

Circumscriptive Approaches to Paraconsistent Reasoning

Ofer Arieli and Marc Denecker

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We introduce a general method for paraconsistent reasoning in the context of classical logic. A standard technique for paraconsistent reasoning on inconsistent classical theories is by shifting to multiple-valued logics. We show how these multiple-valued theories can be “shifted back” to two-valued classical theories, and how preferential reasoning based on multiple-valued logic can be represented by classical circumscription-like axioms. By applying this process we manage to overcome the shortcoming of classical logic in properly handling inconsistent data, and provide new ways of implementing multiple-valued paraconsistent reasoning. Standard multiple-valued reasoning can thus be performed through theorem provers for classical logic, and multiple-valued preferential reasoning can be implemented using algorithms for processing circumscriptive theories (such as DLS and SCAN).

Circumscriptive Approaches to Paraconsistent Reasoning

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Abstract

We introduce a general method for paraconsistent reasoning in the context of classical logic. A standard technique for paraconsistent reasoning on inconsistent classical theories is by shifting to multiple-valued logics. We show how these multiple-valued theories can be “shifted back” to two-valued classical theories, and how preferential reasoning based on multiple-valued logic can be represented by classical circumscription-like axioms. By applying this process we manage to overcome the shortcoming of classical logic in properly handling inconsistent data, and provide new ways of implementing multiple-valued paraconsistent reasoning. Standard multiple-valued reasoning can thus be performed through theorem provers for classical logic, and multiple-valued preferential reasoning can be implemented using algorithms for processing circumscriptive theories (such as DLS and SCAN).

Keywords: preferential semantics, paraconsistent reasoning, circumscription, multiple-valued logics.

1 Introduction

It is well-known that classical logic is inappropriate for imitating “common-sense” reasoning in general, and for reasoning with uncertainty in particular. Indeed, on one hand classical logic is too cautious in drawing conclusions from incomplete theories. This is so since classical logic is monotonic, thus it does not allow to retract previously drawn conclusions in light of new, more accurate information. On the other hand, classical logic is too liberal in drawing conclusions from inconsistent theories. This is explained by the fact that classical logic is not paraconsistent [7], therefore everything classically follows from a contradictory set of premises.

Preferential reasoning [26] is an elegant way to overcome classical logic’s shortcoming for reasoning on uncertainty. It is based on the idea that in order to draw conclusions from a given theory one should not consider all the models of that theory, but only a subset of *preferred models*. This subset is usually determined according to some preference criterion, which is often defined in terms of partial orders on the space of valuations. This method of preferring some models and disregarding

the others yields robust formalisms that successfully handle inconsistent data, and allow to draw intuitive conclusions from partial knowledge (see, e.g., [1, 2, 3, 4, 14, 15, 23, 24]).

In the context of classical logic, preferential semantics cannot help to overcome the problem of trivial reasoning with contradictory theories. Indeed, if a certain theory has no (two-valued) models, then it has no preferred models as well. A useful way of reasoning on contradictory classical theories is therefore by embedding them in multiple-valued logics in general, and Belnap’s four-valued logic [5, 6] in particular (which is the underlying multiple-valued semantics used here). There are several reasons for using this setting. The most important ones for our purposes are the following:

- In the context of four-valued semantics it is possible to define consequence relations that are not degenerated w.r.t. *any* theory (see, e.g., [2, 3, 23, 24, 27]); The fact that every theory has a nonempty set of four-valued models implies that four-valued reasoning may be useful for properly handling inconsistent theories. As it is shown e.g. in [2, 3], this indeed is the case.
- Analysis of four-valued models can be instructive to pinpoint the causes of the inconsistency and/or the incompleteness of the theory under consideration. (See [2, 3, 5, 6] for a detailed discussion on this property, as well as some relevant results).

However, Belnap four-valued logic has its own shortcomings:

- As in classical logic, many theories have too many models, and as a consequence the entailment relation is often too weak. In fact, since Belnap logic is weaker than classical logic w.r.t. consistent theories¹, we are even in a worse situation than in classical logic!

A (partial) solution to this problem is by using preferential reasoning in the context of multiple-valued logic.

- At the computational level, implementing paraconsistent reasoning based on four-valued semantics poses important challenges. An effective implementation of theorem provers for one of the existing proof systems for Belnap’s logic requires a major effort. The problem is even worse in the context of four-valued *preferential* reasoning, for which currently no proof systems are known.

Our goal in this paper is to show a way in which these problems can be avoided (or at least alleviated) altogether. In particular, we present a transformation back from four-valued theories to two-valued theories such that reasoning in preferential four-valued semantics can be implemented by standard theorem proving in two-valued logic. Moreover, preference criteria on four-valued theories are translated into ‘circumscriptive-like’ formulae [20, 21], and thus paraconsistent reasoning may be automatically computed by some specialized methods for compiling circumscriptive theories (such as those described in [13, 25]), and incorporated into algorithms such as SCAN [22] and DLS [8, 9], for reducing second-order formulae to their first-order equivalents.

¹That is, everything that follows in Belnap four-valued logic from a given theory also classically follows from that theory, but not vice-versa. For instance, the rule of excluded middle (either ψ or $\neg\psi$ should hold for every ψ) and the Disjunctive Syllogism (from $\psi \vee \phi$ and $\neg\phi$ conclude ψ) are not sound in Belnap four-valued logic.

In the last part of this paper we show that our approach of representing preferential considerations by higher-order formulae can be generalized to cases in which *arbitrarily many* truth values are needed (such as in probabilistic reasoning or fuzzy logics). For that we use Ginsberg/Fitting’s *bilattices* [11, 12], which are algebraic structures that naturally extend Belnap four-valued structure. It is shown that within the bilattice-based semantics one can use the same methods for syntactically representing preferences in many-valued logics.

The rest of this paper is organized as follows: In the next section we review some basic notions regarding preferential reasoning, define the current framework for such reasoning, and consider some useful preferential orders that can be defined within it. In Section 3 we describe the process of expressing preferences by second-order formulae, and in Section 4 we extend this method to general multy-valued formalisms. In Section 5 we conclude.

2 Preliminaries

2.1 Preferential reasoning

Preferential reasoning [26] is a general model theory for non-monotonic inferences. In this approach, the set of the semantical objects that describe a given theory is equipped with a preference relation that intuitively reflects some preference criterion among the given semantical objects. Inferences are then made only according to those elements that are the most-preferred ones w.r.t. the preference relation. Formally,²

Definition 2.1 A *preferential model* (w.r.t. a language Σ) is a triple $\mathcal{M} = (M, \models, \leq)$, where M is a set (of semantical objects, sometimes called *states*), \models is a relation on $M \times \Sigma$ (sometimes called the *satisfaction relation*), and \leq is a binary relation on the elements of M (sometimes called the *preference relation*).

Note that Definition 2.1 is a very general one. Some formalisms make more specific assumptions on the nature of the components of a preferential model. For instance, in the original definition of Shoham [26], each preferential model corresponds to a theory Γ , the underlying semantical objects (i.e., the elements in M) are the models of Γ w.r.t. the satisfaction relation \models , and the preferential relation \leq is a partial order on M .

Definition 2.2 Let $\mathcal{M} = (M, \models, \leq)$ be a preferential model, Γ a set of formulae in a language Σ , and $m \in M$. Then m *satisfies* Γ (notation: $m \models \Gamma$) if $m \models \gamma$ for every $\gamma \in \Gamma$. m *preferentially satisfies* Γ (alternatively, m is a *\leq -most preferred model* of Γ) if m satisfies Γ and there is no other $n \in M$ s.t. $n \leq m$ and n satisfies Γ . The set of the elements in M that preferentially satisfy Γ is denoted by $!(\Gamma, \leq)$.

Now we can define the preferential entailment relations:

²Several similar ways of defining preferential reasoning are given in the literature. Here we follow the definitions in [19].

Definition 2.3 Let $\mathcal{M} = (M, \models, \leq)$ be a preferential model, Γ a set of formulae in Σ , and ψ a formula in Σ . We say that ψ (preferentially) *follows* from Γ (alternatively, Γ preferentially *entails* ψ), if every element of $\!(\Gamma, \leq)$ satisfies ψ . We denote this by $\Gamma \models_{\leq} \psi$.

In case that M consists of the models of Γ , Definition 2.3 simply says that Γ preferentially entails ψ if every \leq -preferred model of Γ is a model of ψ .

The idea that a non-monotonic deduction should be based on some preference criterion that reflects some normality relation among the relevant semantical objects is a very natural one, and may be traced back to [20]. Furthermore, this approach is the semantical basis of some well-known general patterns for non-monotonic reasoning, introduced in [16, 17, 18, 19], and it is a key concept behind many formalisms for nonmonotonic and paraconsistent reasoning, such as Kifer and Lozinskii's RI [14, 15], Arieli and Avron's bilattice-based logics [1, 4], and Priest's LPm [23, 24]. Our purpose in this paper is to propose techniques of expressing some of the preferential relations used in these formalisms by formulae in the underlying language. Next we define the framework for doing so.

2.2 The underlying semantical structure

The formalism that we consider here is based on Belnap's four-valued algebraic structure [5, 6], denoted by $FOUR$.³ This structure is composed of four elements $FOUR = \{t, f, \perp, \top\}$, arranged in the following two lattice structures:

- $(FOUR, \leq_t)$, in which t is the maximal element, f is the minimal one, and \top, \perp are two intermediate and incomparable elements.
- $(FOUR, \leq_k)$, in which \top is the maximal element, \perp is the minimal one, and t, f are two intermediate and incomparable elements.

Here, t and f correspond to the classical truth values. The two other truth values may intuitively be understood as representing different cases of uncertainty: \top corresponds to a contradictory knowledge, and \perp corresponds to an incomplete knowledge. This interpretation of the meaning of the truth values will be useful in what follows for modeling paraconsistent reasoning.⁴ According to this interpretation, the partial order \leq_t reflects differences in the amount of *truth* that each element represents, and the partial order \leq_k reflects differences in the amount of *knowledge* that each element exhibits. A double-Hasse diagram of $FOUR$ is given in Figure 1.

In what follows we shall denote by \wedge and \vee the meet and join operations on $(FOUR, \leq_t)$, and by \otimes and \oplus the meet and the join operations on $(FOUR, \leq_k)$. A negation, \neg , is a unary operation on $FOUR$, defined by $\neg t = f$, $\neg f = t$, $\neg \top = \top$, and $\neg \perp = \perp$. As usual in such cases, we take t and \top as the *designated elements* in $FOUR$ (i.e., the elements that represent true assertions).

³See [2, 3] for some arguments in favour of using this structure as a semantical background of formalisms for common-sense reasoning.

⁴This was also the original motivation of Belnap when he introduced $FOUR$.

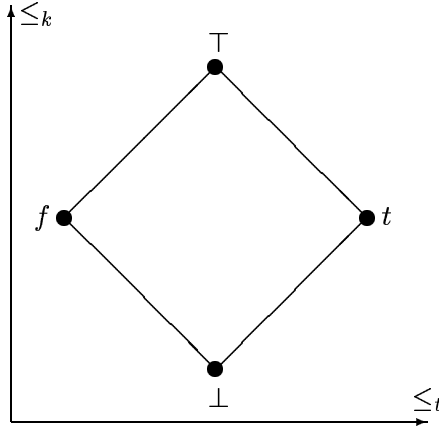


Figure 1: Belnap four-valued structure, *FOUR*

In the rest of this paper we denote by Σ a language with a finite alphabet, in which the connectives are \vee, \wedge, \neg . These connectives correspond to the operations on *FOUR* with the same notations. ν and μ denote arbitrary four-valued *valuations*, i.e., functions that assign a value in *FOUR* to every atom in Σ . The extension to complex formulae in Σ is defined in the usual way. The space of the four-valued valuations is denoted by \mathcal{V}^4 . A valuation $\nu \in \mathcal{V}^4$ is a *model* of a formula ψ (alternatively, ν *satisfies* ψ) if $\nu(\psi) \in \{t, \top\}$. ν is a model of a set Γ of formulae if ν is a model of every $\psi \in \Gamma$. The set of the models of Γ is denoted by $mod(\Gamma)$.

2.3 Four-valued preferential reasoning

A natural definition of a consequence relation on *FOUR* is the following:

Definition 2.4 Let Γ be a set of formulae and ψ a formula in Σ . Denote $\Gamma \models^4 \psi$ if every four-valued model of Γ is a four-valued model of ψ .

In [3] it is shown that \models^4 is a consequence relation in the sense of Tarski. It is also shown there that \models^4 is paraconsistent, compact, and has cut-free, sound and complete Hilbert-type and Gentzen-type proof systems. However, the fact that \models^4 is a Tarskian consequence relation means, in particular, that it is monotonic, and as such it is “over-cautious” in drawing conclusions from incomplete theories. In what follows we therefore refine the reasoning process by using the preferential techniques discussed above. Below are some useful criteria for the preferential reasoning.

Definition 2.5 [3] Let $\nu, \mu \in \mathcal{V}^4$. Denote:

- $\nu \leq_k \mu$ if $\nu(p) \leq_k \mu(p)$ for every atom p .
- $\nu \leq_{\{\top\}} \mu$ if for every atom p , $\mu(p) = \top$ whenever $\nu(p) = \top$.
- $\nu \leq_{\{\top, \perp\}} \mu$ if for every atom p , $\mu(p) \in \{\top, \perp\}$ whenever $\nu(p) \in \{\top, \perp\}$.

It is easy to check that \leq_k is a partial order and $\leq_{\{\top\}}$, $\leq_{\{\top, \perp\}}$ are pre-orders on \mathcal{V}^4 . In what follows we shall write $\nu <_k \mu$ to denote that $\nu \leq_k \mu$ and $\mu \not\leq_k \nu$; Similarly for $<_{\{\top\}}$ and $<_{\{\top, \perp\}}$.

Each one of these preference orders has its own rationality: According to \leq_k , for instance, one prefers valuations with as minimal information as reasonably possible. This is a common criterion for making preferences among different semantics of a given theory.⁵ This criterion may as well be viewed as an argumentation for consistency preserving, since as long as one keeps the amount of information (or belief) as minimal as possible, the tendency of getting into conflicts decreases.

The pre-order $\leq_{\{\top\}}$ states a somewhat more explicit preference of inconsistency minimization: It prefers those valuations that minimize the amount of inconsistent assignments. The last order given in Definition 2.5, $\leq_{\{\top, \perp\}}$, prefers those valuations that are as classical as possible. I.e., those ones that assign classical truth values whenever possible.

Given a set Γ of formulae in Σ , the minimal elements in $mod(\Gamma)$ w.r.t. \leq_k are called the *k-minimal models* of Γ .⁶ Similarly, the minimal elements of $mod(\Gamma)$ w.r.t. $\leq_{\{\top\}}$ are called the *most consistent models* of Γ , and the minimal elements of $mod(\Gamma)$ w.r.t. $\leq_{\{\top, \perp\}}$ are called the *most classical models* of Γ .

Example 2.6 Let $\Gamma = \{p, \neg p \vee q, \neg q, r \vee q\}$. The ten four-valued models of Γ are given in Table 1.

Table 1: The elements in $mod(\Gamma)$

Model No.	p	q	r
$M_1 - M_2$	\top	f	t, \top
$M_3 - M_6$	t	\top	\perp, f, t, \top
$M_7 - M_{10}$	\top	\top	\perp, f, t, \top

Thus, the *k*-minimal models of Γ are $\{M_1, M_3\}$, the most consistent ones are $\{M_1, M_3, M_4, M_5\}$, and the most classical ones are $\{M_1, M_4, M_5\}$.

Each one of the preference criteria considered in Definition 2.5 induces a corresponding preferential consequence relation. Next we define these relations:

Definition 2.7 [3] Let Γ be a set of formulae and ψ a formula in Σ . Denote:

- $\Gamma \models_k^4 \psi$ if every *k*-minimal model of Γ is a model of ψ .
- $\Gamma \models_{\{\top\}}^4 \psi$ if every most consistent model of Γ is a model of ψ .
- $\Gamma \models_{\{\top, \perp\}}^4 \psi$ if every most classical model of Γ is a model of ψ .

⁵Notable examples of formalisms that are based on the idea of \leq_k -minimization are the well-founded semantics [28] and Fitting's fixpoint semantics [10] for general logic programs.

⁶I.e., $\nu \in mod(\Gamma)$ is a *k*-minimal model of Γ if there is no $\mu \in mod(\Gamma)$ s.t. $\mu <_k \nu$.

Example 2.8 Consider again the set Γ of Example 2.6, and let $\psi = r \vee \neg r$. Then $\Gamma \models_{\{\top, \perp\}}^4 \psi$, while $\Gamma \not\models_k^4 \psi$ and $\Gamma \not\models_{\{\top\}}^4 \psi$.

Clearly, the consequence relations defined in 2.7 are particular cases of the preferential entailment relations \models_{\leq} , defined in 2.3 (see also the note after Definition 2.3). Some important properties of these relations are listed in the next proposition:

Proposition 2.9 [3] Denote by \models^2 the two-valued classical consequence relation. For every set Γ of formulae and a formula ψ in Σ ,

1. $\Gamma \models_k^4 \psi$ iff $\Gamma \models^4 \psi$.
2. If Γ is classically consistent and ψ is a formula in CNF, non of its disjuncts is a tautology, then $\Gamma \models_{\{\top\}}^4 \psi$ iff $\Gamma \models^2 \psi$.
3. If Γ is classically consistent then $\Gamma \models_{\{\top, \perp\}}^4 \psi$ iff $\Gamma \models^2 \psi$.
4. \models_k^4 , $\models_{\{\top\}}^4$, and $\models_{\{\top, \perp\}}^4$ are paraconsistent.

Note 2.10 Proposition 2.9 demonstrates the usefulness of the consequence relations considered in Definition 2.7:

- Item 1 implies that \models_k^4 is a compact representation of \models^4 ; It is sufficient to consider only the k -minimal models of a given theory in order to simulate reasoning with \models^4 .
- By item 2 it follows that in order to check whether a formula classically follows from a consistent theory Γ , it is sufficient to convert it to a conjunctive normal form, drop all the conjuncts that are tautologies, and check the remaining formula only w.r.t. the most consistent models of Γ .
- By items 3 and 4 it follows that $\models_{\{\top, \perp\}}^4$ is equivalent to classical logic on consistent theories and is nontrivial w.r.t. inconsistent theories.

A more detailed discussion on the consequence relations defined in 2.7 and some related ones appears in [3, 4]. In the next section we will show how to express the semantical considerations behind such relations by second-order formulae.

3 Paraconsistent classical reasoning

This section shows how to simulate paraconsistent reasoning by classical reasoning. We propose a transformation such that four-valued entailment for theories can be defined in terms of classical two-valued entailment for the transformed theories. Moreover, we show that four-valued preferential entailment can be defined in terms of classical entailment for the transformed theories augmented with circumscriptive axioms.

In Section 3.1 we define the relevant transformation, in Section 3.2 we use this transformation for introducing the main results, and in Section 3.3 we present some experimental results.

3.1 An alternative representation of semantical concepts

The elements of \mathcal{FOUR} can be represented by pairs of components from the two-valued lattice $(\{0, 1\}, 0 < 1)$ as follows: $t = (1, 0)$, $f = (0, 1)$, $\top = (1, 1)$, $\perp = (0, 0)$. In this representation the negation operator is defined in \mathcal{FOUR} by $\neg(x, y) = (y, x)$, and the corresponding partial orders in \mathcal{FOUR} are represented by the following rules: For every $x_1, x_2, y_1, y_2 \in \{0, 1\}$,

$$(x_1, y_1) \leq_t (x_2, y_2) \text{ iff } x_1 \leq x_2 \text{ and } y_1 \geq y_2, \quad (x_1, y_1) \leq_k (x_2, y_2) \text{ iff } x_1 \leq x_2 \text{ and } y_1 \leq y_2.$$

It follows, in particular, that in the representation by pairs of two-valued components, the basic binary operations on \mathcal{FOUR} are defined as follows:

$$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \wedge y_2), \quad (x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \vee y_2),$$

$$(x_1, y_1) \oplus (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2), \quad (x_1, y_1) \otimes (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2).$$

It is obvious that there is a one-to-one correspondence between four-valued valuations and pairs of two-valued valuations. We shall denote these pairs of two-valued components by $\nu = (\nu_1, \nu_2)$. So if, for instance, $\nu(\psi) = t$, then $\nu_1(\psi) = 1$ and $\nu_2(\psi) = 0$.

The preference criteria considered in the previous section may now be reformulated as follows:

Lemma 3.1 Let $\nu, \mu \in \mathcal{V}^4$. Then:

- $\nu \leq_k \mu$ iff for every atom p , $\nu_1(p) \leq \mu_1(p)$ and $\nu_2(p) \leq \mu_2(p)$.
- $\nu \leq_{\{\top\}} \mu$ iff for every atom p , whenever $\nu_1(p) \wedge \nu_2(p) = 1$, $\mu_1(p) \wedge \mu_2(p) = 1$ as well.
- $\nu \leq_{\{\top, \perp\}} \mu$ iff for every atom p , whenever $(\nu_1(p) \wedge \nu_2(p)) \vee (\neg \nu_1(p) \wedge \neg \nu_2(p)) = 1$, we have that $(\mu_1(p) \wedge \mu_2(p)) \vee (\neg \mu_1(p) \wedge \neg \mu_2(p)) = 1$ as well.

Proof: Immediately follows from the corresponding definitions. □

Let ψ be a formula in a language Σ . Denote by $\check{\psi}$ the formula in a language Σ^\pm , obtained from ψ by first translating ψ to its negation normal form, ψ' (where the negation operator precedes atomic formulae only),⁷ then substituting every atomic formula p that appears in a positive context in ψ' ⁸ by a new predicate symbol p^+ , and replacing every negative occurrence of an atomic formula p in ψ' , together with the negation that precedes it, by a new predicate symbol p^- . For instance, if $\psi = \neg(p \vee \neg q)$, then $\check{\psi} = p^- \wedge q^+$. Given a theory Γ , we shall write $\check{\Gamma}$ for the set $\{\check{\psi} \mid \psi \in \Gamma\}$. Note that for every Γ , $\check{\Gamma}$ is a positive theory, and hence it is consistent.

Given a four-valued valuation $\nu = (\nu_1, \nu_2)$, defined on the atoms of a language Σ , $\check{\nu}$ denotes the two-valued valuation on the atoms of Σ^\pm , defined by $\check{\nu}(p^+) = \nu_1(p)$ and $\check{\nu}(p^-) = \nu_2(p)$. Extensions to complex formulae in Σ^\pm are defined in the usual way.⁹

⁷It is easy to verify that as in the two-valued case, ψ and ψ' are logically equivalent in \mathcal{FOUR} .

⁸I.e., p is not preceded by a negation.

⁹Clearly, the converse construction is also possible: Every two-valued valuation ν on Σ^\pm corresponds to a unique four-valued valuation ν' on Σ defined, for every atom p , by $\nu'(p) = (\nu(p^+), \nu(p^-))$.

Proposition 3.2 For $\nu = (\nu_1, \nu_2)$, denote $\bar{\nu} = (-\nu_2, -\nu_1)$. Then $\nu(\psi) = (\check{\nu}(\check{\psi}), \check{\bar{\nu}}(\check{\psi}))$.

Proof: Let ψ' be the negation normal form of ψ . Since ψ and ψ' are logically equivalent in \mathcal{FOUR} , $\nu(\psi)$ is the same as $\nu(\psi')$. The rest of the proof is by an induction on the structure of ψ' :

$$\begin{aligned} \psi' = p: \quad & (\check{\nu}(\check{p}), \check{\bar{\nu}}(\check{p})) = ((\nu_1, \check{\nu}_2)(p^+), \neg(-\nu_2, \check{\nu}_1)(p^+)) = (\nu_1(p), \neg\nu_2(p)) = \\ & (\nu_1(p), \nu_2(p)) = \nu(p). \end{aligned}$$

$$\begin{aligned} \psi' = \neg p: \quad & (\check{\nu}(\check{\neg p}), \check{\bar{\nu}}(\check{\neg p})) = ((\nu_1, \check{\nu}_2)(p^-), \neg(-\nu_2, \check{\nu}_1)(p^-)) = (\nu_2(p), \neg\nu_1(p)) = \\ & (\nu_2(p), \nu_1(p)) = \neg(\nu_1(p), \nu_2(p)) = \neg\nu(p) = \nu(\neg p). \end{aligned}$$

$$\begin{aligned} \psi' = \phi_1 \vee \phi_2: \quad & \nu(\phi_1 \vee \phi_2) = \nu(\phi_1) \vee \nu(\phi_2) = (\check{\nu}(\check{\phi}_1), \check{\bar{\nu}}(\check{\phi}_1)) \vee (\check{\nu}(\check{\phi}_2), \check{\bar{\nu}}(\check{\phi}_2)) = \\ & (\check{\nu}(\check{\phi}_1) \vee \check{\nu}(\check{\phi}_2), \check{\bar{\nu}}(\check{\phi}_1) \wedge \check{\bar{\nu}}(\check{\phi}_2)) = (\check{\nu}(\check{\phi}_1 \vee \check{\phi}_2), \check{\bar{\nu}}(\check{\phi}_1 \vee \check{\phi}_2)) = \\ & (\check{\nu}(\check{\phi}_1 \vee \check{\phi}_2), \check{\bar{\nu}}(\check{\phi}_1 \vee \check{\phi}_2)) = (\check{\nu}(\phi_1 \check{\vee} \phi_2), \check{\bar{\nu}}(\phi_1 \check{\vee} \phi_2)). \end{aligned}$$

The case $\psi' = \phi_1 \wedge \phi_2$ is similar to that of $\phi_1 \vee \phi_2$. □

3.2 Simulating (preferential) four-valued reasoning by classical logic

In what follows we use the pairwise representations, considered in the previous section, for the following goals:

1. Showing that four-valued reasoning can be simulated by classical reasoning,
2. Constructing circumscribing formulae for defining four-valued preferential reasoning.

For item (1) above we first need the following lemma:

Lemma 3.3 For every four-valued valuation ν and a formula ψ in Σ , $\nu(\psi)$ is designated iff $\check{\nu}(\check{\psi}) = 1$.

Proof: $\nu(\psi)$ is designated iff $\nu_1(\psi) = 1$, iff (Proposition 3.2) $\check{\nu}(\check{\psi}) = 1$. □

The following result is an immediate corollary of Lemma 3.3:

Theorem 3.4 $\Gamma \models^4 \psi$ iff $\check{\Gamma} \models^2 \check{\psi}$.

It follows, therefore, that four-valued reasoning may be implemented by two-valued theorem provers (by using, e.g., the pointwise transformation of the previous section). Another immediate consequence of this theorem is the next well-known result:

Corollary 3.5 In positive logic (i.e., in the language without negations), $\Gamma \models^4 \psi$ iff $\Gamma \models^2 \psi$.

We turn now to the preferential case. Denote by $\mathcal{A}(\phi)$ the set of atomic formulae that appear in ϕ , and let ψ be a formula in Σ for which $\mathcal{A}(\psi) = \{p_1, \dots, p_n\}$. We shall write \vec{p}^\pm to denote the set $\{p_1^+, p_1^-, \dots, p_n^+, p_n^-\}$, and $\Psi(\vec{p}^\pm)$ denotes a formulae Ψ such that the elements in \vec{p}^\pm are its only atomic formulae.

Given a four-valued valuation $\nu = (\nu_1, \nu_2)$, denote by $(\vec{p}^\pm : \nu)$ the two-valued valuation on \vec{p}^\pm , defined for every $1 \leq i \leq n$ by $(\vec{p}^\pm : \nu)(p_i^+) = \nu_1(p_i)$ and $(\vec{p}^\pm : \nu)(p_i^-) = \nu_2(p_i)$.

Definition 3.6 A preferential order \leq is *represented* by a formula Ψ_{\leq} if for every four-valued valuations ν and μ we have that $\nu \leq \mu$ iff $(\vec{p}^{\pm} : \nu, \vec{P}^{\pm} : \mu)$ satisfies $\Psi_{\leq}(\vec{p}^{\pm}, \vec{P}^{\pm})$.

Proposition 3.7 Let $\Psi_{\leq}(\vec{p}^{\pm}, \vec{P}^{\pm})$ be a formula that represents a preferential order \leq . Then ν is a \leq -most preferred model of ψ (that is, $\nu \in !(\psi, \leq)$) iff $\check{\nu}$ satisfies $\check{\psi}$ and the following formula:

$$\text{Circ}_{\leq}(\vec{p}^{\pm}) = \forall(\vec{P}^{\pm}) \{ \check{\psi}(\vec{P}^{\pm}) \rightarrow (\Psi_{\leq}(\vec{P}^{\pm}, \vec{p}^{\pm}) \rightarrow \Psi_{\leq}(\vec{p}^{\pm}, \vec{P}^{\pm})) \}.$$

Proof: By Corollary 3.3, ν is a model of ψ iff $\check{\nu}$ satisfies $\check{\psi}$. It remains to show that the fact that $\check{\nu}$ satisfies Circ_{\leq} is a necessary and sufficient condition for assuring that ν is a \leq -minimal element in $\text{mod}(\psi)$. Indeed, $\check{\nu}$ satisfies Circ_{\leq} iff for every valuation μ that satisfies ψ and for which $\mu \leq \nu$, it is also true that $\nu \leq \mu$. Thus, for every $\mu \in \text{mod}(\psi)$, we have that $(\mu \leq \nu) \rightarrow (\nu \leq \mu)$. This is equivalent to the fact that for every $\mu \in \text{mod}(\psi)$, $\neg[(\mu \leq \nu) \wedge \neg(\nu \leq \mu)]$, i.e., there is no $\mu \in \text{mod}(\psi)$, s.t. $(\mu \leq \nu) \wedge \neg(\nu \leq \mu)$. But $\mu < \nu$ iff $(\mu \leq \nu) \wedge \neg(\nu \leq \mu)$, and so we have that there is no $\mu \in \text{mod}(\psi)$ s.t. $\mu < \nu$. I.e., $\nu \in !(\psi, \leq)$. \square

Note 3.8 Let $\Psi_{<}(\vec{P}^{\pm}, \vec{p}^{\pm}) = \Psi_{\leq}(\vec{P}^{\pm}, \vec{p}^{\pm}) \wedge \neg\Psi_{\leq}(\vec{p}^{\pm}, \vec{P}^{\pm})$,¹⁰ and denote by $\vec{p}^{\pm} = \vec{P}^{\pm}$ the formula $\bigwedge_{i=1}^n ((p_i^+ = P_i^+) \wedge (p_i^- = P_i^-))$. Then

a) The formula Circ_{\leq} of Proposition 3.7 may be rewritten as follows:

$$\text{Circ}_{\leq}(\vec{p}^{\pm}) = \forall(\vec{P}^{\pm}) \{ \check{\psi}(\vec{P}^{\pm}) \rightarrow \neg\Psi_{<}(\vec{P}^{\pm}, \vec{p}^{\pm}) \}$$

b) In case that \leq is a partial order, Circ_{\leq} can be rewritten as follows:

$$\text{Circ}_{\leq}(\vec{p}^{\pm}) = \forall(\vec{P}^{\pm}) \{ [\check{\psi}(\vec{P}^{\pm}) \wedge \Psi_{<}(\vec{P}^{\pm}, \vec{p}^{\pm})] \rightarrow \vec{p}^{\pm} = \vec{P}^{\pm} \}$$

The following theorem is an immediate corollary of Proposition 3.7:

Theorem 3.9 Let Γ be a set of formula and ψ a formula in Σ . Let Circ_{\leq} be the formula given in Proposition 3.7 for a preferential relation \leq . Then $\Gamma \models_{\leq} \psi$ iff $\check{\Gamma} \cup \text{Circ}_{\leq} \models^2 \check{\psi}$.

Proposition 3.7 gives a general characterization in terms of “formula circumscription” [21] of the preferred models of a given theory: Given a preferential relation \leq , in order to express \leq -preferential satisfaction of a theory, one should first formulate a corresponding formula Ψ_{\leq} that represents \leq , and then integrate Ψ_{\leq} with Circ_{\leq} as in Proposition 3.7.

Next we define formulae that represent the preferential relations considered above. For that, we shall need the following notations:

Notation 3.10 In what follows we shall write $x \preceq y$ for $x \rightarrow y$, and $x \prec y$ for $(x \rightarrow y) \wedge \neg(y \rightarrow x)$.¹¹

Lemma 3.11 Let n be the number of different atomic formulae in Σ . Then:

¹⁰It is easy to see that for every four-valued valuations μ and ν , $\mu < \nu$ iff $(\vec{P}^{\pm} : \mu, \vec{p}^{\pm} : \nu)$ satisfies $\Psi_{<}(\vec{P}^{\pm}, \vec{p}^{\pm})$.

¹¹Thus, $x \prec y = (x \preceq y) \wedge \neg(y \preceq x)$.

a) The preferential relation \leq_k is represented by the following formula:

$$\Psi_{\leq_k}(\vec{p}^\pm, \vec{P}^\pm) = \bigwedge_{i=1}^n ((p_i^+ \preceq P_i^+) \wedge (p_i^- \preceq P_i^-))$$

b) The preferential relation $\leq_{\{\top\}}$ is represented by the following formula:

$$\Psi_{\leq_{\{\top\}}}(\vec{p}^\pm, \vec{P}^\pm) = \bigwedge_{i=1}^n ((p_i^+ \wedge p_i^-) \preceq (P_i^+ \wedge P_i^-))$$

c) The preferential relation $\leq_{\{\top, \perp\}}$ is represented by the following formula:

$$\Psi_{\leq_{\{\top, \perp\}}}(\vec{p}^\pm, \vec{P}^\pm) = \bigwedge_{i=1}^n (((p_i^+ \wedge p_i^-) \vee (\neg p_i^+ \wedge \neg p_i^-)) \preceq ((P_i^+ \wedge P_i^-) \vee (\neg P_i^+ \wedge \neg P_i^-)))$$

Proof: We show only part (a); The proof of the other parts is similar. By Proposition 3.2,

$$\begin{aligned} \nu \leq_k \mu &\iff \forall 1 \leq i \leq n \quad \nu(p_i) \leq_k \mu(p_i) \\ &\iff \forall 1 \leq i \leq n \quad (\check{\nu}(\check{p}_i), \neg \check{\nu}(\check{p}_i)) \leq_k (\check{\mu}(\check{p}_i), \neg \check{\mu}(\check{p}_i)) \\ &\iff \forall 1 \leq i \leq n \quad \check{\nu}(\check{p}_i) \leq \check{\mu}(\check{p}_i) \text{ and } \neg \check{\nu}(\check{p}_i) \leq \neg \check{\mu}(\check{p}_i) \\ &\iff \forall 1 \leq i \leq n \quad \nu_1(p_i) \leq \mu_1(p_i) \text{ and } \nu_2(p_i) \leq \mu_2(p_i) \\ &\iff (\vec{p}^\pm : \nu, \vec{P}^\pm : \mu) \text{ satisfies } \bigwedge_{i=1}^n ((p_i^+ \preceq P_i^+) \wedge (p_i^- \preceq P_i^-)) \\ &\iff (\vec{p}^\pm : \nu, \vec{P}^\pm : \mu) \text{ satisfies } \Psi_{\leq_k}(\vec{p}^\pm, \vec{P}^\pm) \quad \square \end{aligned}$$

By Proposition 3.7, Lemma 3.11(a), and Note 3.8(b), we have the following corollary:

Corollary 3.12 A valuation $\nu = (\nu_1, \nu_2)$ is a k -minimal model of ψ iff $\check{\nu}$ satisfies $\check{\psi}$ and $\text{Circ}_{\leq_k}(\vec{p}^\pm)$, where $\text{Circ}_{\leq_k}(\vec{p}^\pm)$ is the following formula:¹²

$$\forall(\vec{P}^\pm) \{ [\check{\psi}(\vec{P}^\pm) \wedge \bigwedge_{i=1}^n ((P_i^+ \preceq p_i^+) \wedge (P_i^- \preceq p_i^-))] \rightarrow [\bigwedge_{i=1}^n ((P_i^+ = p_i^+) \wedge (P_i^- = p_i^-))] \}$$

As in Corollary 3.12, the most consistent models and the most classical models of a given theory can be represented by formulae of the form $\text{Circ}_{\leq_{\{\top\}}}(\vec{p}^\pm)$ and $\text{Circ}_{\leq_{\{\top, \perp\}}}(\vec{p}^\pm)$, obtained by respectively integrating the formulae given in parts (b) and (c) of Lemma 3.11 with Circ_{\leq} , given in Proposition 3.7.

In what follows we consider a uniform way of representing $\text{Circ}_{\leq_{\{\top\}}}(\vec{p}^\pm)$, $\text{Circ}_{\leq_{\{\top, \perp\}}}(\vec{p}^\pm)$, and some other formulae that correspond to preferential criteria like $\leq_{\{\top\}}$ and $\leq_{\{\top, \perp\}}$. For this, let $\Delta \subseteq \text{FOUR}$. Define an order relation $<_\Delta$ on FOUR by $x <_\Delta y$ iff $x \notin \Delta$ while $y \in \Delta$. A corresponding pre-order on \mathcal{V}^4 may now be defined as follows: For every $\nu, \mu \in \mathcal{V}^4$, $\nu \leq_\Delta \mu$ iff for every atom p ,

¹²Note that $\text{Circ}_{\leq_k}(\vec{p}^\pm)$ is a standard circumscriptive axiom in the sense of [20].

the fact that $\nu(p) \in \Delta$ entails that $\mu(p) \in \Delta$ as well. The \leq_{Δ} -most preferred models of Γ are the \leq_{Δ} -minimal elements in $mod(\Gamma)$, and $\Gamma \models_{\Delta}^4 \psi$ if every \leq_{Δ} -most preferred model of Γ is a model of ψ .

Clearly, $\leq_{\{\top\}}$ and $\leq_{\{\top, \perp\}}$ are particular cases of \leq_{Δ} , where $\Delta = \{\top\}$ and $\Delta = \{\top, \perp\}$, respectively. Now, the \leq_{Δ} -most preferred models of a given theory can be represented by a circumscriptive formula in the following way:

Notation 3.13 For $\Delta \subseteq FOUR$, let $\Lambda_{\Delta}(p^+, p^-) = \bigvee_{x \in \Delta} \Lambda_x(p^+, p^-)$, where $\Lambda_i(p^+, p^-) = p^+ \wedge \neg p^-$, $\Lambda_f(p^+, p^-) = \neg p^+ \wedge p^-$, $\Lambda_{\perp}(p^+, p^-) = \neg p^+ \wedge \neg p^-$, and $\Lambda_{\top}(p^+, p^-) = p^+ \wedge p^-$.

Similar arguments as those in Lemma 3.11 show that the formula

$$\Psi_{\leq_{\Delta}}(\vec{p}^{\pm}, \vec{P}^{\pm}) = \bigwedge_{i=1}^n (\Lambda_{\Delta}(p_i^+, p_i^-) \preceq \Lambda_{\Delta}(P_i^+, P_i^-))$$

represents the preferential relation \leq_{Δ} . Therefore, by Proposition 3.7,

Proposition 3.14 A valuation ν is a \leq_{Δ} -preferred model of ψ iff $\check{\nu}$ satisfies $\check{\psi}$ and the following formula:

$$\text{Circ}_{\leq_{\Delta}}(\vec{p}^{\pm}) = \forall(\vec{P}^{\pm}) \{ \check{\psi}(\vec{P}^{\pm}) \rightarrow (\Psi_{\leq_{\Delta}}(\vec{P}^{\pm}, \vec{p}^{\pm}) \rightarrow \Psi_{\leq_{\Delta}}(\vec{p}^{\pm}, \vec{P}^{\pm})) \}.$$

3.3 Experimental study

As we have already noted, all the formulae that are obtained by our method have a circumscriptive form. It is therefore possible to apply, for instance, the formula Circ_{\leq_k} , given in Corollary 3.12, in algorithms for reducing circumscriptive axioms. Below are some simple results obtained by experimenting with such algorithm (We have used Doherty, Lukaszewicz and Szalas DLS algorithm [8, 9], available at <http://www.ida.liu.se/labs/kplab/projects/dls/circ.html>).¹³

- Consider the theory $\Gamma = \{Q(a), Q(b), \neg Q(a)\}$, where Q denotes some predicate, and a, b are two constants. In our context, this theory is translated to $\check{\Gamma} = \{Q^+(a), Q^+(b), Q^-(a)\}$. Circumscribing $\check{\Gamma}$ where Q^+ and Q^- are simultaneously minimized, yields the following result:

$$\forall x \{ (Q^-(x) \rightarrow x = a) \wedge (Q^+(x) \rightarrow (x = a \vee x = b)) \}.$$

It follows, then, that a is the only object for which both $Q^+(x)$ and $Q^-(x)$ hold (i.e., a is the only object that is inconsistent w.r.t. Q), and b is the only object for which only $Q^+(x)$ holds. For all the other objects neither $Q^+(x)$ nor $Q^-(x)$ holds. I.e., if $c \notin \{a, b\}$ then $Q(c)$ corresponds to \perp . This indeed is exactly the k -minimal semantics of Γ .

¹³In what follows we deliberately consider some very simple cases. Our experience is that for more complex theories the output quickly becomes more complicated. Although this is useful for automated computations, it is much less comprehensible by humans.

- Suppose that in the previous example one wants to impose a three-valued semantics. It is possible to do so by adding to Γ the restriction $\psi = \forall x(Q(x) \vee \neg Q(x))$, which is translated to $\check{\psi} = \forall x(Q^+(x) \vee Q^-(x))$. Circumscribing $\check{\Gamma} \cup \{\check{\psi}\}$ yields

$$\forall x \{ [(Q^+(x) \wedge x \neq a \wedge x \neq b) \rightarrow \neg Q^-(x)] \wedge [(Q^-(x) \wedge x \neq a) \rightarrow \neg Q^+(x)] \},$$

which has almost the same meaning as before, except that this time if $c \notin \{a, b\}$ then either $Q^+(c)$ or $Q^-(c)$ holds, but not both. It follows, then, that for such c , $Q(c)$ must have some classical value. Again, this corresponds to what one expects when k -minimizing $\Gamma \cup \{\psi\}$.

4 Using more than four values

In this section we extend the results obtained in Section 3 to cases in which the underlying semantical structures may contain more than four elements. The basic idea is to consider structures in which the truth values are represented by pairs of elements, and each element in one out of *arbitrarily many* possible values (rather than $\{0, 1\}$, as in the case of *FOUR*).

Definition 4.1 [12] Let $\mathcal{L} = (L, \leq_L)$ be a complete lattice. The structure $\mathcal{L} \odot \mathcal{L} = (L \times L, \leq_t, \leq_k, \neg)$ consists of pairs of elements from L that are arranged in two lattice structures as follows:

- $(L \times L, \leq_t)$, where $(y_1, y_2) \geq_t (x_1, x_2)$ iff $y_1 \geq_L x_1$ and $y_2 \leq_L x_2$
- $(L \times L, \leq_k)$, where $(y_1, y_2) \geq_k (x_1, x_2)$ iff $y_1 \geq_L x_1$ and $y_2 \geq_L x_2$

The unary operation \neg is defined on $L \times L$ by $\neg(x_1, x_2) = (x_2, x_1)$.

The structure that $\mathcal{L} \odot \mathcal{L}$ forms is called a *bilattice* [11, 12]. A truth value $(x, y) \in \mathcal{L} \odot \mathcal{L}$ may intuitively be understood so that x represents the amount of evidence *for* an assertion, while y represents the amount of evidence *against* it. It is easy to verify that the \leq_k -minimal element of $\mathcal{L} \odot \mathcal{L}$ is $(\inf(L), \inf(L))$, the \leq_k -maximal one is $(\sup(L), \sup(L))$, the \leq_t -minimal element is $(\inf(L), \sup(L))$, and the \leq_t -maximal one is $(\sup(L), \inf(L))$.

Example 4.2 Belnap four-valued lattice *FOUR*, considered in the previous sections, is a particular case of the algebraic structures defined in 4.1, since $\text{FOUR} = \text{TWO} \odot \text{TWO}$, where *TWO* is the two-valued classical lattice. For another example, consider the three-valued lattice *THREE* = $(\{0, \frac{1}{2}, 1\}, 0 < \frac{1}{2} < 1)$. Figure 2 contains a double-Hasse diagram of $\text{THREE} \odot \text{THREE}$.

In what follows we shall continue to use the symbols \vee, \wedge, \oplus , and \otimes for denoting, respectively, the \leq_t -join, \leq_t -meet, \leq_k -join, and the \leq_k -meet operations in $\mathcal{L} \odot \mathcal{L}$. By Definition 4.1 it follows that these operations are computed as in *FOUR*. I.e., for every $x_1, x_2, y_1, y_2 \in L$,

$$\begin{aligned} (x_1, y_1) \vee (x_2, y_2) &= (x_1 \vee x_2, y_1 \wedge y_2), & (x_1, y_1) \wedge (x_2, y_2) &= (x_1 \wedge x_2, y_1 \vee y_2), \\ (x_1, y_1) \oplus (x_2, y_2) &= (x_1 \vee x_2, y_1 \vee y_2), & (x_1, y_1) \otimes (x_2, y_2) &= (x_1 \wedge x_2, y_1 \wedge y_2). \end{aligned}$$

As noted in Example 4.2, Definition 4.1 is a natural extension of Belnap four-valued structure. The notion of the designated values in *FOUR* can also be generalized in a natural way in $\mathcal{L} \odot \mathcal{L}$:

Definition 4.3 [1] Let $\mathcal{L} \odot \mathcal{L}$ be the bilattice defined in 4.1.

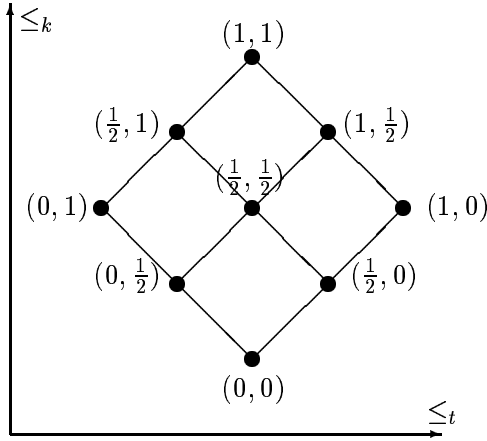


Figure 2: $THREE \odot THREE$

- a) A *bifilter* \mathcal{D} of $\mathcal{L} \odot \mathcal{L}$ is a nonempty proper subset of $L \times L$, such that:
 (i) $x \wedge y \in \mathcal{D}$ iff $x \in \mathcal{D}$ and $y \in \mathcal{D}$, (ii) $x \otimes y \in \mathcal{D}$ iff $x \in \mathcal{D}$ and $y \in \mathcal{D}$.
- b) A bifilter \mathcal{D} is called *prime*, if it also satisfies the following conditions:
 (i) $x \vee y \in \mathcal{D}$ iff $x \in \mathcal{D}$ or $y \in \mathcal{D}$, (ii) $x \oplus y \in \mathcal{D}$ iff $x \in \mathcal{D}$ or $y \in \mathcal{D}$.

Given a bilattice of the form $\mathcal{L} \odot \mathcal{L}$, we fix some prime bifilter \mathcal{D} in $\mathcal{L} \times \mathcal{L}$. This set consists of the *designated elements* of the underlying logic (i.e., those that represent true assertions), and as usual it is used for defining validity of formulae: A valuation ν on $\mathcal{L} \times \mathcal{L}$ is a *model* of a set Γ of formulae if $\nu(\psi) \in \mathcal{D}$ for every $\psi \in \Gamma$.

Example 4.4 The set $\mathcal{D} = \{t, \top\}$ of the designated elements in $FOUR$ is indeed a prime bifilter in $FOUR$ (and, moreover, it is the only prime bifilter in this bilattice). In $THREE \odot THREE$ there are two prime bifilters:

- $\mathcal{D}_1 = \{(1, x) \mid x \in \{0, \frac{1}{2}, 1\}\} = \{(x_1, x_2) \mid (x_1, x_2) \geq_k (1, 0)\}$, and
- $\mathcal{D}_2 = \{(x_1, x_2) \mid x_1 \geq \frac{1}{2}, x_2 \in \{0, \frac{1}{2}, 1\}\} = \{(x_1, x_2) \mid (x_1, x_2) \geq_k (\frac{1}{2}, 0)\}$.

Proposition 4.5 [4]

- a) \mathcal{D} is a bifilter in $\mathcal{L} \odot \mathcal{L}$ iff $\mathcal{D} = \mathcal{D}_{\mathcal{L}} \times \mathcal{L}$, where $\mathcal{D}_{\mathcal{L}}$ is a filter in \mathcal{L} .
- b) \mathcal{D} is a prime bifilter in $\mathcal{L} \odot \mathcal{L}$ iff $\mathcal{D} = \mathcal{D}_{\mathcal{L}} \times \mathcal{L}$, where $\mathcal{D}_{\mathcal{L}}$ is a prime filter in \mathcal{L} .

Corollary 4.6 [4] Let $x_0 \in L$, $x_0 \neq \inf(L)$. Denote: $\mathcal{D}(x_0) = \{(y_1, y_2) \mid y_1 \geq_L x_0, y_2 \in L\}$, and $\mathcal{D}_{\mathcal{L}}(x_0) = \{y \in L \mid y \geq_L x_0\}$. Then:

- a) $\mathcal{D}(x_0)$ is a prime bifilter in $\mathcal{L} \odot \mathcal{L}$ iff $\mathcal{D}_{\mathcal{L}}(x_0)$ is a prime filter in \mathcal{L} .
- b) $\mathcal{D}(\sup(L))$ is a prime bifilter in $\mathcal{L} \odot \mathcal{L}$ iff $\sup(L)$ is join irreducible (i.e., iff $x \vee_{\mathcal{L}} y = \sup(L)$ implies that $x = \sup(L)$ or $y = \sup(L)$).

- c) If the condition of item (b) is met, then $\mathcal{D}(\text{sup}(L))$ is minimal among the (prime) bifilters of $\mathcal{L} \odot \mathcal{L}$.

For constructing the circumscribing formulae in the L -valued case we still have to define some implication connectives on L , and describe the process of “splitting” L -valued valuations. This is what we do next.

Definition 4.7 Let $\mathcal{L}=(L, \leq_L)$ be a lattice and $\mathcal{D}_{\mathcal{L}}$ a (prime) filter in it. For every $x, y \in L$ define:

- a) $x \rightarrow y = t$ if either $x \notin \mathcal{D}_{\mathcal{L}}$ or $y \in \mathcal{D}_{\mathcal{L}}$, otherwise $x \rightarrow y = f$.
b) $x \preceq y = t$ if $x \leq_L y$, otherwise $x \preceq y = f$.
c) $x \prec y = t$ if $x <_L y$, otherwise $x \prec y = f$.¹⁴

In order to construct a formula of the form $\check{\psi}$ from a given formula ψ in Σ , as we have done in the four-valued case, we first have to make sure that ψ is logically equivalent to its negation normal form ψ' . This is the case, for instance, when $\mathcal{L} \odot \mathcal{L}$ is distributive (i.e., all the twelve distributive laws concerning \wedge, \vee, \otimes , and \oplus hold). In [12] it is shown that if \mathcal{L} is distributive then so is $\mathcal{L} \odot \mathcal{L}$. So from now on we shall assume that \mathcal{L} is distributive. As a result, for every formula ψ in Σ we can define its “splitted form”, $\check{\psi}$, in a way that is completely analogous to that in the four-valued case. Also, since every valuation ν on $\mathcal{L} \odot \mathcal{L}$ can be represented by a pair (ν_1, ν_2) of L -valued components, then $\check{\nu}$ is an L -valued valuation defined (just as in the four-valued case) by $\check{\nu}(p^+) = \nu_1(p)$ and $\check{\nu}(p^-) = \nu_2(p)$.

Using the notations above we can now generalize Proposition 3.2 to the case of $\mathcal{L} \odot \mathcal{L}$ as follows:

Proposition 4.8 For a valuation $\nu = (\nu_1, \nu_2)$ on $\mathcal{L} \odot \mathcal{L}$, let $\bar{\nu} = (-\nu_2, -\nu_1)$. Then $\nu(\psi) = (\check{\nu}(\check{\psi}), -\check{\bar{\nu}}(\check{\psi}))$.

The proof of Proposition 4.8 is similar to that of Proposition 3.2, using $\mathcal{L} \odot \mathcal{L}$ instead of \mathcal{FOUR} . The next two corollaries immediately follow from Proposition 4.8.

Corollary 4.9 Let $\mathcal{D}_{\mathcal{L}}$ be a prime filter in \mathcal{L} , and let $\mathcal{D} = \mathcal{D}_{\mathcal{L}} \times \mathcal{L}$ be the set of the designated elements in $\mathcal{L} \odot \mathcal{L}$.¹⁵ For every valuation ν on $\mathcal{L} \odot \mathcal{L}$ and a formula ψ in Σ , $\nu(\psi) \in \mathcal{D}$ iff $\check{\nu}(\check{\psi}) \in \mathcal{D}_{\mathcal{L}}$.

Proof: $\nu(\psi)$ is designated iff $\nu_1(\psi) \in \mathcal{D}_{\mathcal{L}}$, iff (Proposition 4.8) $\check{\nu}(\check{\psi}) \in \mathcal{D}_{\mathcal{L}}$. □

In particular, since by Proposition 4.5(b) every prime bifilter in $\mathcal{L} \times \mathcal{L}$ is of the form $\mathcal{D}_{\mathcal{L}} \times \mathcal{L}$, where $\mathcal{D}_{\mathcal{L}}$ is a prime filter in \mathcal{L} , we have that whenever $\nu(\psi)$ is designated in $\mathcal{L} \times \mathcal{L}$, there is a prime filter in \mathcal{L} with respect to which $\check{\nu}(\check{\psi})$ is designated in \mathcal{L} , and vice-versa.

Corollary 4.10 Suppose that $\text{sup}(L)$ is join irreducible in \mathcal{L} , and let $\mathcal{D} = \mathcal{D}(\text{sup}(L))$ be the set of the designated elements in $\mathcal{L} \odot \mathcal{L}$.¹⁶ For every valuation ν in $\mathcal{L} \odot \mathcal{L}$ and a formula ψ in Σ , $\nu(\psi) \in \mathcal{D}$ iff $\check{\nu}(\check{\psi}) = \text{sup}(L)$.

¹⁴Note that this is a generalization of the definition of the operators with the same notations, given in Notation 3.10. In particular, when L is the two-valued lattice, \rightarrow and \preceq are the same as the classical implication.

¹⁵By Proposition 4.5, \mathcal{D} is indeed a prime bifilter in $\mathcal{L} \odot \mathcal{L}$.

¹⁶By Corollary 4.6, \mathcal{D} is indeed a prime bifilter in $\mathcal{L} \odot \mathcal{L}$.

Proof: $\nu(\psi)$ is designated iff $\nu_1(\psi) = \text{sup}(L)$, iff (Proposition 4.8) $\check{\nu}(\check{\psi}) = \text{sup}(L)$. \square

Let $\mathcal{L} = (L, \leq_L)$ be a distributive lattice with a prime filter $\mathcal{D}_{\mathcal{L}}$. Using the same method as that of Section 3 for the four-valued case, it is now possible to define second-order formulae in an L -valued semantics in order to represent preferential reasoning in $\mathcal{L} \odot \mathcal{L}$ with $\mathcal{D} = \mathcal{D}_{\mathcal{L}} \times \mathcal{L}$. Below are few examples:

- As in Proposition 3.12, the set of the k -minimal models of ψ in $\mathcal{L} \odot \mathcal{L}$ can be represented by $\text{Circ}_{\leq k}$.¹⁷ A valuation ν in $\mathcal{L} \odot \mathcal{L}$ is a k -minimal model of ψ iff $\check{\nu}(\check{\psi}) \in \mathcal{D}_{\mathcal{L}}$ and $\check{\nu}(\text{Circ}_{\leq k}) \in \mathcal{D}_{\mathcal{L}}$.

- Let $\mathcal{I}_{\top} = \{x \in \mathcal{L} \times \mathcal{L} \mid x \in \mathcal{D}, \neg x \in \mathcal{D}\}$ be the set of the *inconsistent* values in $\mathcal{L} \odot \mathcal{L}$. A valuation ν_1 is (strictly) *more consistent* than a valuation ν_2 if the set of atoms p_i s.t. $\nu_1(p_i) \in \mathcal{I}_{\top}$ is (strictly) subsumed in the set of atoms p_j s.t. $\nu_2(p_j) \in \mathcal{I}_{\top}$. A valuation $\nu \in \text{mod}(\psi)$ is a *most consistent model* of ψ if there is no other model of ψ that is strictly more consistent than ν .

As in the four-valued case, the set of the most consistent models of ψ can be represented by $\text{Circ}_{\leq \{\top\}}$: A valuation ν in $\mathcal{L} \odot \mathcal{L}$ is a most consistent model of ψ iff $\check{\nu}$ satisfies $\check{\psi}$ and $\text{Circ}_{\leq \{\top\}}$.

- Let \mathcal{I}_{\top} be the set of the inconsistent elements in $\mathcal{L} \odot \mathcal{L}$ as in the previous item, and let $\mathcal{I}_{\perp} = \{x \in \mathcal{L} \times \mathcal{L} \mid x \notin \mathcal{D}, \neg x \notin \mathcal{D}\}$ be the set of the *incomplete values* in $\mathcal{L} \odot \mathcal{L}$. A valuation ν_1 is (strictly) *more classical* than a valuation ν_2 if the set of atoms p_i s.t. $\nu_1(p_i) \in \mathcal{I}_{\top} \cup \mathcal{I}_{\perp}$ is (strictly) subsumed in the set of atoms p_j s.t. $\nu_2(p_j) \in \mathcal{I}_{\top} \cup \mathcal{I}_{\perp}$. A valuation $\nu \in \text{mod}(\psi)$ is a *most classical model* of ψ if there is no other model of ψ that is strictly more classical than ν .

As in the four-valued case, the set of the most classical models of ψ can be represented by $\text{Circ}_{\leq \{\top, \perp\}}$: A valuation ν in $\mathcal{L} \odot \mathcal{L}$ is a most classical model of ψ iff $\check{\nu}$ satisfies $\check{\psi}$ and $\text{Circ}_{\leq \{\top, \perp\}}$.

The proof of each one of the cases above is similar to that of Proposition 3.12, using Proposition 4.8 and Corollary 4.9 instead of Proposition 3.2 and Corollary 3.3.

Particular cases in which the representations above may be used are the bilattice-based logics introduced in [2, 3, 4], and the annotated logic [27] RI, introduced in [14, 15], provided that the underlying many-valued structure is of the form $\mathcal{L} \odot \mathcal{L}$.

5 Conclusion

In this paper we have introduced a method of representing paraconsistent reasoning by classical second-order formulae. This method touches upon several different aspects:

- Firstly, it demonstrates the usefulness of circumscription not only as a general method for non-monotonic reasoning, but also as an appealing technique for implementing paraconsistent reasoning.
- Secondly, this approach shows that two-valued reasoning may be useful for simulating inference procedures in the context of many-valued semantics.

¹⁷Where \rightarrow , \preceq and \prec are interpreted as in Definition 4.7.

- On the other hand, our approach uses many-valued semantics as a tool for dealing with the shortcoming of classical logic in properly handling inconsistent and incomplete information.
- Finally, this is another evidence to the fact that in many cases concepts that are defined in a “meta-language” (such as preference criteria, etc.) can be expressed in the language itself (using, e.g., higher-order formulae). This enables a potentially wide area for practical implementations. For instance, we have shown that preferential reasoning can be reduced to a level in which it can be incorporated with practical applications for automated reasoning and theorem proving.

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