On the relation between iterated function systems and partitioned iterated function systems

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Abstract

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AMS(MOS) Classification : Primary : 41A99, Secondary : 58F11.

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In this paper, we give a theoretically founded transition from Iterated Function Systems based on affine transformations to Partitioned Iterated Function Systems. We show that there are two essential steps, namely, restricting the affine transformations, and solving the evoked problem of ink dissipation.

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1 Introduction

In the area of image processing, especially when based on fractals, there are two common ways of formalizing a monochrome image, namely, as a measure describing the ink distribution, and as a brightness function. The former one is used in the paradigm of fractal image synthesis and compression, the Iterated Function System (IFS) [1, 9], and in its extensions [2, 7]. The latter one is found in several recent compression strategies [5, 8, 4], one of which is based on Partitioned Iterated Function Systems (PIFSs).

In [3] this concept is introduced from a practical point of view in order to meet the need for IFSs in which each function is only iterated on some part of the image instead of the entire image. For the same reason, in [7] an analogue one is considered using the measure formalization of an image. However, the precise form proposed is not theoretically founded, nor is it clear in [3] what assumptions are implicitly made in using it. That is what we intend to investigate in this paper, by making the transition from IFSs based on affine transformations to PIFSs. We shall see that there are two major steps, the first one being the restricting of the affine transformations, and the second one a particular way of solving the problem of ink dissipation caused by the first step.

After fixing our notation in section 2, we introduce the concepts of an IFS and a PIFS, and state precisely the formalizations of an image as a measure and as a function in section 3. Section 4 discusses the transition from IFSs to PIFSs in detail. Since comparing both concepts is only feasible when using the same formalization, before doing so, IFS theory is translated in the function formalism, and conversely, PIFS theory in the measure formalism.

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2 Notation

Let I be the unit interval $[0,1]$ of the real axis, and I^n the n-th Cartesian product of I for $n \in \mathbb{N}_0$. Unless otherwise stated, d denotes the Euclidean metric on I^n, which is always understood, when no metric on I^n is explicitly mentioned.

For any metric space X, $\mathcal{B}(X)$ is the set of Borel subsets of X, $\mathcal{M}(X)$ the set of positive Borel measures on X, and $\mathcal{P}(X)$ the set of Borel probability measures on X.
If X is compact, $\mathcal{P}(X)$ can be metrized by the Hutchinson metric d_H, defined as follows:
$d_H : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}^+ : (\mu, \nu) \to$
$\sup \{ |\int f\, d\mu - \int f\, d\nu| \mid f : X \to \mathbb{R} \text{ such that } \forall\, x_1, x_2 \in X : |f(x_1) - f(x_2)| \leqslant d(x_1, x_2) \}$,
where d represents the metric on X [6, section 4.3(1) p. 732].
λ denotes the Lebesgue measure on I^2, defined on the set $\mathcal{L}(I^2)$ consisting of the Lebesgue subsets of I^2.

For any space X, let $\mathcal{F}(X)$ be the set of functions $f : X \to I$. $\mathcal{F}_{\mathcal{L}}(I^2)$ is the set of equivalence classes of the Lebesgue measurable functions $f : I^2 \to I$ modulo λ. For any $f \in \mathcal{F}(X)$, graph(f) is $\{(x,y) \in X \times I \mid f(x) = y\}$.

3 Definition of IFSs and PIFSs

In this section, we shall introduce the concepts of an IFS and a PIFS on a general metric space X. In case $X = I^2$, these concepts can be used to compress images formalized as measures or functions respectively.

3.1 IFSs and the measure formalism

In [1] an iterated function system is defined for a general metric space as follows.

Definition 1 *Let X be a space with metric d, $n \in \mathbb{N}_0$, $v_i : X \to X$ a continuous transformation for $i = 1(1)n$, and $p_i \in [0,1]$ for $i = 1(1)n$ satisfying $\sum_{i=1}^{n} p_i = 1$. Then $S = ((X,d); v_1, v_2, \ldots, v_n; p_1, p_2, \ldots, p_n)$ is an iterated function system (IFS).*

S is called hyperbolic iff all transformations v_i are contractive.
An operator on $\mathcal{P}(X)$, called the Markov operator, can be associated with S.

Definition 2 *Let $S = ((X,d); v_1, v_2, \ldots, v_n; p_1, p_2, \ldots, p_n)$ be an IFS. The Markov operator associated with S is the operator*

$$M : \mathcal{P}(X) \to \mathcal{P}(X) : \mu \to \sum_{i=1}^{n} p_i \cdot \mu v_i^{-1}.$$

In case X is compact and S is hyperbolic, M has a unique fixed point, because M can be shown to be contractive w.r.t. the Hutchinson metric, under which $\mathcal{P}(X)$ is complete. This fixed point is called the invariant measure of S.

Definition 3 *Let X be a compact metric space, and S a hyperbolic IFS on X. The invariant measure of S is the unique fixed point of the Markov operator associated with S.*

By the term 'measure formalism', we mean the formalization of a monochrome image as a positive Borel measure μ on I^2. For any Borel set $B \in \mathcal{B}(I^2)$, $\mu(B)$ is the amount of ink on the corresponding part of the image, the carrier of which is I^2. Mostly, the total amount of ink is normalized to 1, so that $\mu \in \mathcal{P}(I^2)$. Using this formalism, an image can be approximated as the invariant measure of a hyperbolic IFS on I^2, and hence lossily compressed to the parameters describing the IFS. In practice, the transformations v_i of the IFS are chosen affine, and therefore can be described compactly.

3.2 PIFSs and the function formalism

In [3] the concept of a partitioned iterated function system on I^2 is introduced. It can be generalized for arbitrary metric spaces as follows.

Definition 4 *Let X be a space with metric d, $n \in \mathbb{N}_0$, $v_i : D_i \subseteq X \to X$ a contractive one-to-one transformation with domain D_i, $s_i, o_i \in \mathbb{R}$, and*

$$w_i : D_i \times I \subseteq X \times I \to X \times I : (x, z) \to (v_i(x), s_i z + o_i)$$

for $i = 1(1)n$. If $\{v_i(D_i)\}_{i=1}^n$ is a partition of X, then $S = ((X, d); w_1, w_2, \ldots, w_n)$ is a partitioned iterated function system (PIFS).

S is called hyperbolic iff $|s_i| < 1$ for $i = 1(1)n$.
The following operator on $\mathcal{F}(X)$ can be associated with S.

Definition 5 *Let $S = ((X, d); w_1, w_2, \ldots, w_n)$ be a PIFS as in definition 4. The operator associated with S is*

$$W : \mathcal{F}(X) \to \mathcal{F}(X) : f \to W(f),$$

$W(f)$ being defined by the condition

$$graph(W(f)) = \bigcup_{i=1}^{n} (graph(f) \cap (D_i \times I)). \tag{1}$$

Since $\{v_i(D_i)\}_{i=1}^n$ forms a partition of X, and v_i is one-to-one for $i = 1(1)n$, the right-hand side of (1) is the graph of a function of X in I. The condition (1) can be expressed equivalently as

$$W(f)|_{v_i(D_i)} = s_i \cdot (f \circ v_i^{-1}) + o_i \tag{2}$$

for $i = 1(1)n$.
It is clear that W is contractive w.r.t. the supremum metric on $\mathcal{F}(X)$ iff S is hyperbolic. In that case, $\mathcal{F}(X)$ being complete w.r.t. the supremum metric, W has a unique fixed point: the invariant function of S.

Definition 6 *Let S be a hyperbolic PIFS. The invariant function of S is the unique fixed point of the operator associated with S.*

By 'function formalism', we understand the formalization of a monochrome image as a nonnegative function f defined on I^2. For any point $x \in I^2$, $f(x)$ gives the brightness at x of the image with carrier I^2. Mostly, the maximum brightness is normalized to 1, so that $f \in \mathcal{F}(I^2)$. In analogy with the IFS case, using this formalism, an image can be lossily compressed to the parameters of a PIFS whose invariant function approximates the given image. In practice, the transformations v_i of the PIFS are restrictions of affine transformations (to simple subsets of I^2, e.g. rectangles), so that the w_i's are also restricted affine transformations: if

$$v_i : D_i \subseteq I^2 \to I^2 : \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e_i \\ f_i \end{bmatrix},$$

where $a_i, b_i, c_i, d_i, e_i, f_i \in \mathbb{R}$, then

$$w_i : D_i \times I \to I^3 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \begin{bmatrix} a_i & b_i & 0 \\ c_i & d_i & 0 \\ 0 & 0 & s_i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} e_i \\ f_i \\ o_i \end{bmatrix}. \tag{3}$$

The conditions stipulated in the definition of a PIFS on I^2, ensure that W induces an operator on $\mathcal{F}_{\mathcal{L}}(I^2)$, provided the domains D_i are Lebesgue sets.

Proposition 1 *Let $S = ((I^2, d); w_1, w_2, \ldots, w_n)$ be a PIFS as in definition 4, such that $D_i \in \mathcal{L}(I^2)$ for $i = 1(1)n$, and let W be the operator associated with S.*

- *If $f \in \mathcal{F}(I^2)$ is Lebesgue measurable, then so is $W(f)$.*
- *For any pair $f, g \in \mathcal{F}(I^2)$ of Lebesgue measurable functions,*

$$f = g \ a.e.(\lambda) \Rightarrow W(f) = W(g) \ a.e.(\lambda)$$

holds.

Proof.

- The former property follows form two observations.
The first is that for any $f \in \mathcal{F}(I^2)$ and any Borel subset B of I^2,

$$\begin{aligned} W(f)^{-1}(B) &= \cup_{i=1}^n \{x \in v_i(D_i) \mid s_i \cdot f(v_i^{-1}(x)) + o_i \in B\} \\ &= \cup_{i=1}^n \{x \in v_i(D_i) \mid f(v_i^{-1}(x)) \in B'\} \\ &= \cup_{i=1}^n v_i(f^{-1}(B') \cap D_i), \end{aligned}$$

where B' represents another Borel subset of I^2.
The second is that v_i^{-1} is Borel measurable, because it is continuous as the inverse of a continuous one-to-one function on a compact space, and that v_i, being contractive, transforms a zero measure set into a zero measure set.

- The key for the proof of the latter property is that for any $f, g \in \mathcal{F}(I^2)$,

$$\{x \in I^2 \mid W(f)(x) \neq W(g)(x)\} \subseteq \cup_{i=1}^n v_i(\{x \in I^2 \mid f(x) \neq g(x)\})$$

holds.

□

Based on this property, W can be defined on $\mathcal{F}_{\mathcal{L}}(I^2)$ in the usual way. Moreover, $\mathcal{F}_{\mathcal{L}}(I^2)$ being closed w.r.t. the supremum metric, proposition 1 implies that the invariant function of a PIFS satisfying its premisses, is Lebesgue measurable. These properties are important for the sequel and theoretically support the use of the RMS metric instead of the supremum metric in practical calculations [3].

4 Transition from IFSs to PIFSs

In this section, it will be shown how PIFSs can be seen as straightforward adaptations of IFSs, meeting the need for restricting the domains of the affine transformations on I^2 involved. In doing so, we shall first reformulate IFS theory in the function formalism, and second, PIFS theory in the measure formalism. Eventually, these reformulations will allow us to indicate precisely what extra modification to the measure formalism of an IFS, beside restricting the affine transformations, has to be introduced, in order for the function formalism to become a PIFS.

The relation that has to hold in going from the measure formalism to the function formalism and vice versa, is that for a measure μ and a function f describing the same image, $f = \frac{d\mu}{d\lambda}$, the right-hand side denoting a Radon-Nikodym derivative of μ w.r.t. λ, or equivalently, that $\mu(B) = \int_B f\, d\mu$ for any $B \in \mathcal{B}(I^2)$. It is clear that the translation of a μ-formalism into an f-formalism is only possible if $\mu \ll \lambda$, and the converse is only possible if f is Lebesgue measurable.

4.1 Translation of IFSs in the function formalism

The Markov operator M associated with an IFS can be translated in an operator T on nonnegative functions on I^2 as follows.

Theorem 1 *Let $S = ((I^2, d); v_1, v_2, \ldots, v_n; p_1, p_2, \ldots, p_n)$ be an IFS with transformations*

$$v_i : I^2 \to I^2 : \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e_i \\ f_i \end{bmatrix}, \tag{4}$$

where $a_i, b_i, c_i, d_i, e_i, f_i \in \mathbb{R}$, and $\det A_i \neq 0$, where $A_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}$ for $i = 1(1)n$. Let M be the Markov operator associated with S, and $\mu \in \mathcal{P}(I^2)$. Let

$$T : \{f : I^2 \to \mathbb{R}^+\} \to \{f : I^2 \to \mathbb{R}^+\} : f \to T(f),$$

$T(f)$ being defined as

$$T(f) : I^2 \to \mathbb{R}^+ : x \to \begin{cases} \sum_{i \in I(x)} \frac{p_i}{|\det A_i|} \cdot f(v_i^{-1}(x)) & \text{if } I(x) \neq \emptyset \\ 0 & \text{if } I(x) = \emptyset, \end{cases} \tag{5}$$

where $I(x) \triangleq \{i \in \{1, 2, \ldots, n\} \mid x \in v_i(I^2)\}$.
If $\mu \ll \lambda$, then $M(\mu) \ll \lambda$, and

$$\frac{dM(\mu)}{d\lambda} = T\left(\frac{d\mu}{d\lambda}\right).$$

Proof. Let $f = \frac{d\mu}{d\lambda}$, and $B \in \mathcal{B}(I^2)$, then for any $i \in \{1, 2, \ldots, n\}$

$$\begin{aligned} \mu v_i^{-1}(B) &= \mu\left(v_i^{-1}(B \cap v_i(I^2))\right) \\ &= \int \chi_{v_i^{-1}(B \cap v_i(I^2))}(x) \cdot f(x)\, d\lambda(x) \end{aligned}$$

$$
\begin{aligned}
&= \int \chi_{v_i^{-1}(B\cap v_i(I^2))}(v_i^{-1}(y)) \cdot f(v_i^{-1}(y))\, d\lambda v_i^{-1}(y) \\
&= \int \chi_{B\cap v_i(I^2)}(y) \cdot f(v_i^{-1}(y))\, d\lambda v_i^{-1}(y) \\
&= \int \chi_B \cdot \left[\chi_{v_i(I^2)} \cdot \frac{1}{|\det A_i|} \cdot (f \circ v_i^{-1})\right] d\lambda, \qquad (6)
\end{aligned}
$$

where we have used the fact that for the affine transformation v_i, $\frac{d\lambda v_i^{-1}}{d\lambda} = \frac{1}{|\det A_i|}$ holds. So, $M(\mu)(B) = \sum_{i=1}^n p_i \cdot \mu v_i^{-1}(B) = \int \chi_B \cdot T(f)\, d\lambda$. □

It should be noted that the set $\{\mu \in \mathcal{P}(I^2) \mid \mu \ll \lambda\}$, although closed under M, is not closed w.r.t. the Hutchinson metric. Correspondingly, the set $\{f : I^2 \to \mathbb{R}^+ \mid \sup_{x\in I^2} f(x) < \infty\}$, although closed under T, is not closed w.r.t. the supremum metric.
A sufficient condition for the contractiveness of T, is that $\frac{p_i}{|\det A_i|} < 1$ for $i = 1(1)n$. In case the sets $v_i(I^2)$ are disjoint, the sum in (5) reduces to one term, so that this condition becomes necessary too.

4.2 Translation of PIFSs in the measure formalism

The operator W on $\mathcal{F}_{\mathcal{L}}(I^2)$ associated with a PIFS can be translated in an operator τ on Borel measures on I^2 in the following way.

Theorem 2 *Let $S = ((I^2, d); w_1, w_2, \ldots, w_n)$ be a PIFS as in definition (4), such that*

$$
v_i : D_i \subseteq I^2 \to I^2 : \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e_i \\ f_i \end{bmatrix}, \qquad (7)
$$

where $a_i, b_i, c_i, d_i, e_i, f_i \in \mathbb{R}$, $A_i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}$, and $D_i \in \mathcal{L}(I^2)$ for $i = 1(1)n$. Let W be the operator on $\mathcal{F}_{\mathcal{L}}(I^2)$ associated with S, and $f \in \mathcal{F}_{\mathcal{L}}(I^2)$. Let

$$
\tau : \mathcal{M}(I^2) \to \mathcal{M}(I^2) : \mu \to \sum_{i=1}^n s_i |\det A_i| \cdot \mu v_i^{-1} + \sum_{i=1}^n o_i \cdot \lambda|_{v_i(D_i)}. \qquad (8)
$$

If $\mu \in \mathcal{M}(I^2)$ satisfies $\frac{d\mu}{d\lambda} = f$, then

$$
\frac{d\tau(\mu)}{d\lambda} = W(f).
$$

Proof. Let $B \in \mathcal{B}(I^2)$, then

$$
\begin{aligned}
\tau(\mu)(B) &= \sum_{i=1}^n s_i |\det A_i| \cdot \mu v_i^{-1}(B) + \sum_{i=1}^n o_i \cdot \lambda(B \cap v_i(D_i)) \\
&\overset{(6)}{=} \sum_{i=1}^n s_i \cdot \int \chi_B \cdot \chi_{v_i(D_i)} \cdot (f \circ v_i^{-1})\, d\lambda + \sum_{i=1}^n o_i \cdot \int \chi_B \cdot \chi_{v_i(D_i)}\, d\lambda \\
&= \int \sum_{i=1}^n \chi_B \cdot \chi_{v_i(D_i)} \cdot W(f)\, d\lambda \\
&= \int \chi_B \cdot W(f)\, d\lambda.
\end{aligned}
$$

□

The set $\mathcal{F}_{\mathcal{L}}(I^2)$ is closed under the operator W as well as w.r.t. the supremum metric. The set $\mathcal{M}(I^2)$ is closed under τ, but cannot be metrized by the Hutchinson metric.

The operator τ matches the general pattern of operators used in [7], in which a PIFS-like concept is developed within the measure formalism. In the terminology of [7][1], the terms of the first sum in the right-hand side of (8) are called the geometric parts of τ, and the terms of the second sum its massic parts.

4.3 Conclusion

Let $S = ((I^2, d); v_1, v_2, \ldots, v_n; p_1, p_2, \ldots, p_n)$ be an IFS with affine transformations v_i as defined by (4) for $i = 1(1)n$.
The major reason for introducing the IFS related concept of a PIFS, was that restricting the domains of the affine transformations v_i greatly improved on the practical capability of compressing images. However, it appears that this modification on its own does not transform the IFS S into a PIFS. Indeed, restricting the domain of v_i to $D_i \subseteq I^2$ as in (7) does not turn the Markov operator $M = \sum_{i=1}^{n} p_i \cdot \mu v_i^{-1}$ into an operator τ of the form (8): the massic parts are missing. Equivalently, the corresponding operator T in the function formalism does not match the definition of the operator W associated with a PIFS. By an analogue argument as in the proof of theorem 1, T turns out to satisfy (5), where $I(x) \triangleq \{i \in \{1, 2, \ldots, n\} \mid x \in v_i(D_i)\}$. In case $\{v_i(D_i)\}_{i=1}^{n}$ is a partition of I^2, (5) reduces to

$$T(f)|_{v_i(D_i)} = \frac{p_i}{|\det A_i|} \cdot (f \circ v_i^{-1}) \tag{9}$$

for $i = 1(1)n$, which does not agree with (2).

However, as a consequence of restricting the v_i's, the Markov operator is no longer guaranteed to preserve the amount of ink on an image. For any $\mu \in \mathcal{M}(I^2)$,

$$M(\mu)(I^2) = \sum_{i=1}^{n} p_i \mu(D_i) \leqslant \mu(I^2), \tag{10}$$

and the equality holds iff

$$\mu\left((\bigcap_{p_i > 0} D_i)^c \right) = 0. \tag{11}$$

First, this implies that, in general, $\mathcal{P}(I^2)$ is no longer closed under the Markov operator, which obviously can still be defined as an operator on $\mathcal{M}(I^2)$

$$M : \mathcal{M}(I^2) \to \mathcal{M}(I^2) : \mu \to \sum_{i=1}^{n} p_i \cdot \mu v_i^{-1}, \tag{12}$$

and, because of (10), as an operator on $\{\mu \in \mathcal{M}(I^2) \mid \mu(I^2) \leqslant 1\}$. Second, and more important, (11) puts unwanted limitations on the invariant measure of the IFS, since the equality in (10) obviously has to hold for μ equal to the invariant measure of the IFS.

[1] At least in the general terminology of [7, section 2.4.3]. In the specific setting of [7, section 5.3.4], the geometric parts only consist of contractions and translations.

Therefore, some extra ink has to be added: some massic parts have to be introduced. Comparing (12) with (8) in the measure formalism, or (9) with (2) in the function formalism, it is clear that the remedy leading to a PIFS, consists in uniformly spreading or extracting some ink ($|o_i|$ to be precise) over each of the regions $v_i(D_i)$. This way, the contractivity of the operators M and T is not affected, and, in case $\{v_i(D_i)\}_{i=1}^n$ forms a partition of I^2, the operator in the function formalism admits a practically interesting evaluation procedure based on (1), where the w_i's become 3D restricted affine transformations of the form (3) [3]. Although this modification does not prevent the dissipation of ink, and hence, does not close $\mathcal{P}(I^2)$ under the operator in the measure formalism, it does provide adequate extra degrees of freedom to enrich the set of invariant measures by compensating for $\sum_{i=1}^n o_i \cdot \lambda(v_i(D_i))$ of the ink loss [7, 3].

So, PIFSs can be seen as IFSs with restricted affine transformations in which the problem of ink dissipation is solved by uniformly spreading some ink on the transformed domains.

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