it follows that in the first place

$$\overline{B_{n+1}(\zeta)} B_{n+1}(z) = \frac{(1 - \alpha_{n+1} \bar{\zeta})(1 - \overline{\alpha_{n+1}} z)}{1 - |\alpha_{n+1}|^2} \overline{\phi_{n+1}(\zeta)} \phi_{n+1}(z) \quad (3.3)$$

and secondly, since

$$\overline{B_{n+2}} \left(\frac{1}{z} \right) = B_{n+2}^{-1}(z),$$

we have

$$1 \equiv \frac{(1 - \alpha_{n+1} \bar{\zeta})(1 - \overline{\alpha_{n+1}} z)}{1 - |\alpha_{n+1}|^2} \left\{ \frac{\overline{B_{n+2}(\zeta)}}{\zeta} \overline{\phi_{n+1}} \left(\frac{1}{\zeta} \right) \right\} \frac{B_{n+2}(z)}{z} \overline{\phi_{n+1}} \left(\frac{1}{z} \right). \quad (3.4)$$

If we substitute the values (3.3) and (3.4) in the right-hand side of formula (3.2), we have

$$\begin{aligned} S_n(\zeta; z) &= \frac{(1 - \overline{\alpha_{n+1} \zeta})(1 - \overline{\alpha_{n+1}} z)}{1 - |\alpha_{n+1}|^2} \times \\ &\times \frac{\left\{ \frac{\overline{B_{n+2}(\zeta)}}{\zeta} \overline{\phi_{n+1}} \left(\frac{1}{\zeta} \right) \right\} \frac{B_{n+2}(z)}{z} \overline{\phi_{n+1}} \left(\frac{1}{z} \right) - \overline{\phi_{n+1}(\zeta)} \phi_{n+1}(z)}{1 - \bar{\zeta} z}. \end{aligned} \quad (3.5)$$

3.2. Let us prove that formula (3.5) remains true for the kernel $S_n(\zeta; z)$ of an arbitrary distribution $(2\pi)^{-1}d\alpha(x)$.

T h e o r e m 2. *For arbitrary $n \geq 0$, z and ζ , the formula*

$$S_n(\zeta; z) = \frac{\overline{(1 - \overline{\alpha_{n+1}}\zeta)}(1 - \overline{\alpha_{n+1}}z)}{1 - |\alpha_{n+1}|^2} \times \\ \times \frac{\left\{ \frac{\overline{B_{n+2}(\zeta)}\overline{\Phi_{n+1}}\left(\frac{1}{\zeta}\right) \right\} \frac{B_{n+2}(z)\overline{\Phi_{n+1}}\left(\frac{1}{z}\right) - \overline{\Phi_{n+1}(\zeta)}\Phi_{n+1}(z)}{1 - \bar{\zeta}z}}{1 - \bar{\zeta}z}}{1 - \bar{\zeta}z} \quad (3.6)$$

holds for the kernel $S_n(\zeta; z)$ of an arbitrary distribution $(2\pi)^{-1}d\alpha(x)$.

Proof. First note that by (2.40):

$$\frac{B_{n+2}(z)\overline{\Phi_{n+1}}\left(\frac{1}{z}\right)}{z} = -\{k_{n+1}\phi_0^{(n+1)}(z) + \dots + \overline{l_{n+1}}\phi_{n+1}^{(n+1)}(z)\}, \quad (3.7)$$

where $\{\phi_k^{(n+1)}(z)\}_0^{n+1}$ is the Malmquist system associated with the ordered group of numbers $\{\alpha_{n+1}, \alpha_n, \dots, \alpha_0\}$.

In view of the definition of the system of functions $\{\phi_k^{(n+1)}(z)\}_0^{n+1}$ (see formula (2.21)) it follows immediately from (3.7), that the rational function

$$\frac{B_{n+2}(z)\overline{\Phi_{n+1}}\left(\frac{1}{z}\right)}{z}$$

has poles only for values $z = 1/\overline{\alpha_k}$ ($k = 0, 1, \dots, n+1$) which are outside of the unit disc.

Defining

$$U_n(\zeta; z) = \overline{(1 - \overline{\alpha_{n+1}}\zeta)}(1 - \overline{\alpha_{n+1}}z) \times \\ \times \left\{ \left[\frac{\overline{B_{n+2}(\zeta)}\overline{\Phi_{n+1}}\left(\frac{1}{\zeta}\right) \right] \frac{B_{n+2}(z)\overline{\Phi_{n+1}}\left(\frac{1}{z}\right) - \overline{\Phi_{n+1}(\zeta)}\Phi_{n+1}(z)}{1 - \bar{\zeta}z} \right\}, \quad (3.8)$$

we assert, that the rational function $U_n(\zeta; z)$ is continuous on the unit circle $z = e^{ix}$ ($-\pi \leq x \leq \pi$), if the restrictions

$$|\zeta| \neq 1, \quad \zeta \neq 1/\overline{\alpha_k} \quad (k = 0, 1, 2, \dots, n). \quad (3.9)$$

are imposed on the parameter ζ .

With the assumption (3.9), we define further

$$V_n(\zeta; g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{U_n(\zeta; g)}{1 - \bar{\zeta}z} g(z) d\alpha(x), \quad z = e^{ix}, \quad (3.10)$$

where

$$g(z) = \sum_{k=0}^n \alpha_k \phi_k(z) \quad (3.11)$$

is an arbitrary "polynomial of degree n " of Malmquist functions.

However, the identity

$$\overline{g(z)} = \frac{\overline{g(\zeta)}}{1 - \bar{\zeta}z} + \left\{ \frac{\overline{g(z) - g(\zeta)}}{z - \zeta} \right\} \bar{z},$$

holds for $z = e^{ix}$ ($-\pi \leq x \leq \pi$), from which we get the representation for functions $V_n(\zeta; g)$:

$$V_n(\zeta; g) = \overline{g(\zeta)} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{U_n(\zeta; z)}{1 - \overline{\zeta}z} d\alpha(x) + \frac{1}{2\pi} \int_{-\pi}^{\pi} U_n(\zeta; z) \left\{ \frac{\overline{g(z) - g(\zeta)}}{z - \zeta} \right\} \overline{z} d\alpha(x) \equiv V_n^{(1)}(\zeta; g) + V_n^{(2)}(\zeta; g). \quad (3.10')$$

Hence, first one can see, that

$$V_n^{(1)}(\zeta; g) = C_n^{(1)}(\zeta) \overline{g(\zeta)}, \quad (3.12)$$

where $C_n^{(1)}(\zeta)$ is independent from the function $g(\zeta)$ [§]. Further, in view of definition (3.8) of the function $U_n(\zeta; z)$, we have:

$$\begin{aligned} V_n^{(2)}(\zeta; g) &= \left\{ (1 - \overline{\alpha_{n+1}}\zeta) \frac{B_{n+2}(\zeta)}{\zeta} \Phi_{n+1} \left(\frac{1}{\zeta} \right) \right\} \times \\ &\times \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 - \overline{\alpha_{n+1}}z) \frac{B_{n+2}(z)}{z} \overline{\Phi_{n+1}} \left(\frac{1}{z} \right) \left\{ \frac{\overline{g(z) - g(\zeta)}}{z - \zeta} \right\} d\alpha(x) - \\ &- \left\{ (1 - \overline{\alpha_{n+1}}\zeta) \Phi_{n+1}(\zeta) \right\} \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{n+1}(z) \left\{ (z - \alpha_{n+1}) \frac{\overline{g(z) - g(\zeta)}}{z - \zeta} \right\} d\alpha(x) \equiv \\ &\equiv V_n^{(3)}(\zeta; g) + V_n^{(4)}(\zeta; g). \end{aligned} \quad (3.13)$$

To simplify the further steps of the proof, we assume now, that all numbers of the group $\{\alpha_k\}_0^n$ are different from each other and different from zero.

With this assumption, it follows from (3.11), that the function $g(z)$ can be represented in the form

$$g(z) = \frac{P_n(z)}{\prod_{k=0}^n (1 - \overline{\alpha_k}z)}, \quad (3.14)$$

where $P_n(z)$ is an ordinary polynomial of powers not higher than n .

Assuming again that the parameter ζ satisfies the condition (3.9), from the representation (3.14), we have:

$$\frac{g(z) - g(\zeta)}{z - \zeta} = \frac{P_n(z) \prod_{k=0}^n (1 - \overline{\alpha_k}\zeta) - P_n(\zeta) \prod_{k=0}^n (1 - \overline{\alpha_k}z)}{(z - \zeta) \prod_{k=0}^n (1 - \overline{\alpha_k}z)(1 - \overline{\alpha_k}\zeta)}, \quad (3.15)$$

where it is obviously, that the numerator of the quotient in the right-hand side is dividable by $z - \zeta$.

However, for our assumption about the collection of numbers $\{\alpha_k\}_0^n$, the numerator of the quotient (3.15) is a polynomial of degree $n + 1$ in z of the form

$$\left[(-1)^n \prod_{k=0}^n \overline{\alpha_k} \right] P_n(\zeta) z^{n+1} + b_n z^n + \dots + b_0.$$

[§]Later on such functions of ζ will be denoted by $C_n^{(j)}(\zeta)$ ($j = 1, 2, \dots$).

Therefore, dividing it by $z - \zeta$, we have by (3.14):

$$\frac{g(z) - g(\zeta)}{z - \zeta} = \frac{Q_n(z)}{\prod_{k=0}^n (1 - \overline{\alpha_k}z)(1 - \overline{\alpha_k}\zeta)}, \quad (3.16)$$

where $Q_n(z)$ is a polynomial of degree n of the form

$$Q_n(z) = \left[(-1)^n \prod_{k=0}^n \overline{\alpha_k} \right] P_n(\zeta) z^n + c_{n-1} z^{n-1} + \cdots + c_0. \quad (3.16')$$

From (3.16) and (3.16'), it follows that

$$(z - \alpha_{n+1}) \frac{g(z) - g(\zeta)}{z - \zeta} = -\frac{P_n(\zeta)}{\prod_{k=0}^n (1 - \overline{\alpha_k}\zeta)} + \frac{\Omega_n(z; \zeta)}{\prod_{k=0}^n (1 - \overline{\alpha_k}z)}, \quad (3.17)$$

where $\Omega_n(z; \zeta)$ is a certain polynomial in z of degree at most n .

Now observe that the regular rational quotient

$$\Omega_n(z; \zeta) \prod_{k=0}^n (1 - \overline{\alpha_k}z)^{-1}$$

has a partial fraction decomposition of the form

$$\frac{\Omega_n(z; \zeta)}{\prod_{k=0}^n (1 - \overline{\alpha_k}z)} = \sum_{k=0}^n \frac{A_k(\zeta)}{1 - \overline{\alpha_k}z}. \quad (3.18)$$

On the other hand, the system of functions $\{\Phi_k(z)\}_0^n$ results by taking the ordered system of functions $\left\{ \frac{1}{1 - \overline{\alpha_k}z} \right\}_0^n$ and orthogonalizing it on the unit circle with respect to the weight $(2\pi)^{-1} d\alpha(x)$. Therefore it is obvious, that together with (3.18), the representation

$$\frac{\Omega_n(z; \zeta)}{\prod_{k=0}^n (1 - \overline{\alpha_k}z)} = \sum_{k=0}^n B_k(\zeta) \Phi_k(z) \quad (3.18')$$

holds true.

Finally, from (3.14), (3.18') and (3.17), the representation

$$(z - \alpha_{n+1}) \frac{g(z) - g(\zeta)}{z - \zeta} = -g(\zeta) + \sum_{k=0}^n B_k(\zeta) \Phi_k(z) \quad (3.17')$$

follows. As

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{n+1}(z) \overline{\Phi_k(z)} d\alpha(x) = 0 \quad (k = 0, 1, \dots, n),$$

we get by substituting the value (3.17') in the expression for the function $V_n^{(4)}(\zeta; g)$:

$$V_n^{(4)}(\zeta; g) = \overline{g(\zeta) \left\{ (1 - \overline{\alpha_{n+1}}\zeta) \Phi_{n+1}(\zeta) \right\}} \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{n+1}(z) d\alpha(x).$$

Thus we have

$$V_n^{(4)}(\zeta; g) = C_n^{(2)}(\zeta) \overline{g(\zeta)}. \quad (3.19)$$

Finally, we take up the function $V_n^{(3)}(\zeta; g)$.

Therefore we first observe that

$$(1 - \overline{\alpha_{n+1}}z) \frac{B_{n+2}(z)}{z} = -\frac{|\alpha_{n+1}|}{\alpha_{n+1}} (1 - \alpha_{n+1}\bar{z}) B_{n+1}(z)$$

for $z = e^{ix}$. Thus it follows that

$$\begin{aligned} V_n^{(3)}(\zeta; g) &= C_n^{(3)}(\zeta) \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{\Phi_{n+1}(z)} B_{n+2}(z) \bar{z} \times \\ &\times \left\{ (1 - \overline{\alpha_{n+1}}z) \frac{g(z) - g(\zeta)}{z - \zeta} \right\} d\alpha(x). \end{aligned} \quad (3.20)$$

But from (3.16) and (3.16'), we have:

$$(1 - \overline{\alpha_{n+1}}z) \frac{g(z) - g(\zeta)}{z - \zeta} = \overline{\alpha_{n+1}} \frac{P_n(\zeta)}{\prod_{k=0}^n (1 - \overline{\alpha_k}\zeta)} + \frac{\omega_n(z; \zeta)}{\prod_{k=0}^n (1 - \overline{\alpha_k}z)},$$

where $\omega_n(z; \zeta)$ is a polynomial in z of degree n at most.

Therefore, with calculation (3.14), we have for $z = e^{ix}$

$$\left\{ (1 - \overline{\alpha_{n+1}}z) \frac{g(z) - g(\zeta)}{z - \zeta} \right\} = \alpha_{n+1} \overline{g(\zeta)} - \frac{z r_n(z; \zeta)}{\prod_{k=0}^n (\alpha_k - z)}, \quad (3.21)$$

where $r_n(z; \zeta) = (-1)^{n+1} z^n \overline{\omega_n(\frac{1}{z}; \bar{\zeta})}$ is a polynomial in z of degree at most n .

By substituting the value (3.21) in integral (3.20), we get:

$$\begin{aligned} V_n^{(3)}(\zeta; g) &= C_n^{(4)}(\zeta) \overline{g(\zeta)} + \\ &+ C_n^{(3)}(\zeta) \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{\Phi_{n+1}(z)} B_{n+1}(z) \frac{r_n(z; \zeta)}{\prod_{k=0}^n (\alpha_k - z)} d\alpha(x). \end{aligned} \quad (3.22)$$

Further, it is obvious that

$$B_{n+1}(z) \frac{r_n(z; \zeta)}{\prod_{k=0}^n (\alpha_k - z)} = \frac{\gamma_n(z; \zeta)}{\prod_{k=0}^n (1 - \overline{\alpha_k}z)},$$

where $\gamma_n(z; \zeta)$ is a polynomial in z of degree not higher than n .

Therefore we have the representation:

$$V_n^{(3)}(\zeta; g) = C_n^{(4)}(\zeta) \overline{g(\zeta)}. \quad (3.23)$$

Finally, taking the formulas (3.10') and (3.13) into account, and by virtue of (3.12), (3.19) and (3.23), we get in the end:

$$V_n(\zeta; g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{U_n(\zeta; z)}{1 - \bar{\zeta}z} \overline{g(z)} d\alpha(x) = C_n(\zeta) \overline{g(\zeta)}, \quad (3.24)$$

where the function $C_n(\zeta)$ certainly satisfies the conditions (3.9) for the values of parameter ζ . Moreover, it does not depend on the function $g(\zeta)$.

Now let us prove, that $C_n(\zeta) \neq 0$.

Therefore, we first remark that

$$(1 - \overline{\alpha_{n+1}z})\Phi_{n+1}(z) = \frac{\tau_{n+1}(z)}{\prod_{k=0}^n (1 - \overline{\alpha_k z})}$$

and

$$(1 - \overline{\alpha_{n+1}z})\frac{B_{n+2}(z)}{z}\overline{\Phi_{n+1}}\left(\frac{1}{z}\right) = \frac{\chi_{n+1}(z)}{\prod_{k=0}^n (1 - \overline{\alpha_k z})},$$

in the result of which, we have the representation:

$$U_n(\zeta; z) = \frac{u_{n+1}(z; \zeta)}{\prod_{k=0}^n (1 - \overline{\alpha_k z})}, \quad (3.25)$$

where $\tau_{n+1}(z)$, $\chi_{n+1}(z)$ and $u_{n+1}(z; \zeta)$ are polynomials in z of degree not higher than $n + 1$.

On the other hand, as

$$\overline{B_{n+2}(\zeta)}B_{n+2}\left(\frac{1}{\zeta}\right) \equiv 1,$$

it follows from definition (3.8) for the function $U_n(\zeta; z)$ itself, that $U_n(\zeta; \frac{1}{\zeta}) \equiv 0$.

This means that the rational quotient (3.25) is dividable by $1 - \zeta z$ for arbitrary ζ . Therefore we have:

$$\frac{U_n(\zeta; z)}{1 - \zeta z} = \frac{v_{n+1}(z; \zeta)}{\prod_{k=0}^n (1 - \overline{\alpha_k z})} = \sum_{k=0}^n \gamma_k(\zeta)\Phi_k(z), \quad (3.26)$$

where $v_{n+1}(z; \zeta)$ is a polynomial in z of degree not higher than n .

Assuming $C_n(\zeta) \equiv 0$ and taking formula (3.24) for the successive functions

$$g(z) = \Phi_k(z) \quad (k = 0, 1, \dots, n),$$

we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{U_n(\zeta; z)}{1 - \zeta z} \overline{\Phi_k(z)} d\alpha(x) \equiv 0 \quad (k = 0, 1, \dots, n).$$

From this and from representation (3.26), it follows that $\gamma_k(\zeta) = 0$ ($k = 0, 1, \dots, n$) and consequently, $U_n(\zeta; z) \equiv 0$.

But then, in view of definition (3.8) of the function $U_n(\zeta; z)$, we have identically (with respect to the variables z and ζ)

$$\left[\frac{B_{n+2}(\zeta)}{\zeta} \overline{\Phi_{n+1}}\left(\frac{1}{\zeta}\right) \right] \frac{B_{n+2}(z)}{z} \overline{\Phi_{n+1}}\left(\frac{1}{z}\right) \equiv \overline{\Phi_{n+1}(\zeta)}\Phi_{n+1}(z).$$

Hence, setting specifically $\zeta = z$, we have

$$\left| \frac{B_{n+2}(z)}{z} \overline{\Phi_{n+1}}\left(\frac{1}{z}\right) \right|^2 = |\Phi_{n+1}(z)|^2. \quad (3.27)$$

Now observe, that it follows from formulas (2.34) and (2.34') of Lemma 6, that

$$\begin{aligned} \lim_{z \rightarrow \alpha_{n+1}} \left| \frac{B_{n+2}(z)}{z} \overline{\Phi_{n+1}} \left(\frac{1}{z} \right) \right|^2 &= S_{n+1}^2(\alpha_{n+1}; \alpha_{n+1}) \frac{1 - |\alpha_{n+1}|^2}{k_{n+1}^2} = \\ &= S_{n+1}(\alpha_{n+1}; \alpha_{n+1}) . \end{aligned} \quad (3.28)$$

Therefore, when taking the limit for $z \rightarrow \alpha_{n+1}$ in the identity (3.27), we get

$$S_{n+1}(\alpha_{n+1}; \alpha_{n+1}) = |\Phi_{n+1}(\alpha_{n+1})|^2 ,$$

from which it follows that

$$S_{n+1}(\alpha_{n+1}; \alpha_{n+1}) = \sum_{k=0}^n |\Phi_k(\alpha_{n+1})|^2 = 0 . \quad (3.29)$$

But according to (1.9) and (1.10),

$$\Phi_0(z) = \frac{\phi_0(z)}{\sqrt{D_0}} = (\phi_0, \phi_0)^{-1/2} \frac{(1 - |\alpha_0|^2)^{1/2}}{1 - \overline{\alpha_0}z} .$$

But in view of $|\Phi_0(\alpha_{n+1})|^2 > 0$, we have $S_n(\alpha_{n+1}; \alpha_{n+1}) > 0$, contradicting equality (3.29).

Thus the assumption $C_n(\zeta) \equiv 0$ leads us to contradiction.

On the other hand, from definition (3.8) of the function $U_n(\zeta; z)$, it also follows, that the expression

$$\frac{U_n(\zeta; z)}{1 - \overline{\zeta}z}$$

is a rational function in $\overline{\zeta}$ with poles in points $\overline{\zeta} = 1/\alpha_k$ ($k = 0, 1, \dots, n$) at $|z| = 1$.

Therefore it also follows from the integral formula (3.24), that the product $C_n(\zeta) \overline{g(\zeta)}$ is a rational function of $\overline{\zeta}$ whose poles can only be in the points $\overline{\zeta} = 1/\alpha_k$ ($k = 0, 1, \dots, n$).

As $C_n(\zeta) \not\equiv 0$, we have $C_n(\zeta) \neq 0$ everywhere, except for a finite number of points. For such a ζ , formula (3.24) can be written in the form

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{U_n(\zeta; z)}{C_n(\zeta)(1 - \overline{\zeta}z)} \overline{g(z)} d\alpha(x) \equiv \overline{g(\zeta)} .$$

But then, according to Lemma 4, we have:

$$\frac{U_n(\zeta; z)}{1 - \overline{\zeta}z} \equiv C_n(\zeta) S_n(\zeta; z) \quad (3.30)$$

which holds identically with respect to z and ζ .

Now we verify that $C_n(\zeta)$ is independent of ζ .

Therefore noting that

$$\overline{S_n(\zeta; z)} = S_n(z; \zeta) , \quad \overline{U_n(\zeta; z)} = U_n(z; \zeta)$$

and taking conjugates in (3.30), we get:

$$\frac{U_n(z; \zeta)}{1 - \overline{z}\zeta} \equiv \overline{C_n(\zeta)} S_n(z; \zeta) .$$

Interchanging the positions of variables z and ζ in this identity leads us to the identity

$$\frac{U_n(\zeta; z)}{1 - \bar{\zeta}z} \equiv \overline{C_n(z)} S_n(\zeta; z) . \quad (3.31)$$

Comparing identities (3.30) and (3.31), we get

$$C_n(\zeta) \equiv \overline{C_n(z)} ,$$

from which it follows that $C_n(\zeta) \equiv c_n = \text{const}$.

Thus, from (3.30), we have

$$\frac{U_n(\zeta; z)}{1 - \bar{\zeta}z} \equiv c_n S_n(\zeta; z) . \quad (3.30')$$

where c_n is does not dependent on z or ζ .

In order to determine the value of the constant c_n , we note that by definition (3.8) of the function $U_n(\zeta; z)$ and by (3.28), we have:

$$\begin{aligned} U_n(\alpha_{n+1}; \alpha_{n+1}) &= \lim_{z \rightarrow \alpha_{n+1}} U_n(z; z) = (1 - |\alpha_{n+1}|^2)^2 \{S_{n+1}(\alpha_{n+1}; \alpha_{n+1}) - \\ &\quad - |\Phi_{n+1}(\alpha_{n+1})|^2\} = (1 - |\alpha_{n+1}|^2)^2 S_n(\alpha_{n+1}; \alpha_{n+1}) . \end{aligned}$$

Therefore, setting $\zeta = z$ in (3.30'), we get for $z \rightarrow \alpha_{n+1}$

$$(1 - |\alpha_{n+1}|^2) S_n(\alpha_{n+1}; \alpha_{n+1}) = c_n S_n(\alpha_{n+1}; \alpha_{n+1}) .$$

Hence, because

$$S_n(\alpha_{n+1}; \alpha_{n+1}) \geq |\Phi_0(\alpha_{n+1})|^2 > 0 ,$$

we get

$$c_n = (1 - |\alpha_{n+1}|^2) . \quad (3.32)$$

Substituting the value (3.32) in (3.30') and having the meaning (3.8) of the function $U_n(\zeta; z)$ in mind, we arrive at formula (3.6) of the theorem. However, for the moment it is shown only under the condition that all the numbers of the group $\{\alpha_k\}_0^n$ are different from each other and differ from zero.

Finally, we want to release this restriction. To this end, we proceed in the same way as we did already in the proof of Theorem 1. Namely, having an arbitrary group of numbers $\{\alpha_k\}_0^{n+1}$, we consider a new group $\{\tilde{\alpha}_k\}_0^{n+1}$ ($0 < |\tilde{\alpha}_k| < 1$) satisfying the restrictions formulated above.

Let $\{\tilde{\phi}_k(z)\}_0^{n+1}$ be a Malmquist system associated with the ordered group of numbers $\{\tilde{\alpha}_k\}_0^{n+1}$, and let $\{\tilde{\Phi}_k(z)\}_0^{n+1}$ be the result of orthogonalization of this system on the unit circle with respect to the weight $(2\pi)^{-1} d\alpha(x)$.

Finally let

$$\tilde{S}_n(\zeta; z) = \sum_{k=0}^n \overline{\tilde{\Phi}_k(\zeta)} \tilde{\Phi}_k(z)$$

be the kernel of the system $\{\tilde{\Phi}_k(z)\}_0^n$.

As all numbers of the group $\{\tilde{\alpha}_k\}_0^{n+1}$ are different from each other and different from zero, formula (3.6) of the theorem is valid for the kernel $\tilde{S}_n(\zeta; z)$, i.e.,

$$\begin{aligned} \tilde{S}_n(\zeta; z) &= \frac{(1 - \tilde{\alpha}_{n+1}\bar{\zeta})(1 - \overline{\tilde{\alpha}_{n+1}z})}{1 - |\tilde{\alpha}_{n+1}|^2} \times \\ &\times \frac{\left[\frac{\tilde{B}_{n+2}(\zeta)\overline{\tilde{\Phi}_{n+1}\left(\frac{1}{\zeta}\right)}}{\zeta} \right] \frac{\tilde{B}_{n+2}(z)\overline{\tilde{\Phi}_{n+1}\left(\frac{1}{z}\right)}}{z} - \overline{\tilde{\Phi}_{n+1}(\zeta)}\tilde{\Phi}_{n+1}(z)}{1 - \bar{\zeta}z}, \end{aligned} \quad (3.6')$$

where

$$\tilde{B}_{n+2}(z) = \prod_{k=0}^{n+1} \frac{\tilde{\alpha}_k - z}{1 - \tilde{\alpha}_k z} \frac{|\tilde{\alpha}_k|}{\tilde{\alpha}_k}.$$

But, if[¶]

$$\lim \tilde{\alpha}_k = \alpha_k \quad (k = 0, 1, \dots, n+1),$$

then, it is easy to see that

$$\lim \tilde{\phi}_k(z) = \phi_k(z) \quad (k = 0, 1, \dots, n+1),$$

$$\lim \tilde{B}_{n+2}(z) = B_{n+2}(z),$$

and therefore

$$\lim \tilde{\Phi}_k(z) = \Phi_k(z) \quad (k = 0, 1, \dots, n+1).$$

Hence, passing to the limit in identity (3.6'), we already get the desired formula (3.6) of the theorem without any restriction on the collection of numbers $\{\alpha_k\}_0^{n+1}$. Thus, the theorem is proved completely.

In the particular case that $\alpha_k = 0$ ($k = 0, 1, \dots$), we get the Szegő formula from this theorem.

C o r o l l a r y. *For the kernel $S_n(\zeta; z)$ of the system of Szegő polynomials $\{P_k(z)\}_0^\infty$, the formula*

$$S_n(\zeta; z) = \frac{\overline{\zeta^{n+1} P_{n+1}\left(\frac{1}{\zeta}\right)} z^{n+1} \overline{P_{n+1}\left(\frac{1}{z}\right)} - \overline{P_{n+1}(\zeta)} P_{n+1}(z)}{1 - \bar{\zeta}z} \quad (3.33)$$

holds.

4 Recurrence relations, important example of distribution

4.1. Now we deduce recurrence formulas for orthogonal systems $\{\Phi_k(z)\}_0^\infty$.

To this end, we write identity (3.6) of the theorem in the form

$$\begin{aligned} R_n(\zeta; z) &\equiv \frac{\overline{B_{n+2}(\zeta)\overline{\Phi_{n+1}\left(\frac{1}{\zeta}\right)}}}{\zeta} \frac{B_{n+2}(z)\overline{\Phi_{n+1}\left(\frac{1}{z}\right)}}{z} - \overline{\Phi_{n+1}(\zeta)}\Phi_{n+1}(z) = \\ &= (1 - |\alpha_{n+1}|^2) \frac{1 - \bar{\zeta}z}{(1 - \alpha_{n+1}\bar{\zeta})(1 - \overline{\alpha_{n+1}z})} S_n(\zeta; z). \end{aligned} \quad (4.1)$$

[¶]Note, that we can take again all $\tilde{\alpha}_{n+1} = \alpha_{n+1}$.

We consider the integral

$$J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} R_n(\zeta; z) \phi_{n+1}(\zeta) dt, \quad \zeta = e^{it} \quad (4.2)$$

and compute it in two ways by using the identity (4.1).

First note that

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{\Phi_{n+1}(\zeta)} \phi_{n+1}(\zeta) dt = \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{\{k_{n+1} \phi_{n+1}(\zeta) + \cdots + l_{n+1} \phi_0(\zeta)\}} \phi_{n+1}(\zeta) dt = k_{n+1} \end{aligned} \quad (4.3)$$

in view of the orthogonality of the Malmquist system.

Furthermore observe that

$$\begin{aligned} & \overline{\frac{B_{n+2}(\zeta)}{\zeta} \overline{\Phi_{n+1}} \left(\frac{1}{\zeta} \right)} \phi_{n+1}(\zeta) = \zeta \overline{B_{n+2}(\zeta)} \overline{\Phi_{n+1}(\zeta)} \phi_{n+1}(\zeta) = \\ &= \frac{\zeta(1 - |\alpha_{n+1}|^2)^{1/2}}{1 - \overline{\alpha_{n+1}}\zeta} \overline{B_{n+2}(\zeta)} B_{n+1}(\zeta) \overline{\Phi_{n+1}(\zeta)} = \\ &= -\zeta \frac{|\alpha_{n+1}|}{\alpha_{n+1}} (1 - |\alpha_{n+1}|^2)^{1/2} \frac{\overline{\Phi_{n+1}(\zeta)}}{\zeta - \alpha_{n+1}}, \end{aligned}$$

for $\zeta = e^{it}$, and therefore

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{\frac{B_{n+2}(\zeta)}{\zeta} \overline{\Phi_{n+1}} \left(\frac{1}{\zeta} \right)} \phi_{n+1}(\zeta) dt = \\ &= -\frac{|\alpha_{n+1}|}{\alpha_{n+1}} (1 - |\alpha_{n+1}|^2)^{1/2} \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\overline{\Phi_{n+1}(\zeta)}}{\zeta - \alpha_{n+1}} d\zeta = \\ &= -\frac{|\alpha_{n+1}|}{\alpha_{n+1}} (1 - |\alpha_{n+1}|^2)^{1/2} \overline{\Phi_{n+1}(\alpha_{n+1})}. \end{aligned} \quad (4.4)$$

Having in mind the value of the integrals (4.3) and (4.4) and using the right-hand side of formula (4.1), we have

$$\begin{aligned} J_n(z) = & -\frac{|\alpha_{n+1}|}{\alpha_{n+1}} (1 - |\alpha_{n+1}|^2)^{1/2} \overline{\Phi_{n+1}(\alpha_{n+1})} \frac{B_{n+2}(z)}{z} \overline{\Phi_{n+1}} \left(\frac{1}{z} \right) - \\ & -k_{n+1} \overline{\Phi_{n+1}(z)}. \end{aligned} \quad (4.4')$$

In order to compute the same integral in a second way, assume now that $\alpha_{n+1} \neq 0$. Then, in view of the identity

$$\frac{1 - \bar{\zeta}z}{1 - \alpha_{n+1}\bar{\zeta}} = \frac{1}{\alpha_{n+1}} \left\{ z + \frac{\alpha_{n+1} - z}{1 - \alpha_{n+1}\bar{\zeta}} \right\},$$

we have from (4.1)

$$R_n(\zeta; z) = (1 - |\alpha_{n+1}|^2) \frac{z}{\alpha_{n+1}(1 - \overline{\alpha_{n+1}}z)} S_n(\zeta; z) +$$

$$\begin{aligned}
& + (1 - |\alpha_{n+1}|^2) \frac{\alpha_{n+1} - z}{\alpha_{n+1}(1 - \alpha_{n+1}\bar{\zeta})(1 - \bar{\alpha}_{n+1}z)} S_n(\zeta; z) \equiv \\
& \equiv R_n^{(1)}(\zeta; z) + R_n^{(2)}(\zeta; z) .
\end{aligned} \tag{4.5}$$

Since obviously

$$S_n(\zeta; z) = \sum_{k=0}^n d_k(z) \overline{\phi_k(\zeta)},$$

we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} R_n^{(1)}(\zeta; z) \phi_{n+1}(\zeta) dt = 0, \quad \zeta = e^{it}. \tag{4.6}$$

Furthermore, noting that

$$\overline{\left(\frac{B_{n+1}(\zeta)}{\zeta} \right)} \phi_{n+1}(\zeta) = \zeta \frac{(1 - |\alpha_{n+1}|^2)^{1/2}}{1 - \bar{\alpha}_{n+1}\zeta}$$

for $\zeta = e^{it}$ and using identity (2.14) of Theorem 1, we get from definition (4.5) of the function $R_n^{(2)}(\zeta; z)$ that

$$R_n^{(2)}(\zeta; z) \phi_{n+1}(\zeta) = (1 - |\alpha_{n+1}|^2)^{3/2} \frac{B_{n+2}(z)}{|\alpha_{n+1}|z} \frac{\zeta S_n\left(\frac{1}{\bar{z}}; \zeta\right)}{(1 - \alpha_{n+1}\bar{\zeta})(1 - \bar{\alpha}_{n+1}\zeta)}.$$

Hence, from (4.6), (4.2), and (4.5), we have

$$\begin{aligned}
J_n(z) &= \frac{(1 - |\alpha_{n+1}|^2)^{3/2} B_{n+2}(z)}{|\alpha_{n+1}|} \frac{1}{z} \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\zeta S_n\left(\frac{1}{\bar{z}}; \zeta\right)}{1 - \bar{\alpha}_{n+1}\zeta} \frac{d\zeta}{\zeta - \alpha_{n+1}} = \\
&= \frac{|\alpha_{n+1}|}{\alpha_{n+1}} (1 - |\alpha_{n+1}|^2)^{1/2} \frac{B_{n+2}(z)}{z} S_n\left(\frac{1}{\bar{z}}; \alpha_{n+1}\right)
\end{aligned} \tag{4.7}$$

under the restriction $\alpha_{n+1} \neq 0$. By taking the limit, we can remove the restriction on α_{n+1} as usual.

Comparing the two representations (4.4) and (4.7) of the function $J_n(z)$, we arrive at the following formula for systems of functions $\{\Phi_k\}_0^\infty$:

$$\begin{aligned}
& - \frac{|\alpha_{n+1}|}{\alpha_{n+1}} \frac{k_{n+1}}{(1 - |\alpha_{n+1}|^2)^{1/2}} \Phi_{n+1}(z) = \\
& = \frac{B_{n+2}(z)}{z} \left\{ \Phi_{n+1}(\alpha_{n+1}) \overline{\Phi_{n+1}\left(\frac{1}{z}\right)} + \sum_{k=0}^n \Phi_k(\alpha_{n+1}) \overline{\Phi_k\left(\frac{1}{z}\right)} \right\} \\
& \quad (k = 0, 1, 2, \dots) .
\end{aligned} \tag{4.8}$$

If $\alpha_{n+1} \neq 0$, then formula (4.7) can be transformed, using the functional identity (2.14) for the kernel $S_n(\zeta; z)$.

Thus, after an appropriate transformation, we have under the restriction $\alpha_{n+1} \neq 0$ that

$$\begin{aligned}
& - \frac{|\alpha_{n+1}|}{\alpha_{n+1}} \frac{k_{n+1}}{(1 - |\alpha_{n+1}|^2)^{1/2}} \Phi_{n+1}(z) = \Phi_{n+1}(\alpha_{n+1}) \frac{B_{n+2}(z)}{z} \overline{\Phi_{n+1}\left(\frac{1}{z}\right)} + \\
& + \frac{|\alpha_{n+1}|}{\alpha_{n+1}} \frac{B_{n+2}(\alpha_{n+1})}{\alpha_{n+1}} \frac{\alpha_{n+1} - z}{1 - \bar{\alpha}_{n+1}z} \sum_{k=0}^n \overline{\Phi_k\left(\frac{1}{\alpha_{n+1}}\right)} \Phi_k(z)
\end{aligned}$$

$$(n = 0, 1, 2, \dots) . \quad (4.9)$$

In the limiting case, when $\alpha_k = 0$ ($k = 0, 1, 2, \dots$), it follows from (4.8), that for a system of Szegő polynomials $\{P_k(z)\}_0^\infty$, we have

$$k_{n+1}P_{n+1}(z) = z^{n+1} \left\{ P_{n+1}(0)\overline{P_{n+1}}\left(\frac{1}{z}\right) + \sum_{k=0}^n P_k(0)\overline{P_k}\left(\frac{1}{z}\right) \right\} . \quad (4.10)$$

However, using the first of formula (2.44), we get

$$\sum_{k=0}^n P_k(0)\overline{P_k}\left(\frac{1}{z}\right) = S_n\left(0; \frac{1}{z}\right) = k_n z^{-n} P_n(z) .$$

But according to the definition of the numbers l_k , we have in the considered case that $P_{n+1}(0) = l_{n+1}$. Therefore, from (4.10) we come to the first recurrence formula of Szegő

$$k_{n+1}P_{n+1}(z) = l_{n+1}z^{n+1}\overline{P_{n+1}}\left(\frac{1}{z}\right) + k_n z P_n(z) \quad (n = 0, 1, 2, \dots) . \quad (4.11)$$

Substituting z for $\frac{1}{z}$ in (4.11) and taking complex conjugates, and eliminating $z^{n+1}\overline{P_{n+1}}\left(\frac{1}{z}\right)$, gives the second recurrence formula of Szegő:

$$k_{n+1}P_{n+1}(z) = k_{n+1}P_n(z) + l_{n+1}z^n \overline{P_n}\left(\frac{1}{z}\right) . \quad (4.11')$$

We can give analog formulas for general orthogonal systems of rational functions $\{\Phi_k(z)\}_0^\infty$, but we will not do this here.

4.2. To conclude, we establish explicit formulas for orthogonal systems $\{\Phi_k(z)\}_0^\infty$ and for corresponding kernels $S_n(\zeta; z)$ in the case of special but important classes of distribution functions.

Assume that the constants

$$A_p > 0 \quad \text{and} \quad \{\gamma_k\}_0^p \quad (0 \leq \gamma_k < 1) \quad (4.12)$$

are arbitrary. For given $p \geq 0$, we consider the functions

$$\omega_p(z) = A_p^{-1/2} \prod_{k=0}^p \frac{z - \gamma_k}{1 - \overline{\alpha_k} z}, \quad D_p(z) = A_p^{1/2} \prod_{k=0}^p \frac{1 - \overline{\alpha_k} z}{1 - \gamma_k z}, \quad (4.13)$$

and also the distribution function $d\alpha(x) = w_p(x)dx$, where

$$w_p(x) = |D_p(e^{ix})|^2 \quad (-\pi \leq x \leq \pi) . \quad (4.14)$$

Let us prove

Theorem 3. *1°. An orthogonal system of rational functions $\{\Phi_k(z)\}_0^\infty$ associated with sequence $\{\alpha_k\}_0^\infty$ ($|\alpha_k| < 1$) and with distribution $(2\pi)^{-1}w_p(x)dx$ admits the representation*

$$\Phi_n(z) = e^{ix_n} \omega_p(z) \frac{(1 - |\alpha_n|^2)^{1/2}}{1 - \overline{\alpha_n} z} \prod_{k=p+1}^{n-1} \frac{\alpha_k - z}{1 - \overline{\alpha_k} z} \frac{|\alpha_k|}{\alpha_k} \quad (n = p + 1, p + 2, \dots) , \quad (4.15)$$

where χ_n ($\text{Im}\chi_n = 0$) is a constant.

2°. The formula

$$\begin{aligned} S_n(\zeta; z) &= \sum_{k=0}^n \overline{\Phi_k(\zeta)} \Phi_k(z) = \\ &= \frac{1}{(1 - \bar{\zeta}z) \overline{D_p(\zeta)} D_p(z)} - \frac{\overline{B_{n+1}(\zeta)} B_{n+1}(z)}{(1 - \bar{\zeta}z) \overline{D_p\left(\frac{1}{\zeta}\right)} D_p\left(\frac{1}{z}\right)} \\ &\quad (n = p, p+1, \dots). \end{aligned} \quad (4.16)$$

is valid.

Proof. 1°. Observe that, according to (4.13) and (4.14),

$$w_p(x) |\omega_p(z)|^2 \equiv 1 \quad (z = e^{ix}, -\pi \leq x \leq \pi). \quad (4.17)$$

Therefore, denoting

$$\tilde{\Phi}_n(z) = \omega_p(z) \frac{(1 - |\alpha_n|^2)^{1/2}}{1 - \bar{\alpha}_n z} \prod_{k=p+1}^{n-1} \frac{\alpha_k - z}{1 - \bar{\alpha}_k z} \frac{|\alpha_k|}{\alpha_k} \quad (n = p+1, p+2, \dots), \quad (4.18)$$

for $n, m \geq p+1$, we have:

$$\begin{aligned} &\frac{1}{2\pi} \int_{-\pi}^{\pi} w_p(x) \tilde{\Phi}_n(z) \overline{\tilde{\Phi}_m(z)} dx = \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \frac{(1 - |\alpha_n|^2)^{1/2}}{1 - \bar{\alpha}_n z} \prod_{k=p+1}^{n-1} \frac{\alpha_k - z}{1 - \bar{\alpha}_k z} \frac{|\alpha_k|}{\alpha_k} \right\} \times \\ &\times \overline{\left\{ \frac{(1 - |\alpha_m|^2)^{1/2}}{1 - \bar{\alpha}_m z} \prod_{k=p+1}^{m-1} \frac{\alpha_k - z}{1 - \bar{\alpha}_k z} \frac{|\alpha_k|}{\alpha_k} \right\}} dx = \delta_{n,m} \quad (z = e^{ix}), \end{aligned} \quad (4.19)$$

in view of the orthonormality of a Malmquist system associated with the sequence $\{\alpha_k\}_{p+1}^{\infty}$.

Thus, the system of functions $\{\tilde{\Phi}_k(z)\}_{p+1}^{\infty}$ is orthonormal on the unit circle with respect to the weight $(2\pi)^{-1} w_p(x) dx$

Now we show that for arbitrary $n \geq p+1$ and $0 \leq m \leq n-1$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} w_p(x) \tilde{\Phi}_n(z) \overline{\tilde{\Phi}_m(z)} dx = 0 \quad (z = e^{ix}). \quad (4.20)$$

Indeed, from (4.17) and (4.18), we have for $n \geq p+1$

$$\begin{aligned} J_m^{(n)} &\equiv \frac{1}{2\pi} \int_{-\pi}^{\pi} w_p(x) \tilde{\Phi}_n(z) \overline{\tilde{\Phi}_m(z)} dx = \\ &= \frac{1}{2\pi i} \int_{|z|=1} \frac{(1 - |\alpha_n|^2)^{1/2}}{1 - \bar{\alpha}_n z} \prod_{k=p+1}^{n-1} \frac{\alpha_k - z}{1 - \bar{\alpha}_k z} \frac{|\alpha_k|}{\alpha_k} \overline{\left\{ z \frac{\tilde{\Phi}_m(z)}{\omega_p(z)} \right\}} dz. \end{aligned} \quad (4.21)$$

On the other hand, since the representation

$$\Phi_m(z) = \sum_{k=0}^m c_{k,m} \phi_k(z), \quad c_{m,m} > 0 \quad (4.22)$$

holds true, where $\{\phi_k(z)\}_0^\infty$ is a system of Malmquist functions, we also have

$$\Phi_m(z) = \frac{P_m(z)}{\prod_{k=0}^n (1 - \overline{\alpha_k}z)},$$

where $P_m(z)$ is a polynomial of degree m at most.

Because

$$\left\{ z \frac{\Phi_m(z)}{\omega_p(z)} \right\} = A_p^{1/2} z^m \overline{P_m} \left(\frac{1}{z} \right) \prod_{k=0}^m (z - \alpha_k)^{-1} \prod_{k=0}^p \frac{z - \alpha_k}{1 - \overline{\gamma_k}z}$$

for $|z| = 1$, it follows that for $0 \leq m \leq n - 1$, the integrand of the second integral of (4.21) can be represented in the form

$$Q_m(z) \prod_{k=p+1}^n (1 - \overline{\alpha_k}z)^{-1} \prod_{k=0}^p (1 - \overline{\gamma_k}z)^{-1} \prod_{k=m-1}^{n-1} (z - \alpha_k), \quad (4.23)$$

where $Q_m(z)$ is a polynomial of degree not higher than m . (For $m = n - 1$, the product $\prod_{k=m-1}^{n-1}$ must be replaced by one.)

However, the function (4.23) is holomorphic in the closed disc $|z| \leq 1$, and therefore the correctness of formula (4.20) follows from (4.21).

Finally, from the obvious representation

$$\tilde{\Phi}_n(z) = \sum_{k=0}^n a_{k,n} \Phi_k(z) \quad (n \geq p_1)$$

and in view of condition (4.19), we conclude that

$$a_{k,n} = 0 \quad (k = 0, 1, \dots, n - 1).$$

Thus, for $n \geq p + 1$, we have

$$\tilde{\Phi}_n(z) = a_{n,n} \Phi_n(z),$$

from which we get in view of the equality

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} w_p(x) |\Phi_p(z)|^2 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} w_p(x) |\tilde{\Phi}_p(z)|^2 dx = 1,$$

that $|a_{n,n}| = 1$.

Hence it follows, that

$$\Phi_n(z) = e^{i\chi_n} \tilde{\Phi}_n(z) \quad (n \geq p + 1),$$

where the real constant χ_n is determined from the condition that $c_{n,n} > 0$ in representation (4.22). Thus formula (4.15) is established.

2°. According to formula (4.15), we have in particular

$$\Phi_{p+1}(z) = e^{i\chi_{p+1}} \omega_p(z) \frac{(1 - |\alpha_{p+1}|^2)^{1/2}}{1 - \overline{\alpha_{p+1}}z},$$

and therefore

$$\overline{\Phi_{p+1}}\left(\frac{1}{z}\right) = e^{-i\chi_{p+1}} z \overline{\omega_p}\left(\frac{1}{z}\right) \frac{(1 - |\alpha_{p+1}|^2)^{1/2}}{z - \alpha_{p+1}}.$$

From this formula follow the identities

$$\frac{(1 - \overline{\alpha_{p+1}}\zeta)(1 - \overline{\alpha_{p+1}}z)}{1 - |\alpha_{p+1}|^2} \overline{\Phi_{p+1}(\zeta)} \Phi_{p+1}(z) = \overline{\omega_p(\zeta)} \omega_p(z), \quad (4.24)$$

$$\begin{aligned} & \frac{(1 - \overline{\alpha_{p+1}}\zeta)(1 - \overline{\alpha_{p+1}}z)}{1 - |\alpha_{p+1}|^2} \left\{ \frac{B_{p+2}(\zeta)}{\zeta} \overline{\Phi_{p+1}}\left(\frac{1}{\zeta}\right) \right\} \frac{B_{p+2}(z)}{z} \overline{\Phi_{p+1}}\left(\frac{1}{z}\right) = \\ & = \left\{ \overline{\omega_p}\left(\frac{1}{\zeta}\right) B_{p+1}(\zeta) \right\} \overline{\omega_p}\left(\frac{1}{z}\right) B_{p+1}(z). \end{aligned} \quad (4.25)$$

Having noted that, according to Theorem 2,

$$\begin{aligned} S_p(\zeta; z) &= \frac{(1 - \overline{\alpha_{p+1}}\zeta)(1 - \overline{\alpha_{p+1}}z)}{1 - |\alpha_{p+1}|^2} \times \\ & \times \frac{\left\{ \frac{B_{p+2}(\zeta)}{\zeta} \overline{\Phi_{p+1}}\left(\frac{1}{\zeta}\right) \right\} \frac{B_{p+2}(z)}{z} \overline{\Phi_{p+1}}\left(\frac{1}{z}\right) - \overline{\Phi_{p+1}(\zeta)} \Phi_{p+1}(z)}{1 - \zeta z}, \end{aligned}$$

we get from (4.24) and (4.25)

$$S_p(\zeta; z) = \frac{\left\{ \overline{\omega_p}\left(\frac{1}{\zeta}\right) B_{p+1}(\zeta) \right\} \overline{\omega_p}\left(\frac{1}{z}\right) B_{p+1}(z) - \overline{\omega_p(\zeta)} \omega_p(z)}{1 - \zeta z}. \quad (4.26)$$

We denote further

$$\beta_j = \alpha_{p+1+j} \quad (j = 0, 1, 2, \dots)$$

and we introduce a system $\{\psi_k(z)\}_0^\infty$ associated with the sequence of complex numbers $\{\beta_k\}_0^\infty$:

$$\psi_k(z) = \frac{(1 - |\beta_k|^2)^{1/2}}{1 - \overline{\beta_k}z} \prod_{j=0}^{k-1} \frac{\beta_j - z}{1 - \overline{\beta_j}z} \frac{|\beta_j|}{\beta_j} \quad (k = 0, 1, 2, \dots).$$

Then, according to (4.15),

$$\Phi_{p+1+k}(z) = e^{i\chi_{p+1+k}} \omega_p(z) \psi_k(z) \quad (k = 0, 1, 2, \dots)$$

and therefore

$$\sum_{k=p+1}^{p+n} \overline{\Phi_k(\zeta)} \Phi_k(z) = \overline{\omega_p(\zeta)} \omega_p(z) \sum_{k=0}^{n-1} \overline{\psi_k(\zeta)} \psi_k(z). \quad (4.27)$$

But for a system of functions $\{\psi_k(z)\}_0^\infty$, we have according to Lemma 5:

$$\sum_{k=0}^{n-1} \overline{\psi_k(\zeta)} \psi_k(z) = \frac{1 - \overline{\tilde{B}_n(\zeta)} \tilde{B}_n(z)}{1 - \zeta z}$$

where

$$\tilde{B}_n(z) = \sum_{j=0}^{n-1} \frac{\beta_j - z}{1 - \overline{\beta_j}z} \frac{|\beta_j|}{\beta_j} = \frac{B_{n+p+1}(z)}{B_{p+1}(z)}.$$

Consequently, formula (4.27) can be written in the form

$$S_{n+p}(\zeta; z) - S_p(\zeta; z) = \overline{\omega_p(\zeta)} \omega_p(z) \frac{1 - \overline{\left\{ \frac{B_{n+p+1}(\zeta)}{B_{p+1}(\zeta)} \right\}} \frac{B_{n+p+1}(z)}{B_{p+1}(z)}}{1 - \bar{\zeta}z}. \quad (4.28)$$

Finally, from (4.24) and (4.28), and for all $n = 0, 1, 2, \dots$, we have

$$\begin{aligned} S_{n+p}(\zeta; z) &= \\ &= \frac{\overline{\left\{ B_{n+1}(\zeta) \overline{\omega_p} \left(\frac{1}{\zeta} \right) \right\}} B_{p+1}(z) \overline{\omega_p} \left(\frac{1}{z} \right) - \overline{\left\{ \omega_p(\zeta) \frac{B_{n+p+1}(\zeta)}{B_{p+1}(\zeta)} \right\}} \omega_p(z) \frac{B_{n+p+1}(z)}{B_{p+1}(z)}}{1 - \bar{\zeta}z}. \end{aligned} \quad (4.29)$$

Now we observe that, from definition (4.13) of the functions $\omega_p(z)$ and $D_p(z)$, we have

$$B_{p+1}(z) \overline{\omega_p} \left(\frac{1}{z} \right) = \frac{c_p}{D_p(z)} \quad (|c_p| = 1), \quad (4.30)$$

from which, by virtue of the obvious identity $B_{p+1}(z) \overline{B_{p+1}} \left(\frac{1}{z} \right) \equiv 1$, we also have

$$\omega_p(z) = \frac{\bar{c}_p}{\overline{D_p} \left(\frac{1}{z} \right)} B_{p+1}(z). \quad (4.31)$$

Finally, using identities (4.30) and (4.31), the formulas (4.24) and (4.29) can be combined and written in the form

$$\begin{aligned} S_{n+p}(\zeta; z) &= \frac{1}{(1 - \bar{\zeta}z) \overline{D_p}(\zeta) D_p(z)} - \frac{\overline{B_{n+p+1}(\zeta)} B_{n+p+1}(z)}{(1 - \bar{\zeta}z) \overline{D_p} \left(\frac{1}{\zeta} \right) \overline{D_p} \left(\frac{1}{z} \right)} \\ &\quad (n = 0, 1, 2, \dots) \end{aligned}$$

which is equivalent to the formula (4.16) of the theorem.

4.3. From the previous theorem, it follows further that

T h e o r e m 4. 1°. *If*

$$B = \sum_{k=0}^{\infty} (1 - |\alpha_k|) = +\infty, \quad (4.32)$$

then the formula

$$S_n(\zeta; z) \equiv \sum_{k=0}^{\infty} \overline{\Phi_k(\zeta)} \Phi_k(z) = \frac{1}{(1 - \bar{\zeta}z) \overline{D_p}(\zeta) D_p(z)} \quad (4.33)$$

holds true for arbitrary z and ζ ($|z| < 1$ and $|\zeta| < 1$).

2°. *If*

$$B = \sum_{k=0}^{\infty} (1 - |\alpha_k|) < +\infty, \quad (4.32)$$

then, for arbitrary z and ζ ($|z| \neq 1$ and $|\zeta| \neq 1$) which are different from numbers of the sequence $\{1/\overline{\alpha_k}\}_0^{\infty}$, the formula

$$S_n(\zeta; z) = \sum_{k=0}^{\infty} \overline{\Phi_k(\zeta)} \Phi_k(z) = \frac{1}{(1 - \bar{\zeta}z) \overline{D_p}(\zeta) D_p(z)} - \frac{\overline{B(\zeta)} B(z)}{(1 - \bar{\zeta}z) \overline{D_p} \left(\frac{1}{\zeta} \right) \overline{D_p} \left(\frac{1}{z} \right)}, \quad (4.35)$$

where

$$B(z) = \prod_{k=0}^{\infty} \frac{\alpha_k - z}{1 - \overline{\alpha_k} z} \frac{|\alpha_k|}{\alpha_k} \quad (4.36)$$

is a convergent Blaschke product.

In order to obtain statements 1° and 2° of the theorem, it is sufficient to take the limit in formula (4.16) for $n \rightarrow +\infty$. Here we only need to consider the case, that

$$\lim_{n \rightarrow \infty} B_n(z) = 0 \quad (|z| < 1)$$

for $B = +\infty$ and that

$$\lim_{n \rightarrow \infty} B_n(z) = B(z) \quad (|z| \neq 1, z \neq 1/\overline{\alpha_k}, k = 0, 1, \dots)$$

for the condition $B < +\infty$.

In the considered example of the distribution $(2\pi)^{-1}w_p(x)dx$, there is a clear and essential distinction for the sets of convergence points and the values of the sum of the bilinear series

$$S(\zeta; z) = \sum_{k=0}^{\infty} \overline{\Phi_k(\zeta)} \Phi_k(z)$$

depending on divergence or convergence of the series

$$B = \sum_{k=0}^{\infty} (1 - |\alpha_k|) .$$

It turns out that, in the general case of an arbitrary distribution $(2\pi)^{-1}d\alpha(x)$, the nature of convergence and the value of the sum of series $S(\zeta; z)$ depends essentially on the fact that the values

$$A = \int_{-\pi}^{\pi} \log \alpha'(x) dx \quad \text{and} \quad B = \sum_{k=0}^{\infty} (1 - |\alpha_k|) .$$

are finite or infinite. However, we will not discuss this here but we will address this question in a separate paper.

References

- [1] M.M. Džrbašian. Orthogonal systems of rational functions on a circle with a given set of poles.. *Doklady Akademii Nauk SSSR*, 147, 1962. English translation *Soviet Mathematics Doklady* 3(6):1794–1798, 1962.
- [2] G. Szegő. Orthogonal polynomials. Ch. X and XI. M. 1962. English edition, AMS, Providence, Rhode Island, first edition 1939, fourth edition 1975.
- [3] U. Grenander and G. Szegő. Toeplitz forms and their applications. Ch. 2 and 3. M. 1961. English edition, University of California Press, Berkley, 1958.
- [4] F. Malmquist. Sur la détermination d'une classe de fonctions analytiques par leurs valeurs dans un ensemble donné de poits. In *C.R. 6ième Cong. Math. Scand. (Kopenhagen, 1925)*, pages 253–259, Copenhagen, 1926. Gjellerups.

- [5] J.L. Walsh. Interpolation and approximation. §10.7. M. 1961. English edition, AMS, Providence, Rhode Island, 3rd ed., 1960.
- [6] R. Lagrange. Mémoire sur les séries d'interpolation. *Acta Math.*, 64, 1935.
- [7] E. Lammel. Approximation regulärer Funktionen eines komplexen Argumentes durch rationale Funktionen. *Math. Zeitschr.*, 46, 1940.
- [8] M.M. Džrbašian. On the theory of Fourier series with rational functions. *Izv. Akad. Nauk. Armjan. SSR Ser. Fiz.-Math.* 9(7) 1996.